On conjectures of Reed

Ross J. Kang*

~~Radboud University Nijmegen~~ University of Amsterdam

Reedfest

Oxford 7/2022

*Joint works with Cambie, Davies, Dvořák, Esperet, Hurley, de Joannis de Verclos, Kelly, Ozeki, Pirot, Sereni. Support from NWO and Van Gogh grants.

Brooks' theorem

Theorem (Brooks 1941)

$\chi(G) \leq \Delta(G)$ unless $G = K_{\Delta(G)+1}$ or G is an odd cycle.

Vizing's problem

Vizing, "Some unsolved problems in graph theory" (1968):

Можно ли для любого наперед заданного натурального числа $k \geqslant 2$ построить граф со сколь угодно большим обхватом и с хроматическим числом k? Можно. Это доказал П. Эрдёш [39], основываясь на мощностных соображениях. Удивительно, что до сих пор нет конструктивного доказательства этого факта. В [40] указан способ построения графов с любым хроматическим числом без циклов длины $\leqslant 7$. Это лучшее, что мы имеем на сегодняшний день.

Если $\sigma(L)$ — максимальная степень вершины графа L, то, очевидно, $\gamma(L) \leqslant \sigma(L)+1$. В 1941 г. Р. Брукс [41] доказал, что при $\sigma(L) \geqslant 3$ и $\omega(L) \leqslant \sigma(L)$ справедлива оценка $\gamma(L) \leqslant \sigma(L)$. Дальнейшие исследования можно проводить, учитывая более точно соотношения между σ и ω. Пожалуй, следует начать с оценки хроматического числа графа без треугольников $(\omega = 2)$ с данной максимальной степенью вершины.

Vizing's problem

Vizing, "Some unsolved problems in graph theory" (1968):

Given a natural number $k \geqslant 2$, is it possible to construct a graph with arbitrarily large compass and with chromatic number k? Erdős [39] has proved this; his proof is based on counting arguments. It is astonishing that no constructive proof for this fact has yet been given. In [40] a method is given of constructing graphs of arbitrary chromatic number without having cycles of length $\leqslant 7$. This is the best we have at the present time.

If $\sigma(L)$ is the maximum degree of a vertex in a graph L, it is clear that $\gamma(L) \leqslant \sigma(L) + 1$. Brooks [41] showed in 1941 that $\gamma(L) \leqslant \sigma(L)$ whenever $\sigma(L) \geqslant 3$ and $\omega(L) \leqslant \sigma(L)$. Further investigations could be conducted, taking into account a more exact relation between σ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.

Measures of sparsity/structure

$$\delta \leq \overline{\mathsf{deg}} \leq \Delta$$

Measures of sparsity/structure

$$\delta \leq \overline{\mathsf{deg}} \leq \Delta$$

$$\omega \leq \quad \rho \leq \chi_f \leq \chi \leq \chi_\ell \quad \leq \Delta + 1$$

Measures of sparsity/structure

$$\delta \leq \overline{\mathsf{deg}} \leq \Delta$$

$$\omega \leq \quad \rho \leq \chi_f \leq \chi \leq \chi_\ell \quad \leq \Delta + 1$$

where $\rho = \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)}$

(upper bounds on ρ are like lower bounds on α)

Local to global

What global graph structure arises
from conditions on local structure?

What global graph structure arises
from conditions on local structure?

Examples:

- (Brooks' theorem) $\omega \leq \Delta (> 2) \implies \chi \leq \Delta$

Local to global

What global graph structure arises
from conditions on local structure?

Examples:

- (Brooks' theorem) $\omega \leq \Delta (> 2) \implies \chi \leq \Delta$
- (Off-diagonal Ramsey numbers) $\omega \leq 2 \implies$ small ρ

Local to global

What global graph structure arises
from conditions on local structure?

Examples:

- (Brooks' theorem) $\omega \leq \Delta (> 2) \implies \chi \leq \Delta$
- (Off-diagonal Ramsey numbers) $\omega \leq 2 \implies$ small ρ
- (Vizing's problem) $\omega \leq 2 \implies \chi \ll \Delta$?

Local to global

What global graph structure arises
from conditions on local structure?

Examples:

- (Brooks' theorem) $\omega \le \Delta (> 2) \implies \chi \le \Delta$
- (Off-diagonal Ramsey numbers) $\omega \le 2 \implies$ small ρ
- (Vizing's problem) $\omega \le 2 \implies \chi \ll \Delta$?
- (Ajtai–Erdős–Komlós–Szemerédi) $\omega < r \implies \rho \le C_r \dfrac{\Delta}{\log \Delta}$?

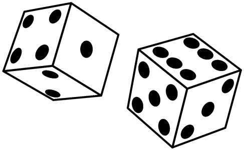
Local to global

What global graph structure arises
from conditions on local structure?

