On conjectures of Reed

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Reedfest

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Brooks’ theorem

Theorem (Brooks 1941)

\[ \chi(G) \leq \Delta(G) \text{ unless } G = K_{\Delta(G)+1} \text{ or } G \text{ is an odd cycle.} \]
Vizing’s problem

Vizing, “Some unsolved problems in graph theory” (1968):

Можно ли для любого наперед заданного натурального числа $k \geq 2$ построить граф со сколь угодно большим обхватом и с хроматическим числом $k$? Можно. Это доказал П. Эрдёш [39], основываясь на мощностных соображениях. Удивительно, что до сих пор нет конструктивного доказательства этого факта. В [40] указан способ построения графов с любым хроматическим числом без циклов длины $\leq 7$. Это лучшее, что мы имеем на сегодняшний день.

Если $\sigma(L)$ — максимальная степень вершины графа $L$, то, очевидно, $\gamma(L) \leq \sigma(L) + 1$. В 1941 г. Р. Брукс [41] доказал, что при $\sigma(L) \geq 3$ и $\omega(L) \leq \sigma(L)$ справедлива оценка $\gamma(L) \leq \sigma(L)$. Дальнейшие исследования можно проводить, учитывая более точно соотношения между $\sigma$ и $\omega$. Пожалуй, следует начать с оценки хроматического числа графа без треугольников ($\omega = 2$) с данной максимальной степенью вершины.
Vizing, "Some unsolved problems in graph theory" (1968):

Given a natural number \( k \geq 2 \), is it possible to construct a graph with arbitrarily large compass and with chromatic number \( k \)? Erdős [39] has proved this; his proof is based on counting arguments. It is astonishing that no constructive proof for this fact has yet been given. In [40] a method is given of constructing graphs of arbitrary chromatic number without having cycles of length \(< 7\). This is the best we have at the present time.

If \( \sigma(L) \) is the maximum degree of a vertex in a graph \( L \), it is clear that \( \gamma(L) \leq \sigma(L) + 1 \). Brooks [41] showed in 1941 that \( \gamma(L) \leq \sigma(L) \) whenever \( \sigma(L) > 3 \) and \( \omega(L) \leq \sigma(L) \). Further investigations could be conducted, taking into account a more exact relation between \( \sigma \) and \( \omega \). Perhaps one should start with estimates of the chromatic number of a graph without triangles (\( \omega = 2 \)) and with given maximal degree for vertices.
Measures of sparsity/structure

\[ \delta \leq \deg \leq \Delta \]
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\[ \delta \leq \bar{\text{deg}} \leq \Delta \]

\[ \omega \leq \rho \leq \chi_f \leq \chi \leq \chi_\ell \leq \Delta + 1 \]
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\[ \omega \leq \rho \leq \chi_f \leq \chi \leq \chi_\ell \leq \Delta + 1 \]

where \( \rho = \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)} \)

(upper bounds on \( \rho \) are like lower bounds on \( \alpha \))
Local to global

What global graph structure arises from conditions on local structure?

- (Brooks' theorem) $\omega \leq \Delta > 2 \Rightarrow \chi \leq \Delta$
- (Off-diagonal Ramsey numbers) $\omega \leq 2 \Rightarrow \text{small} \rho$
- (Vizing's problem) $\omega \leq 2 \Rightarrow \chi \ll \Delta$
- (Ajtai–Erdős–Komlós–Szemerédi) $\omega < \rho = \Rightarrow \rho \leq C \rho \Delta \log \Delta$
- (Reed's) $\chi \leq \frac{1}{2} (\omega + \Delta + 1)$
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What global graph structure arises from conditions on local structure?

Examples:

• (Brooks’ theorem) \( \omega \leq \Delta(> 2) \implies \chi \leq \Delta \)

• (Off-diagonal Ramsey numbers)

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• (Ajtai–Erdős–Komlós–Szemerédi)

• (Reed’s)
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• (Ajtai–Erdős–Komlós–Szemerédi) \( \omega < r \implies \rho \leq C_r \frac{\Delta}{\log \Delta} ? \)
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- (Reed’s) \( \chi \leq \left\lceil \frac{1}{2} (\omega + \Delta + 1) \right\rceil? \)
Probabilistic method

If random object has property with positive probability, then there exists at least one object with that property
Further investigations could be conducted, taking into account a more exact relationship between $\Delta$ and $\omega$. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices. (Vizing 1968)
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$$\omega(G) \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1$$

Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

$$\rho(G) \lesssim \frac{\Delta}{\log \Delta}$$  
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Theorem (Johansson 1996+)

$$\chi_\ell(G) = O \left( \frac{\Delta}{\log \Delta} \right) \quad \text{for any triangle-free } G \text{ of maximum degree } \Delta$$
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NB: these bounds are sharp up to a factor 2 by random $\Delta$-regular graphs.
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Conjecture (Ajtai, Erdős, Komlós, Szemerédi 1981)

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Bounds of Shearer (1995) and Johansson (1996+) are out by a $\log \log \Delta$ factor.
The $\omega$, $\Delta$, $\chi$ conjecture

\[\omega(G) \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1\]
The $\omega$, $\Delta$, $\chi$ Conjecture

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\omega(G) \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_e(G) \leq \Delta(G) + 1
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Conjecture (Reed 1998)

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\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil \text{ for any } G \text{ of clique number } \omega \text{ and maximum degree } \Delta
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†Picture credit: Wikipedia/David Eppstein
The ω, Δ, χ conjecture

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Odd cycles have \( \omega = 2, \Delta = 2, \chi = 3 \)

† Picture credit: Wikipedia/David Eppstein
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Chvátal graph (1970)† has $\omega = 2$, $\Delta = 4$, $\chi = 4$

†Picture credit: Wikipedia/David Eppstein
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- with $\chi_f$ instead of $\chi$ (McDiarmid, cf. Molloy & Reed 2002)

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Bound holds:

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- for $\omega = 2$, $\Delta$ large enough (Johansson 1996+)
- for $\omega \geq \Delta - 1$ (Brooks 1941)

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- (for $\omega \leq \Delta^{1/101}$, $\Delta$ large enough (Davies, Kang, Pirot, Sereni 2020$^+$))
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Theorem (Reed 1998)

\[ \chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil \] for any \( G \) of clique number \( \omega \) and maximum degree \( \Delta \) provided \( \omega \geq (1 - \varepsilon_1)\Delta \) for some \( \varepsilon_1 > 0 \) and \( \Delta \) is large enough
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Corollary
\[ \chi(G) \leq \lceil \varepsilon_2 \omega + (1 - \varepsilon_2)(\Delta + 1) \rceil \text{ for any } G \text{ of clique number } \omega \text{ and maximum degree } \Delta \text{ for some } \varepsilon_2 > 0 \text{ provided } \Delta \text{ is large enough} \]
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- Reed (1998): \( \varepsilon_2 > 0.000000005 \)

NB: \( \varepsilon_2 \) may not be larger than 0.5
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- Reed (1998): \( \varepsilon_2 > 0.000000005 \)
- Bonamy, Perrett, Postle (2022): \( \varepsilon_2 > 0.038 \)
- Delcourt, Postle (2017+): \( \varepsilon_2 > 0.076 \)

NB: \( \varepsilon_2 \) may not be larger than 0.5.
ω, Δ, χ CONJECTURE

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- Hurley, de Joannis de Verclos, Kang (2021): \( \varepsilon_2 > 0.119 \)

NB: \( \varepsilon_2 \) may not be larger than 0.5
Local density

no edge in any neighbourhood
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no edge in any neighbourhood $\iff$ triangle-free
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$\downarrow$

at most a certain proportion of edges per neighbourhood

$\eta < \frac{1}{\Delta^2}$ means triangle-free,
$\eta = 1$ means unrestricted
Local density

no edge in any neighbourhood $\iff$ triangle-free

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$G$ of max degree $\Delta$ has local density $\leq \eta$ if $\leq \eta\left(\frac{\Delta}{2}\right)$ edges per neighbourhood
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$G$ of max degree $\Delta$ has local density $\leq \eta$ if $\leq \eta\binom{\Delta}{2}$ edges per neighbourhood

$\eta < 1/\binom{\Delta}{2}$ means triangle-free, $\eta = 1$ means unrestricted
Largest chromatic number for local density \( \leq \eta \) for \( \eta \) near 0?
Largest chromatic number for local density $\leq \eta$ for $\eta$ near 0?


$$\chi(G) = O\left(\frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}\right) \text{ for any } G \text{ of maximum degree } \Delta$$

and local density at most $\eta$, $\frac{1}{\Delta^2} \leq \eta \leq 1$
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Theorem (Davies, Kang, Pirot, Sereni 2020+)

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$$\chi(G) \preceq \frac{\Delta}{\log \frac{e}{\sqrt{\eta}}} \text{ for any } G \text{ of maximum degree } \Delta$$

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$\eta = \frac{1}{\Delta^2}$ matches Molloy’s;
Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ for $\eta$ near 0?


