Chromatic structure of bipartite graphs

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Discrete Seminar

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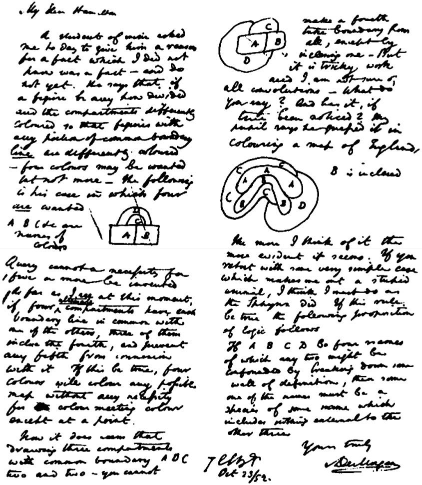
*With Alon, Cambie, Cames van Batenburg, Davies. Support from NWO.

De Morgan's letter†


My dear Hamilton

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted

A B C D are names of colours

Query cannot a necessity for five or more be invented

As far as I see at this moment, if four ultimate compartments have each boundary line in common with one of the others, three of them inclose the fourth, and prevent any fifth from connexion with it. If this be true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

Now it does seem that drawing three compartments with common boundary A B C two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England.

B is inclosed

The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did. If this rule be true the following proposition of logic follows

If A B C D be four names of which any two might be confounded by breaking down some wall of definition, then some one of the names must be a species of some name which includes nothing external to the other three.

Yours truly
A De Morgan

7 [illegible]
Oct 23/52.

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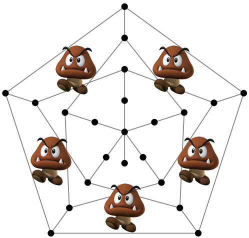
†Picture credit: public domain, see https://en.wikipedia.org/wiki/Four_color_theorem

Chromatic mystery when $\chi = 2$??

List colouring

Imagine *adversaries* to colouring

- that issue arbitrary lists of allowable colours per vertex
- but must give at least k per list



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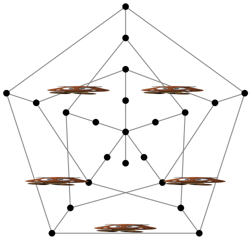
What is least k for which colouring is always possible? (Necessarily $k \geq \chi$.)



List colouring (formally)

Let G be a graph.

A *list-assignment* is some $L : V(G) \to 2^{\mathbb{Z}^+}$;
a *k-list-assignment* is some $L : V(G) \to \binom{\mathbb{Z}^+}{k}$.

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The *list chromatic number* $\mathrm{ch}(G)$ is least k such that G is k-choosable.

Introduced independently by Vizing (1976) and Erdős, Rubin, Taylor (1980).

$\chi = 2$ WHILE $\mathsf{ch} > k$

Theorem (Erdős, Rubin, Taylor 1980)

There are bipartite graphs with arbitrarily large list chromatic number.

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Let $\mathcal{F}$ be a family of finite sets. Consider $K_{|\mathcal{F}|,|\mathcal{F}|}$ with parts A and B.

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Such an X does not exist for $\mathcal{F} = \binom{[2k-1]}{k}$. $\implies \mathrm{ch}\left(K_{\binom{[2k-1]}{k},\binom{[2k-1]}{k}}\right) > k$. □

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OPEN QUESTION

What is the minimum number N(2,k) of nodes in a graph G which is 2-colorable but not k-choosable?

F. Bernstein. Zur theorie der trigonometrische Reihen. *Leipz. Ber.* 60: 325-328, 1908.

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Note $M(k) \leq \binom{2k-1}{k}$.

Theorem (Erdős 1963/4)

$M(k) = \Omega(2^k)$ and $M(k) = O(k^2 2^k)$.

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Theorem (Radhakrishnan & Srinivasan 2000)

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Colour from $L(v) \cap X$ if $v \in A$, from $L(v) \setminus X$ if $v \in B$. □

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Recall a Turán-type classic:

Theorem (Andrásfai, Erdős, Sós 1974)

Any triangle-free graph on n vertices of minimum degree $> 2n/3$ is bipartite.

Erdős, Rubin, Taylor (1980): $\mathrm{ch}(K_{n,n}) \sim \log_2 n$.

Theorem (Alon 2000)

For any G of minimum degree δ, $\mathrm{ch}(G) \gtrsim \frac{1}{2}\log_2 \delta$.