Examples:

- (Brooks' theorem) $\omega \le \Delta (> 2) \implies \chi \le \Delta$
- (Off-diagonal Ramsey numbers) $\omega \le 2 \implies$ small ρ
- (Vizing's problem) $\omega \le 2 \implies \chi \ll \Delta$?

- (Ajtai–Erdős–Komlós–Szemerédi) $\omega < r \implies \rho \le C_r \dfrac{\Delta}{\log \Delta}$?
- (Reed's) $\chi \le \left\lceil \frac{1}{2}(\omega + \Delta + 1) \right\rceil$?



If random object has property with positive probability,
then there exists at least one object with that property

Chromatic number of triangle-free graphs

Further investigations could be conducted, taking into account a more exact relationship between Δ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.

(Vizing 1968)

Chromatic number of triangle-free graphs

`Further investigations could be conducted, taking into account a more exact relationship between` Δ `and` ω. `Perhaps one should start with estimates of the chromatic number of a graph without triangles (`$\omega = 2$`) and with given maximal degree for vertices.` (Vizing 1968)

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

$\rho(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ*

Chromatic number of triangle-free graphs

`Further investigations could be conducted, taking into account a more exact relationship between` Δ `and` ω`. Perhaps one should start with estimates of the chromatic number of a graph without triangles (`$\omega = 2$`) and with given maximal degree for vertices.` (Vizing 1968)

$$\omega(G) \le \quad \rho(G) \le \chi_f(G) \le \chi(G) \le \chi_\ell(G) \quad \le \Delta(G) + 1$$

Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

$\rho(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ*

Theorem (Johansson 1996+)

$\chi_\ell(G) = O\left(\dfrac{\Delta}{\log \Delta}\right)$ *for any triangle-free G of maximum degree Δ*

Chromatic number of triangle-free graphs

`Further investigations could be conducted, taking into account a more exact relationship between` Δ `and` ω`. Perhaps one should start with estimates of the chromatic number of a graph without triangles (`$\omega = 2$`) and with given maximal degree for vertices.`

(Vizing 1968)

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

$\rho(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ*

Theorem (Molloy 2019, cf. Johansson 1996+)

$\chi_\ell(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ*

Chromatic number of triangle-free graphs

`Further investigations could be conducted, taking into account a more exact relationship between` Δ `and` ω`. Perhaps one should start with estimates of the chromatic number of a graph without triangles (`$\omega = 2$`) and with given maximal degree for vertices.` (Vizing 1968)

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

$\rho(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ*

Theorem (Molloy 2019, cf. Johansson 1996+)

$\chi_\ell(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ*

NB: these bounds are sharp up to a factor 2 by random Δ-regular graphs.

Chromatic number of K_r-free graphs

Further investigations could be conducted, taking into account a more exact relationship between Δ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices. (Vizing 1968)

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Chromatic number of K_r-free graphs

Further investigations could be conducted, taking into account a more exact relationship between Δ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices. (Vizing 1968)

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Ajtai, Erdős, Komlós, Szemerédi 1981)

$\rho(G) = O\left(\frac{\Delta}{\log \Delta}\right)$ *for any K_r-free G of maximum degree Δ*

Chromatic number of K_r-free graphs

Further investigations could be conducted, taking into account a more exact relationship between Δ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices. (Vizing 1968)

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Ajtai, Erdős, Komlós, Szemerédi 1981)

$\rho(G) = O\left(\frac{\Delta}{\log \Delta}\right)$ *for any K_r-free G of maximum degree Δ*

Conjecture (Alon, Krivelevich, Sudakov 1999)

$\chi(G) = O\left(\frac{\Delta}{\log \Delta}\right)$ *for any K_r-free G of maximum degree Δ*

Chromatic number of K_r-free graphs

Further investigations could be conducted, taking into account a more exact relationship between Δ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices. (Vizing 1968)

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Ajtai, Erdős, Komlós, Szemerédi 1981)

$\rho(G) = O\left(\frac{\Delta}{\log \Delta}\right)$ *for any K_r-free G of maximum degree Δ*

Conjecture (Alon, Krivelevich, Sudakov 1999)

$\chi(G) = O\left(\frac{\Delta}{\log \Delta}\right)$ *for any K_r-free G of maximum degree Δ*

Bounds of Shearer (1995) and Johansson (1996+) are out by a $\log \log \Delta$ factor.