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$\eta = \frac{1}{\Delta^2}$ matches Molloy’s; the bound is sharp up to a factor of between 2 and 4
Largest chromatic number for local density $\leq \eta$ for $\eta$ near 0?


$$\chi(G) = O \left( \frac{\Delta}{\log e^{\sqrt{\eta}}} \right) \text{ for any } G \text{ of maximum degree } \Delta$$

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$\eta = \frac{1}{\Delta^2}$ matches Molloy’s; the bound is sharp up to a factor of between 2 and 4

NB: $\eta = 1$ should match $\Delta + 1$ bound, but neither gives this...
Largest chromatic number for local density $\leq \eta$ for $\eta$ near 1?

Nontrivial improvement on $\chi \leq \Delta + 1$?
Largest chromatic number for local density $\leq \eta$ for $\eta$ near 1?

Nontrivial improvement on $\chi \leq \Delta + 1$? Yes (asymptotically):

Theorem (Molloy & Reed 1997)

$\chi(G) \leq (1 - \varepsilon)\Delta$ for any $G$ of maximum degree $\Delta$ and local density at most $\eta$ for some $\varepsilon(\eta) > 0$ provided $\Delta$ is large enough
Largest chromatic number for local density $\leq \eta$ for $\eta$ near 1?

Nontrivial improvement on $\chi \leq \Delta + 1$? Yes (asymptotically):

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Lower bounds on $\varepsilon$ key to sharpest results towards Reed’s conjecture (and sharpest results towards the Erdős-Nešetřil conjecture)
Theorem (Molloy & Reed 1997)
\[ \chi(G) \leq (1 - \varepsilon)\Delta \text{ for any } G \text{ of maximum degree } \Delta \text{ and local density at most } \eta \text{ for some } \varepsilon(\eta) > 0 \text{ provided } \Delta \text{ is large enough} \]

- Molloy & Reed (1997): 
  \[ \varepsilon > 0.0238(1 - \eta) \]
Theorem (Molloy & Reed 1997)

\(\chi(G) \leq (1 - \varepsilon)\Delta\) for any \(G\) of maximum degree \(\Delta\) and local density at most \(\eta\) for some \(\varepsilon(\eta) > 0\) provided \(\Delta\) is large enough

- Molloy & Reed (1997): \(\varepsilon > 0.0238(1 - \eta)\)
- Bruhn & Joos (2018): \(\varepsilon > 0.1827(1 - \eta) - 0.0778(1 - \eta)^{3/2}\)
- Bonamy, Perrett, Postle (2022): \(\varepsilon > 0.3012(1 - \eta) - 0.1283(1 - \eta)^{3/2}\)
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- Hurley, de Joannis de Verclos, Kang (2021): \[ \varepsilon > 0.5(1 - \eta) - 0.1667(1 - \eta)^{3/2} \]
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\[ \chi(G) \leq (1 - \varepsilon)\Delta \] for any \( G \) of maximum degree \( \Delta \) and local density at most \( \eta \) for some \( \varepsilon(\eta) > 0 \) provided \( \Delta \) is large enough

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Clique of size \( \sqrt{\eta} \cdot \Delta \) (plus pendant vertices) \( \implies \varepsilon \leq 1 - \sqrt{\eta} \xrightarrow{\eta \to 1} 0.5(1 - \eta) \)
Theorem (Hurley, de Joannis de Verclos, Kang 2021)

\[ \chi(G) \leq (1 - 0.5(1 - \eta) + 0.1667(1 - \eta)^{3/2})\Delta \] for any \( G \) of maximum degree \( \Delta \) and local density at most \( \eta \) provided \( \Delta \) is large enough.

Clique of size \( \sqrt{1 - \sigma} \cdot \Delta \) ( + pendant vertices)
Naïve colouring

Given $G$ and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each $v$ uniformly from $[M]$
Naïve colouring

Given $G$ and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each $v$ uniformly from $[M]$
2. uncolour one or both endpoints of each monochromatic edge

• Molloy & Reed (1997): uncolour both endpoints in

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3. complete partial proper colouring to full one
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Theorem (Davies, Kang, Pirot, Sereni 2020+)

\[ \chi(G) \lesssim \frac{\Delta}{\log \frac{e}{\sqrt{\eta}}} \quad \text{for any } G \text{ of maximum degree } \Delta \]

and local density at most \( \eta, \frac{1}{\Delta^2} \leq \eta \ll 1 \)

