A Precise Condition for Independent Transversals in Bipartite Covers

Ross J. Kang*

University of Amsterdam

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* Joint with Stijn Cambie, Penny Haxell, Ronen Wdowinski. Support from NWO Vidi grant.

Shameless plugs

1. Postdoc opportunities in Amsterdam!

- 12 to 24 months, starting ASAP
- eligibility: ≤ 2 years from PhD, ≤ 12 of previous 48 months based in NL
- affiliated with NETWORKS consortium

2. Launch of *Innovations in Graph Theory* journal!

INNOVATIONS
IN GRAPH THEORY

igt.centre-mersenne.org

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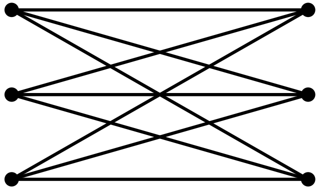
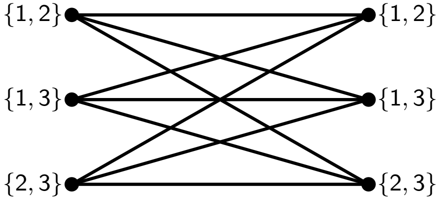
INNOVATIONS IN GRAPH THEORY

How best to colour a bipartite graph?



How best to colour a bipartite graph?



List colouring (formally)

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An L-colouring is some $c : V(G) \to \mathbb{Z}^+$ with $c(v) \in L(v)$ for every $v \in V(G)$.

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The list chromatic number $\chi_\ell(G)$ is least k such that G is k-choosable.

$\chi_\ell(G) \geq \chi(G)$ by considering constant L.

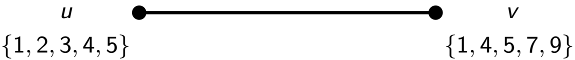
How best to colour a bipartite graph?

Conjecture (Alon & Krivelevich 1998)

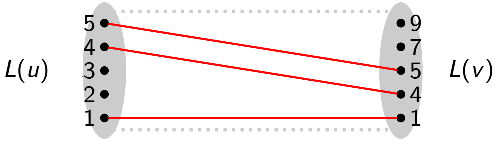
For bipartite G of maximum degree Δ, $\chi_\ell(G) = O(\log \Delta)$ as $\Delta \to \infty$

Best possible if true, but only $O(\Delta / \log \Delta)$ bound known

List-assignments



List-covers


$L(u)$
5
4
3
2
1
9
7
5
4
1
$L(v)$

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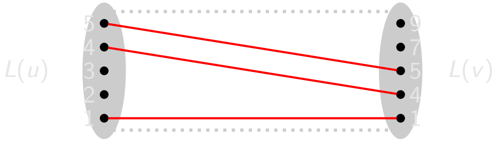
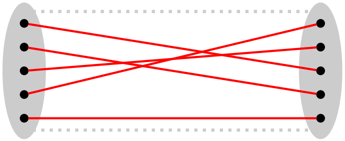
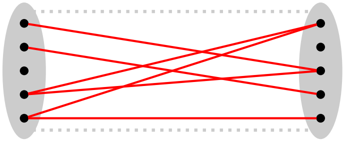


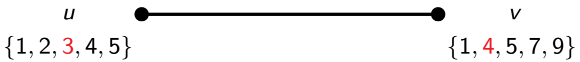


Covers



List-colouring ≡ IT in list-cover

We want independent transversals (ITs) in vertex-partitioned graphs.



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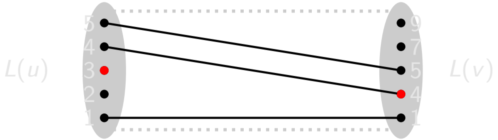

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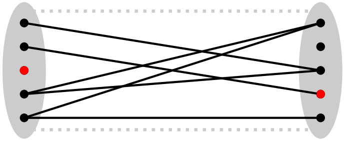

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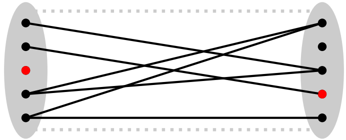

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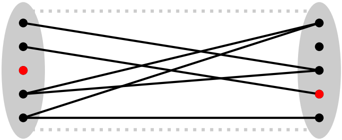

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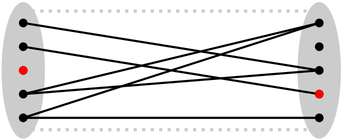

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Theorem (Haxell 2001)

Any graph of maximum degree D with a vertex-partition into parts of size $2D$ admits an IT.