The ω, Δ, χ conjecture

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

† Picture credit: Wikipedia/David Eppstein

The ω, Δ, χ conjecture

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ*

† Picture credit: Wikipedia/David Eppstein

The ω, Δ, χ conjecture

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ*

Odd cycles have $\omega = 2$, $\Delta = 2$, $\chi = 3$

†Picture credit: Wikipedia/David Eppstein

The ω, Δ, χ conjecture

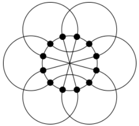
$$\omega(G) \le \quad \rho(G) \le \chi_f(G) \le \chi(G) \le \chi_\ell(G) \quad \le \Delta(G) + 1$$

Conjecture (Reed 1998)

$\chi(G) \le \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ*

Odd cycles have $\omega = 2$, $\Delta = 2$, $\chi = 3$

Chvátal graph (1970)† has $\omega = 2$, $\Delta = 4$, $\chi = 4$



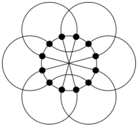
†Picture credit: Wikipedia/David Eppstein

The ω, Δ, χ conjecture

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ*



Odd cycles have $\omega = 2$, $\Delta = 2$, $\chi = 3$

Chvátal graph (1970)† has $\omega = 2$, $\Delta = 4$, $\chi = 4$

Bound holds:

- with χ_f instead of χ (McDiarmid, cf. Molloy & Reed 2002)

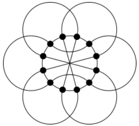
†Picture credit: Wikipedia/David Eppstein

The ω, Δ, χ conjecture

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ*



Odd cycles have $\omega = 2$, $\Delta = 2$, $\chi = 3$

Chvátal graph (1970)† has $\omega = 2$, $\Delta = 4$, $\chi = 4$

Bound holds:

- with χ_f instead of χ (McDiarmid, cf. Molloy & Reed 2002)
- for $\omega = 2$, Δ large enough (Johansson 1996+)
- for $\omega \geq \Delta - 1$ (Brooks 1941)

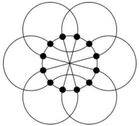
† Picture credit: Wikipedia/David Eppstein

THE ω, Δ, χ CONJECTURE

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

Conjecture (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ*



Odd cycles have $\omega = 2$, $\Delta = 2$, $\chi = 3$

Chvátal graph (1970)† has $\omega = 2$, $\Delta = 4$, $\chi = 4$

Bound holds:

- with χ_f instead of χ (McDiarmid, cf. Molloy & Reed 2002)
- for $\omega = 2$, Δ large enough (Johansson 1996+)
  (for $\omega \leq \Delta^{1/101}$, Δ large enough (Davies, Kang, Pirot, Sereni 2020+))
- for $\omega \geq \Delta - 1$ (Brooks 1941)

† Picture credit: Wikipedia/David Eppstein

Theorem (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ for any G of clique number ω and maximum degree Δ provided $\omega \geq (1 - \varepsilon_1)\Delta$ for some $\varepsilon_1 > 0$ and Δ is large enough

ω, Δ, χ conjecture

Theorem (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any* G *of clique number* ω *and maximum degree* Δ *provided* $\omega \geq (1 - \varepsilon_1)\Delta$ *for some* $\varepsilon_1 > 0$ *and* Δ *is large enough*

Corollary

$\chi(G) \leq \lceil \varepsilon_2 \omega + (1 - \varepsilon_2)(\Delta + 1) \rceil$ *for any* G *of clique number* ω *and maximum degree* Δ *for some* $\varepsilon_2 > 0$ *provided* Δ *is large enough*

Theorem (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ for any G of clique number ω and maximum degree Δ provided $\omega \geq (1 - \varepsilon_1)\Delta$ for some $\varepsilon_1 > 0$ and Δ is large enough

Corollary

$\chi(G) \leq \lceil \varepsilon_2 \omega + (1 - \varepsilon_2)(\Delta + 1) \rceil$ for any G of clique number ω and maximum degree Δ for some $\varepsilon_2 > 0$ provided Δ is large enough

- Reed (1998): $\varepsilon_2 > 0.000000005$

NB: ε_2 may not be larger than 0.5

ω, Δ, χ CONJECTURE

Theorem (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ provided $\omega \geq (1 - \varepsilon_1)\Delta$ for some $\varepsilon_1 > 0$ and Δ is large enough*

Corollary

$\chi(G) \leq \lceil \varepsilon_2 \omega + (1 - \varepsilon_2)(\Delta + 1) \rceil$ *for any G of clique number ω and maximum degree Δ for some $\varepsilon_2 > 0$ provided Δ is large enough*

- Reed (1998): $\varepsilon_2 > 0.000000005$
- Bonamy, Perrett, Postle (2022): $\varepsilon_2 > 0.038$
- Delcourt, Postle (2017+): $\varepsilon_2 > 0.076$

NB: ε_2 may not be larger than 0.5

Theorem (Reed 1998)

$\chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ *for any G of clique number ω and maximum degree Δ provided $\omega \geq (1 - \varepsilon_1)\Delta$ for some $\varepsilon_1 > 0$ and Δ is large enough*

Corollary

$\chi(G) \leq \lceil \varepsilon_2 \omega + (1 - \varepsilon_2)(\Delta + 1) \rceil$ *for any G of clique number ω and maximum degree Δ for some $\varepsilon_2 > 0$ provided Δ is large enough*