Theorem (Hurley, de Joannis de Verclos, Kang 2021)

\[ \chi(G) \leq (1 - 0.5(1 - \eta) + 0.1667(1 - \eta)^{3/2})\Delta \quad \text{for any } G \text{ of maximum degree } \Delta \]

and local density at most \( \eta \) provided \( \Delta \) is large enough.
\[ \omega(G) \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_e(G) \leq \Delta(G) + 1 \]
List colouring (formally)

Introduced independently by Vizing (1976) and Erdős, Rubin, Taylor (1980).
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A list-assignment is some \( L : V(G) \rightarrow 2^{\mathbb{Z}^+} \);
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An \( L \)-colouring is some \( c : V(G) \rightarrow \mathbb{Z}^+ \) with \( c(v) \in L(v) \) for every \( v \in V(G) \). G is \( k \)-choosable if there is a proper \( L \)-colouring for any \( k \)-list-assignment \( L \).
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$G$ is $k$-choosable if there is a proper $L$-colouring for any $k$-list-assignment $L$.

The list chromatic number $\chi_\ell(G)$ is least $k$ such that $G$ is $k$-choosable.

$\chi_\ell(G) \geq \chi(G)$ by considering constant $L$. 
Cover graphs

$u \quad \bullet \quad \bullet \quad \bullet \quad v$
Cover graphs

$L(u)$

10 • 9 • 8 • 7 • 6 • 5 • 4 • 3 • 2 • 1

$L(v)$

17 • 15 • 14 • 11 • 10 • 9 • 7 • 5 • 4 • 1
Cover graphs

\[ L(u) \]

\[ L(v) \]
Cover graphs
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We want independent transversals (ITs) in vertex-partitioned graphs.
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An aside: In 1972 Woodall presented at the open problem session of the 3rd BCC here in Oxford. After some interchange with Erdős, there eventually arose the Bollobás–Erdős–Szemerédi conjecture (1975), solved much later:
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**Theorem (Haxell 2001)**

Any graph of maximum degree $D$ with a partition into parts of size at least $2D$ admits an IT.
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BESz knew hypothetically the sharpness of the factor 2, but a construction of Szabó & Tardos (2005) shows moreover the statement is **exactly** sharp!

Theorem (Szabó & Tardos 2005)

*There is a graph of maximum degree $D$ with a partition into parts of size $2D - 1$ that does **not** admit an IT.*
Trivial bound $(\chi^\ell(G) \leq \Delta(G) + 1)$ redux:

*If $H$ is the cover graph of some graph $G$ with a $(D + 1)$-list-assignment and $G$ has maximum degree $D$, then $H$ admits an IT.*
Trivial bound ($\chi_\ell(G) \leq \Delta(G) + 1$) redux:

*If $H$ is the cover graph of some graph $G$ with a $(D + 1)$-list-assignment and $G$ has maximum degree $D$, then $H$ admits an $IT$."

**Conjecture (Reed 1999)**

*If $H$ is the cover graph of some graph $G$ with a $(D + 1)$-list-assignment and $H$ has maximum degree $D$, then $H$ admits an $IT$."

NB: so $G$ may well be a complete graph!

Bohman & Holzman (2002): counterexample!
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Theorem (Reed & Sudakov 2002)
There is an $f$ satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds. If $H$ is the cover graph of some graph with an $f(D)$-list-assignment and $H$ has maximum degree $D$, then $H$ admits an IT.
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**Still open:** could $f(D) = D + C$ suffice?
Trivial bound $(\chi_c(G) \leq \Delta(G) + 1)$ redux redux:

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Solving a problem of Aharoni & Holzman:

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There is an \( f \) satisfying \( f(D) = D + o(D) \) as \( D \to \infty \) so the following holds.

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**NB:** this is stronger than Reed–Sudakov.
Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If $H$ is a $4D$-fold cover graph and $H$ has maximum part-averaged degree $D$, then $H$ admits an IT.

NB: the maximum degree of $H$ may be much larger than $D!$

NB: Reed & Wood (2012) already knew/used this with $2e$ instead of $4$.

NB: $4$ is necessary (Groenland, Kaiser, Treffers, Wales 2021+).

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Loh–Sudakov (2007) and Kang–Kelly (2022) conjectures
Theorem (Amini & Reed 2008)

There is $C > 0$ such that, if $H$ is a $(C \cdot D / \log D)$-fold list-cover, $H$ is triangle-free, and $H$ has maximum degree $D$, then $H$ admits an IT.

NB: Alon & Assadi (2020+) independently derived a very similar result with the aim of ‘palette sparsification’.
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Conjecture (Cambie & Kang 2020+)

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NB: Anderson, Bernshteyn, Dhawan (2021+) propose an even stronger form.
Questions?