Local graph colouring

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*With Davies, de Joannis de Verclos, Pirot, Sereni. Support from NWO.

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(It epitomises the classic NP-hard problems)

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- many colouring methods we have are ‘still’ local! (push further?)

Brooks' theorem

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Do we have finer bound using finer degree information?

Theorem (Caro 1979, Wei 1981)

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}$$

$$\implies \alpha(G) \geq \frac{n(G)}{\Delta(G) + 1}$$

Encoding colouring more locally

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Recall

- *list-assignment* is some $L : V(G) \rightarrow 2^{\mathbb{Z}^+}$
- *L-colouring* is some $c : V(G) \rightarrow \mathbb{Z}^+$ with $c(v) \in L(v)$ for every $v \in V(G)$
- G is *k-choosable* if there is a proper L-colouring for any list-assignment L with $|L(v)| \geq k$ for every $v \in V(G)$
- *list chromatic number* $\mathrm{ch}(G)$ is least k such that G is k-choosable

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Can allow k to 'vary' with v, or rather, according to quantities local to v

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Theorem (Borodin 1979, Erdős, Rubin, Taylor 1980, cf. Gallai 1963)

G is not degree-choosable if and only if G a Gallai tree

Quantities local to v

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With various local quantities, what 'natural' choices of f suffice?

Systematically more local?

- Bonamy, Kelly, Nelson, Postle 2022 initiated

[†]More fully, the lattice gas with hard-core self-repulsion and nearest-neighbour exclusion.
Picture credit: Wikipedia/Grap-wh

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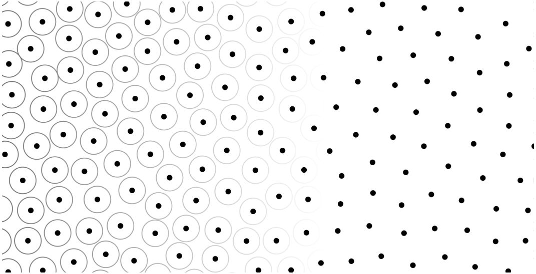
$\rightarrow$ *local occupancy method*



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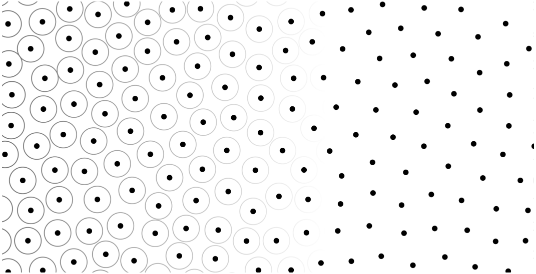

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- today we highlight a particularly elegant setting, the triangle-free case

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Colouring triangle-free graphs: a brief history

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Due to random Δ-regular graphs, this is asymptotically sharp up to a factor 2

Incidentally matches the longstanding bound for $R(3, k)$ due to Shearer 1983

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Fix $\varepsilon > 0$*. For* Δ *sufficiently large and* $\delta = (192 \log \Delta)^{2/\varepsilon}$*, any triangle-free graph of maximum degree* Δ *and minimum degree* δ *is* f*-choosable with*

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In fact there are analogous results for *all* the local quantities mentioned earlier, though always with the caveat of some δ condition

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Proposition (Davies, de Joannis de Verclos, Kang, Pirot 2020)

For any δ, there is a bipartite graph of maximum degree $\Delta := \exp^{\delta-1}(\delta)$ and minimum degree δ that is not f-choosable with

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Problem

What is the largest degree span that suffices for a 'local Molloy's'?

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Questions?