- Reed (1998): $\varepsilon_2 > 0.000000005$
- Bonamy, Perrett, Postle (2022): $\varepsilon_2 > 0.038$
- Delcourt, Postle (2017+): $\varepsilon_2 > 0.076$
- Hurley, de Joannis de Verclos, Kang (2021): $\varepsilon_2 > 0.119$

NB: ε_2 may not be larger than 0.5

Local density

no edge in any neighbourhood

Local density

no edge in any neighbourhood $\iff$ triangle-free

Local density

no edge in any neighbourhood $\iff$ triangle-free

$\downarrow$

at most a certain proportion of edges per neighbourhood

Local density

no edge in any neighbourhood $\iff$ triangle-free

$\downarrow$

at most a certain proportion of edges per neighbourhood

G of max degree Δ has local density $\leq \eta$ if $\leq \eta\binom{\Delta}{2}$ edges per neighbourhood

Local density

no edge in any neighbourhood $\iff$ triangle-free

$\downarrow$

at most a certain proportion of edges per neighbourhood

G of max degree Δ has local density $\leq \eta$ if $\leq \eta\binom{\Delta}{2}$ edges per neighbourhood

$\eta < 1/\binom{\Delta}{2}$ means triangle-free, $\eta = 1$ means unrestricted

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ for η near 0?

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ *for* η *near* 0?

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

$\chi(G) = O\left(\frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}\right)$ *for any* G *of maximum degree* Δ *and local density at most* η, $\frac{1}{\Delta^2} \leq \eta \leq 1$

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ for η near 0?

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

$\chi(G) = O\left(\frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}\right)$ *for any G of maximum degree Δ and local density at most η, $\frac{1}{\Delta^2} \leq \eta \leq 1$*

Theorem (Davies, Kang, Pirot, Sereni 2020+)

$\chi(G) \lesssim \frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}$ *for any G of maximum degree Δ and local density at most η, $\frac{1}{\Delta^2} \leq \eta \ll 1$*

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ for η near 0?

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

$\chi(G) = O\left(\frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}\right)$ *for any G of maximum degree Δ and local density at most η, $\frac{1}{\Delta^2} \leq \eta \leq 1$*

Theorem (Davies, Kang, Pirot, Sereni 2020+)

$\chi(G) \lesssim \frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}$ *for any G of maximum degree Δ and local density at most η, $\frac{1}{\Delta^2} \leq \eta \ll 1$*

$\eta = \frac{1}{\Delta^2}$ matches Molloy's;

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ *for* η *near* 0*?*

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

$\chi(G) = O\left(\frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}\right)$ *for any* G *of maximum degree* Δ *and local density at most* η, $\frac{1}{\Delta^2} \leq \eta \leq 1$

Theorem (Davies, Kang, Pirot, Sereni 2020+)

$\chi(G) \lesssim \frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}$ *for any* G *of maximum degree* Δ *and local density at most* η, $\frac{1}{\Delta^2} \leq \eta \ll 1$

$\eta = \frac{1}{\Delta^2}$ matches Molloy's; the bound is sharp up to a factor of between 2 and 4

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ for η near 0?

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

$\chi(G) = O\left(\frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}\right)$ *for any G of maximum degree Δ and local density at most η, $\frac{1}{\Delta^2} \leq \eta \leq 1$*

Theorem (Davies, Kang, Pirot, Sereni 2020+)

$\chi(G) \lesssim \frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}$ *for any G of maximum degree Δ and local density at most η, $\frac{1}{\Delta^2} \leq \eta \ll 1$*

$\eta = \frac{1}{\Delta^2}$ matches Molloy's; the bound is sharp up to a factor of between 2 and 4

NB : $\eta = 1$ should match $\Delta + 1$ bound, but neither gives this...

Chromatic number under bounded local density

Largest chromatic number for local density $\leq \eta$ for η near 1?

Nontrivial improvement on $\chi \leq \Delta + 1$?

Chromatic number under bounded local density

Largest chromatic number for local density $\leq \eta$ for η near 1?

Nontrivial improvement on $\chi \leq \Delta + 1$? Yes (asymptotically):

Theorem (Molloy & Reed 1997)

$\chi(G) \leq (1-\varepsilon)\Delta$ for any G of maximum degree Δ and local density at most η for some $\varepsilon(\eta) > 0$ provided Δ is large enough

Largest chromatic number for local density $\leq \eta$ for η near 1?