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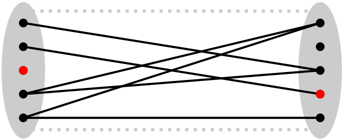

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Theorem (Szabó & Tardos 2005)

There is a graph of maximum degree D with a vertex-partition into parts of size $2D-1$ that does not admit an IT.

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Let degree conditions, list-/part-sizes depend on the side

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Notation: Call the sides A and B, with max degrees Δ_A, Δ_B and sizes k_A, k_B

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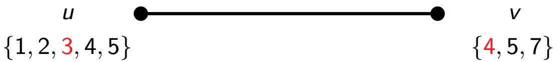
(**Warning:** Parameter soup!)

Asymmetric bipartite list colouring



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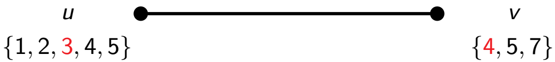

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Given bipartite graph of maximum degree Δ, any list-assignment with A-lists of size $\log \Delta$ and B-lists of size $(1+\varepsilon)\Delta/\log \Delta$ admits a list-colouring.

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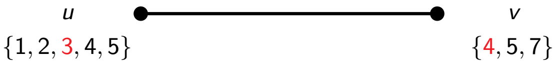

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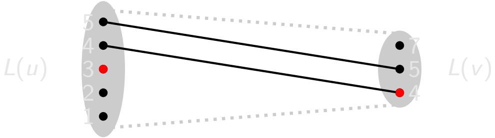
Conjecture (Alon, Cambie, Kang 2021, the ‘crossed’ one)

Given bipartite graph with max A-degree Δ_A, max B-degree Δ_B, any list-assignment with A-lists of size $C \log \Delta_B$, B-lists of size $C \log \Delta_A$ admits a list-colouring.

ITs in asymmetric bipartite list-covers



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Any bipartite list-cover of maximum degree Δ *with*
A-parts of size $\log \Delta$ *and B-parts of size* $(1+\varepsilon)\Delta/\log \Delta$ *admits an IT.*

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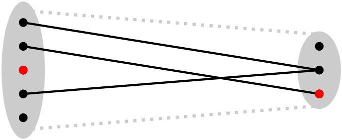

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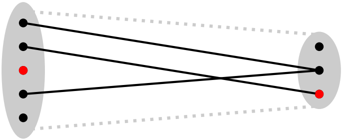

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Without any constraint on the structure between parts (like in a list-cover), what conditions on parameters Δ_A, Δ_B and k_A, k_B guarantee an IT?

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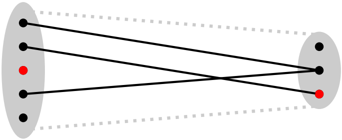

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Any bipartite cover with max A-degree Δ_A, max B-degree Δ_B, A-parts of size k_A, B-parts of size k_B such that

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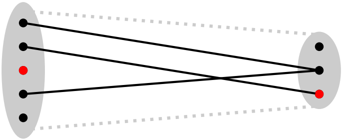

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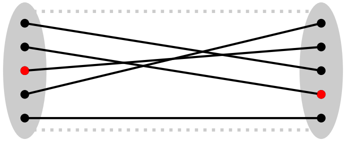
ITs in bipartite correspondence-covers



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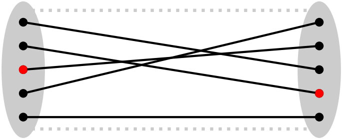

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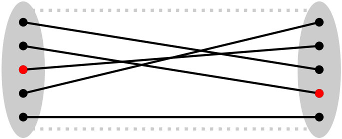

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(**Warning:** Equivalent to Alon–Krivelevich conjecture!)

Questions?