Choice number and minimum degree

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Theorem (Saxton & Thomason 2015)

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If $k \geq (1 + \varepsilon)\Delta / \log \Delta$, then w.p.p. there's spare colour for every $v \in A$. □

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Theorem (Molloy 2019)

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NB2: For bipartite, a persistent **exponential** gap in our knowledge!

A tilted perspective

In the heuristic, the colouring tasks are isolated per part.

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Let's first focus on first.

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Conjecture (Alon & Krivelevich 1998)

For any bipartite G of maximum degree Δ, $\mathrm{ch}(G) = O(\log \Delta)$.

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Lemma

Any bipartite G with parts A and B is (k_A, k_B)-choosable if

$$e\Delta_A(\Delta_B - 1)(1 - (1 - 1/k_B)^{k_B \Delta_A / k_A})^{k_A} \leq 1.$$

Proof of theorem.

$\Delta_A = \Delta_B = \Delta$, $k_A = (1+\varepsilon)\Delta/\log \Delta$, $k_B = \log \Delta$ satisfies the condition.
(In fact, Δ_B superpolynomial in Δ still suffices!) □

Coupon collector again

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$$\leq \prod_{c\in L(v)} (1 - (1 - 1/k_B)^{x(c)}) = \exp\left(k_A \frac{\sum_{c\in L(v)} \log(1 - (1 - 1/k_B)^{x(c)})}{\sum_{c\in L(v)} 1}\right)$$

$$\leq \exp\left(k_A \log(1 - (1 - 1/k_B)^{\sum_{c\in L(v)} x(c)/k_A}\right) \leq (1 - (1 - 1/k_B)^{k_B\Delta_A/k_A})^{k_A}.$$

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The lemma follows by checking mutual independence for the local lemma. □

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What happens for complete bipartite graphs?

Hypergraph Turán numbers

For $2 \leq k_1, k_2 \leq \ell$, a family $\mathcal{F} \subseteq \binom{[\ell]}{k_2}$ has *Property* $A(k_1, k_2, \ell)$ if there is some set $X \in \binom{[\ell]}{k_1}$ that intersects every member of $\mathcal{F}$.

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$\overline{M}(k_1, k_2, \ell)$ is minimum number of edges in k_2-uniform hypergraph on ℓ vertices with no independent set of size $\ell - k_1$.

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The object of Rödl's theorem (erstwhile Erdős–Hanani conjecture).

Among other things, we know

Theorem (Erdős & Spencer 1974)

For $k_1, k_2 \ge 2$ and $\ell \ge k_1 + k_2$,

$$\frac{\ell!(\ell - k_1 - k_2)!}{(\ell - k_2)!(\ell - k_1)!} \le \overline{M}(k_1, k_2, \ell) < \frac{\ell!(\ell - k_1 - k_2)!}{(\ell - k_2)!(\ell - k_1)!} \log \binom{\ell}{k_1}.$$

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For $k_A, k_B \geq 2$ and $\ell = k_1 + k_2 + 1$, $K_{\overline{M}(k_1,k_A,\ell),\overline{M}(k_2,k_B,\ell)}$ isn't (k_A, k_B)-choosable.

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X_A intersects every member of $\mathcal{F}_A$, so $|X_A| > k_1$. Similarly, $|X_B| > k_2$.
But $|X_A| + |X_B| \geq k_1 + k_2 + 2 > \ell$, contradicting that colouring is proper. □

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Not far from extremal for most of 4-parameter space for complete bipartite, and perhaps even for general bipartite graphs. . .

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Asymmetric Alon–Krivelevich conjecture

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3. $\Delta_A = \Delta_B = \Delta$, *and, for some* $C > 0$,

$$k_B \geq C(\Delta/\log\Delta)^{1/k_A} \log\Delta \quad \text{or} \quad k_A \geq C(\Delta/\log\Delta)^{1/k_B} \log\Delta.$$

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This is well-defined and $\mathrm{ch}^\star \geq \mathrm{ch}$.

Conjecture (Cambie, Cames van Batenburg, Davies, Kang 2021+)

$\mathrm{ch}^\star(G) = O(\mathrm{ch}(G))$.

NB: We know $\mathrm{ch}^\star(G) = 2^{O(\mathrm{ch}(G))}$.

Bipartite list packing

Theorem (Cambie, Cames van Batenburg, Davies, Kang 2021+)

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Conjecture (Cambie, Cames van Batenburg, Davies, Kang 2021+)
For any triangle-free G of maximum degree Δ, $\mathrm{ch}^\star(G) = O(\Delta/\log\Delta)$.

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Questions?