Nontrivial improvement on $\chi \leq \Delta + 1$? Yes (asymptotically):

Theorem (Molloy & Reed 1997)

$\chi(G) \leq (1-\varepsilon)\Delta$ for any G of maximum degree Δ and local density at most η for some $\varepsilon(\eta) > 0$ provided Δ is large enough

Lower bounds on ε key to sharpest results towards Reed's conjecture
(and sharpest results towards the Erdős-Nešetřil conjecture)

Theorem (Molloy & Reed 1997)

$\chi(G) \leq (1-\varepsilon)\Delta$ for any G of maximum degree Δ and local density at most η for some $\varepsilon(\eta) > 0$ provided Δ is large enough

- Molloy & Reed (1997): $\varepsilon > 0.0238(1-\eta)$

Theorem (Molloy & Reed 1997)

$\chi(G) \leq (1-\varepsilon)\Delta$ for any G of maximum degree Δ and local density at most η for some $\varepsilon(\eta) > 0$ provided Δ is large enough

- Molloy & Reed (1997): $\varepsilon > 0.0238(1-\eta)$
- Bruhn & Joos (2018): $\varepsilon > 0.1827(1-\eta) - 0.0778(1-\eta)^{3/2}$
- Bonamy, Perrett, Postle (2022): $\varepsilon > 0.3012(1-\eta) - 0.1283(1-\eta)^{3/2}$

Theorem (Molloy & Reed 1997)

$\chi(G) \le (1-\varepsilon)\Delta$ for any G of maximum degree Δ and local density at most η for some $\varepsilon(\eta) > 0$ provided Δ is large enough

- Molloy & Reed (1997): $\varepsilon > 0.0238(1-\eta)$
- Bruhn & Joos (2018): $\varepsilon > 0.1827(1-\eta) - 0.0778(1-\eta)^{3/2}$
- Bonamy, Perrett, Postle (2022): $\varepsilon > 0.3012(1-\eta) - 0.1283(1-\eta)^{3/2}$
- Hurley, de Joannis de Verclos, Kang (2021): $\varepsilon > 0.5(1-\eta) - 0.1667(1-\eta)^{3/2}$

Theorem (Molloy & Reed 1997)

$\chi(G) \le (1-\varepsilon)\Delta$ for any G of maximum degree Δ and local density at most η for some $\varepsilon(\eta) > 0$ provided Δ is large enough

- Molloy & Reed (1997): $\varepsilon > 0.0238(1-\eta)$
- Bruhn & Joos (2018): $\varepsilon > 0.1827(1-\eta) - 0.0778(1-\eta)^{3/2}$
- Bonamy, Perrett, Postle (2022): $\varepsilon > 0.3012(1-\eta) - 0.1283(1-\eta)^{3/2}$
- Hurley, de Joannis de Verclos, Kang (2021): $\varepsilon > 0.5(1-\eta) - 0.1667(1-\eta)^{3/2}$

Clique of size $\sqrt{\eta} \cdot \Delta$ (+ pendant vertices)

Chromatic number under bounded local density

Theorem (Molloy & Reed 1997)

$\chi(G) \leq (1-\varepsilon)\Delta$ for any G of maximum degree Δ and local density at most η for some $\varepsilon(\eta) > 0$ provided Δ is large enough

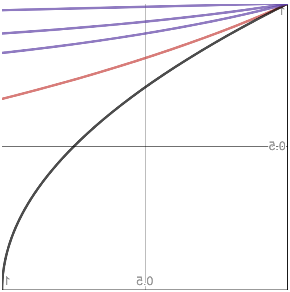
- Molloy & Reed (1997): $\varepsilon > 0.0238(1-\eta)$
- Bruhn & Joos (2018): $\varepsilon > 0.1827(1-\eta) - 0.0778(1-\eta)^{3/2}$
- Bonamy, Perrett, Postle (2022): $\varepsilon > 0.3012(1-\eta) - 0.1283(1-\eta)^{3/2}$
- Hurley, de Joannis de Verclos, Kang (2021): $\varepsilon > 0.5(1-\eta) - 0.1667(1-\eta)^{3/2}$

Clique of size $\sqrt{\eta} \cdot \Delta$ (+ pendant vertices) $\implies \varepsilon \leq 1 - \sqrt{\eta} \underset{\eta \to 1}{\sim} 0.5(1-\eta)$

Theorem (Hurley, de Joannis de Verclos, Kang 2021)

$\chi(G) \leq (1 - 0.5(1-\eta) + 0.1667(1-\eta)^{3/2})\Delta$ for any G of maximum degree Δ and local density at most η provided Δ is large enough

Clique of size $\sqrt{1-\sigma} \cdot \Delta$ (+ pendant vertices)



Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$

Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$
2. uncolour one or both endpoints of each monochromatic edge

Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$
2. uncolour one or both endpoints of each monochromatic edge
3. complete partial proper colouring to full one

Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$
2. uncolour one or both endpoints of each monochromatic edge
3. complete partial proper colouring to full one

- Molloy & Reed (1997): uncolour both endpoints in **??**

Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$
2. uncolour one or both endpoints of each monochromatic edge
3. complete partial proper colouring to full one

- Molloy & Reed (1997): uncolour both endpoints in **??**
- Bruhn & Joos (2018): toss a coin to decide in **??**

Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$
2. uncolour one or both endpoints of each monochromatic edge
3. complete partial proper colouring to full one

- Molloy & Reed (1997): uncolour both endpoints in **??**
- Bruhn & Joos (2018): toss a coin to decide in **??**
- Bonamy, Perrett, Postle (2022): several iterations of **??** and **??**

Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$
2. uncolour one or both endpoints of each monochromatic edge
3. complete partial proper colouring to full one

- Molloy & Reed (1997): uncolour both endpoints in **??**
- Bruhn & Joos (2018): toss a coin to decide in **??**
- Bonamy, Perrett, Postle (2022): several iterations of **??** and **??**
- Hurley, de Joannis de Verclos, Kang (2021):
  random vertex ordering to decide in **??**
  allows better iteration of **??** and **??**

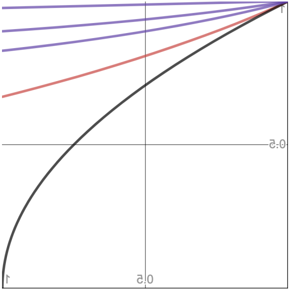
Link up?

Theorem (Davies, Kang, Pirot, Sereni 2020+)

$\chi(G) \lesssim \dfrac{\Delta}{\log \frac{e}{\sqrt{\eta}}}$ *for any G of maximum degree Δ and local density at most η,* $\frac{1}{\Delta^2} \le \eta \ll 1$

Theorem (Hurley, de Joannis de Verclos, Kang 2021)

$\chi(G) \le (1 - 0.5(1-\eta) + 0.1667(1-\eta)^{3/2})\Delta$ *for any G of maximum degree Δ and local density at most η provided Δ is large enough*



Beyond trivial

$$\omega(G) \leq \quad \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_\ell(G) \quad \leq \Delta(G) + 1$$

List colouring (formally)

Introduced independently by Vizing (1976) and Erdős, Rubin, Taylor (1980).

List colouring (formally)

Introduced independently by Vizing (1976) and Erdős, Rubin, Taylor (1980).

A list-assignment is some $L : V(G) \to 2^{\mathbb{Z}^+}$;
a k-list-assignment is some $L : V(G) \to \binom{\mathbb{Z}^+}{k}$.

List colouring (formally)

Introduced independently by Vizing (1976) and Erdős, Rubin, Taylor (1980).

A list-assignment is some $L : V(G) \to 2^{\mathbb{Z}^+}$;
a k-list-assignment is some $L : V(G) \to \binom{\mathbb{Z}^+}{k}$.

An L-colouring is some $c : V(G) \to \mathbb{Z}^+$ with $c(v) \in L(v)$ for every $v \in V(G)$.

G is k-choosable if there is a proper L-colouring for any k-list-assignment L.

List colouring (formally)

Introduced independently by Vizing (1976) and Erdős, Rubin, Taylor (1980).

A list-assignment is some $L : V(G) \to 2^{\mathbb{Z}^+}$;
a k-list-assignment is some $L : V(G) \to \binom{\mathbb{Z}^+}{k}$.

An L-colouring is some $c : V(G) \to \mathbb{Z}^+$ with $c(v) \in L(v)$ for every $v \in V(G)$.

G is k-choosable if there is a proper L-colouring for any k-list-assignment L.

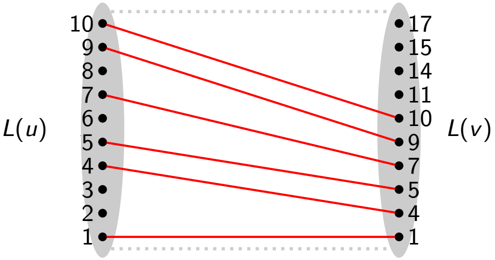
The list chromatic number $\chi_\ell(G)$ is least k such that G is k-choosable.

$\chi_\ell(G) \geq \chi(G)$ by considering constant L.

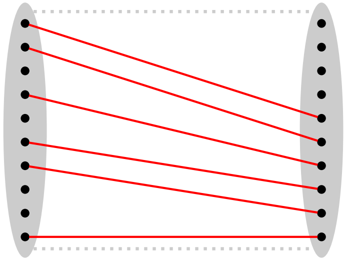
Cover graphs


L(u)
10
9
8
7
6
5
4
3
2
1
17
15
14
11
10
9
7
5
4
1
L(v)

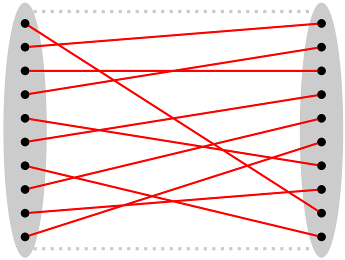
Cover graphs



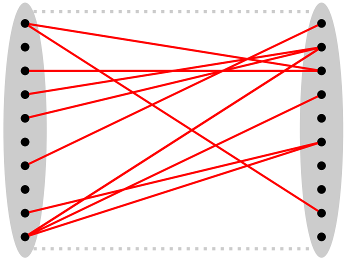
Cover graphs



Cover graphs



Cover graphs



That was a bit too far

We want independent transversals (ITs) in vertex-partitioned graphs.

We want independent transversals (ITs) in vertex-partitioned graphs.

An aside: In 1972 Woodall presented at the open problem session of the 3rd BCC here in Oxford. After some interchange with Erdős, there eventually arose the Bollobás–Erdős–Szemerédi conjecture (1975), solved much later:

That was a bit too far

We want independent transversals (ITs) in vertex-partitioned graphs.

An aside: In 1972 Woodall presented at the open problem session of the 3rd BCC here in Oxford. After some interchange with Erdős, there eventually arose the Bollobás–Erdős–Szemerédi conjecture (1975), solved much later:

Theorem (Haxell 2001)

Any graph of maximum degree D with a partition into parts of size at least $2D$ admits an IT.

That was a bit too far

We want independent transversals (ITs) in vertex-partitioned graphs.

An aside: In 1972 Woodall presented at the open problem session of the 3rd BCC here in Oxford. After some interchange with Erdős, there eventually arose the Bollobás–Erdős–Szemerédi conjecture (1975), solved much later:

Theorem (Haxell 2001)

Any graph of maximum degree D with a partition into parts of size at least $2D$ admits an IT.

BESz knew hypothetically the sharpness of the factor 2, but a construction of Szabó & Tardos (2005) shows moreover the statement is exactly sharp!

Theorem (Szabó & Tardos 2005)

There is a graph of maximum degree D with a partition into parts of size $2D - 1$ that does not admit an IT.

Bruce didn't want to go so far

Trivial bound ($\chi_\ell(G) \le \Delta(G) + 1$) redux:

If H is the cover graph of some graph G with a $(D+1)$-list-assignment and G has maximum degree D, then H admits an IT.

Bruce didn't want to go so far

Trivial bound ($\chi_\ell(G) \leq \Delta(G) + 1$) redux:

> *If H is the cover graph of some graph G with a $(D+1)$-list-assignment and G has maximum degree D, then H admits an IT.*

Conjecture (Reed 1999)

If H is the cover graph of some graph G with a $(D+1)$-list-assignment and H has maximum degree D, then H admits an IT.

Bruce didn't want to go so far

Trivial bound ($\chi_\ell(G) \leq \Delta(G) + 1$) redux:

> *If H is the cover graph of some graph G with a $(D+1)$-list-assignment and G has maximum degree D, then H admits an IT.*

Conjecture (Reed 1999)

If H is the cover graph of some graph G with a $(D+1)$-list-assignment and H has maximum degree D, then H admits an IT.

NB: so G may well be a complete graph!

Bruce didn't want to go so far

Trivial bound ($\chi_\ell(G) \leq \Delta(G) + 1$) redux:

> *If H is the cover graph of some graph G with a $(D+1)$-list-assignment and G has maximum degree D, then H admits an IT.*

Conjecture (Reed 1999)

If H is the cover graph of some graph G with a $(D+1)$-list-assignment and H has maximum degree D, then H admits an IT.

NB: so G may well be a complete graph!

Bohman & Holzman (2002): counterexample!

At least a little bit!

Conjecture (Reed 1999)

If H is the cover graph of some graph G with a $(D+1)$-list-assignment and H has maximum degree D, then H admits an IT.

At least a little bit!

Conjecture (Reed 1999)

If H is the cover graph of some graph G with a $(D+1)$-list-assignment and H has maximum degree D, then H admits an IT.

Theorem (Reed & Sudakov 2002)

There is an f satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds. If H is the cover graph of some graph with an $f(D)$-list-assignment and H has maximum degree D, then H admits an IT.

At least a little bit!

Conjecture (Reed 1999)

If H is the cover graph of some graph G with a $(D+1)$-list-assignment and H has maximum degree D, then H admits an IT.

Theorem (Reed & Sudakov 2002)

There is an f satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds. If H is the cover graph of some graph with an $f(D)$-list-assignment and H has maximum degree D, then H admits an IT.

Still open: could $f(D) = D + C$ suffice?

And a little bit further?

Trivial bound ($\chi_c(G) \leq \Delta(G) + 1$) redux redux:

If H is a $(D+1)$-fold correspondence-cover graph of some graph G and G has maximum degree D, then H admits an IT.

And a little bit further

Trivial bound ($\chi_c(G) \leq \Delta(G) + 1$) redux redux:

> *If H is a $(D+1)$-fold correspondence-cover graph of some graph G and G has maximum degree D, then H admits an IT.*

Solving a problem of Aharoni & Holzman:

Theorem (Loh & Sudakov 2007)

There is an f satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds. If H is an $f(D)$-fold correspondence-cover graph and H has maximum degree D, then H admits an IT.

And a little bit further

Trivial bound ($\chi_c(G) \leq \Delta(G) + 1$) redux redux:

> *If H is a $(D+1)$-fold correspondence-cover graph of some graph G and G has maximum degree D, then H admits an IT.*

Solving a problem of Aharoni & Holzman:

Theorem (Loh & Sudakov 2007)

There is an f satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds.
If H is an $f(D)$-fold correspondence-cover graph
and H has maximum degree D, then H admits an IT.

NB: this is stronger than Reed–Sudakov.

And then even a little further

Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If H is a $4D$-fold cover graph and H has maximum part-averaged degree D, then H admits an IT.

And then even a little further

Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If H is a $4D$-fold cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: the maximum degree of H may be much larger than D!

Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If H is a $4D$-fold cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: the maximum degree of H may be much larger than D!
NB: Reed & Wood (2012) already knew/used this with $2e$ instead of 4.

And then even a little further

Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If H is a $4D$-fold cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: the maximum degree of H may be much larger than D!
NB: Reed & Wood (2012) already knew/used this with $2e$ instead of 4.
NB: 4 is necessary (Groenland, Kaiser, Treffers, Wales 2021+).

And then even a little further

Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If H is a $4D$-fold cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: the maximum degree of H may be much larger than D!
NB: Reed & Wood (2012) already knew/used this with $2e$ instead of 4.
NB: 4 is necessary (Groenland, Kaiser, Treffers, Wales 2021+).

Theorem (Glock & Sudakov 2022, Kang & Kelly 2022)

There is an f satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds.
If H is an $f(D)$-fold correspondence-cover graph and
H has maximum part-averaged degree D, then H admits an IT.

And then even a little further

Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If H is a $4D$-fold cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: the maximum degree of H may be much larger than D!
NB: Reed & Wood (2012) already knew/used this with $2e$ instead of 4.
NB: 4 is necessary (Groenland, Kaiser, Treffers, Wales 2021+).

Theorem (Glock & Sudakov 2022, Kang & Kelly 2022)

There is an f satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds. If H is an $f(D)$-fold correspondence-cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: this is stronger than Loh–Sudakov.

And then even a little further

Proposition (Dvořák, Esperet, Kang, Ozeki 2021, cf. Wanless & Wood 2022)

If H is a $4D$-fold cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: the maximum degree of H may be much larger than D!
NB: Reed & Wood (2012) already knew/used this with $2e$ instead of 4.
NB: 4 is necessary (Groenland, Kaiser, Treffers, Wales 2021+).

Theorem (Glock & Sudakov 2022, Kang & Kelly 2022)

There is an f satisfying $f(D) = D + o(D)$ as $D \to \infty$ so the following holds. If H is an $f(D)$-fold correspondence-cover graph and H has maximum part-averaged degree D, then H admits an IT.

NB: this is stronger than Loh–Sudakov.
NB: incidentally, this confirms a conjecture of Erdős, Gyárfás, Łuczak (1994).

Even further?

Instead of prescribing maximum part-degree (correspondence-cover means 1),

Even further?

Instead of prescribing maximum part-degree (correspondence-cover means 1), prescribe maximum part-averaged part-degree, say $o(D)$?

Even further?

Instead of prescribing maximum part-degree (correspondence-cover means 1), prescribe maximum part-averaged part-degree, say $o(D)$?

Loh–Sudakov (2007) and Kang–Kelly (2022) conjectures

Theorem (Amini & Reed 2008)

There is $C > 0$ such that, if H is a $(C \cdot D/\log D)$-fold list-cover, H is triangle-free, and H has maximum degree D, then H admits an IT.

NB: Alon & Assadi (2020+) independently derived a very similar result with the aim of 'palette sparsification'.

A crossing of the two arcs

Theorem (Amini & Reed 2008)

There is $C > 0$ *such that, if* H *is a* $(C \cdot D/\log D)$*-fold list-cover,*
H *is triangle-free, and* H *has maximum degree* D*, then* H *admits an IT.*

NB: Alon & Assadi (2020+) independently derived a very similar result with the aim of 'palette sparsification'.

Conjecture (Cambie & Kang 2020+)

There is $C > 0$ *such that, if* H *is a* $(C \cdot D/\log D)$*-fold correspondence-cover,*
H *is triangle-free, and* H *has maximum degree* D*, then* H *admits an IT.*

NB: Anderson, Bernshteyn, Dhawan (2021+) propose an even stronger form.

Questions?