2008

What is the fewest independent sets a triangle-free graph can have?

Ross Kang

University of Amsterdam

BGTC, Brussels, 7/2025

Vraagstuk XXVIII.

K 13 a. Er zijn eenige punten gegeven waarvan geen vier in een zelfde vlak liggen. Hoeveel rechten kan men hoogstens tusschen die punten trekken zonder driehoeken te vormen? (W. MANTEL.)

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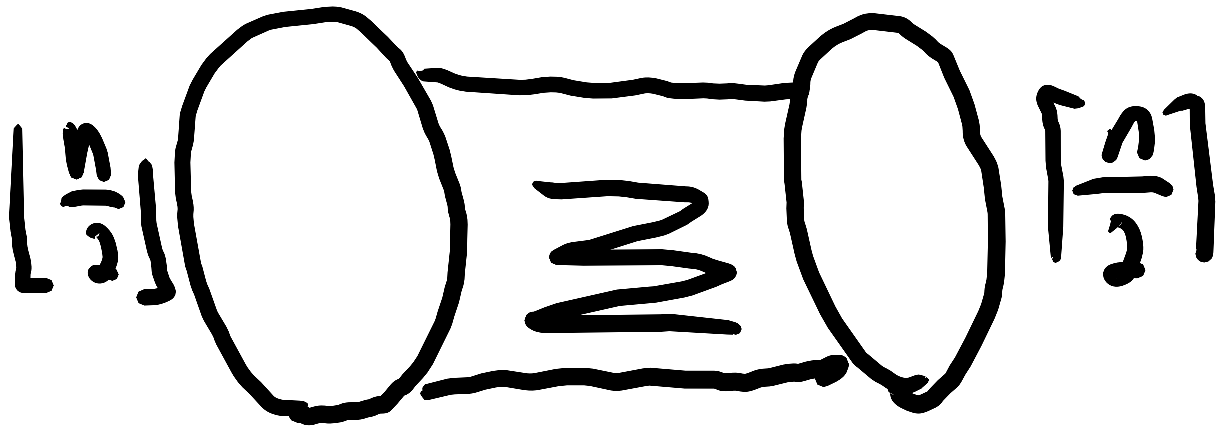

ON A PROBLEM OF FORMAL LOGIC

By F. P. RAMSEY.

<u>Ramsey's theorem</u> (a special case)

There is some (smallest) integer $n = R(3,k)$ such that every graph on n vertices contains a triangle or its complement contains K_k

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Known(!): $\frac{k^2}{3\log k} \lesssim R(3,k) \lesssim \frac{k^2}{\log k}$

Hard-core model

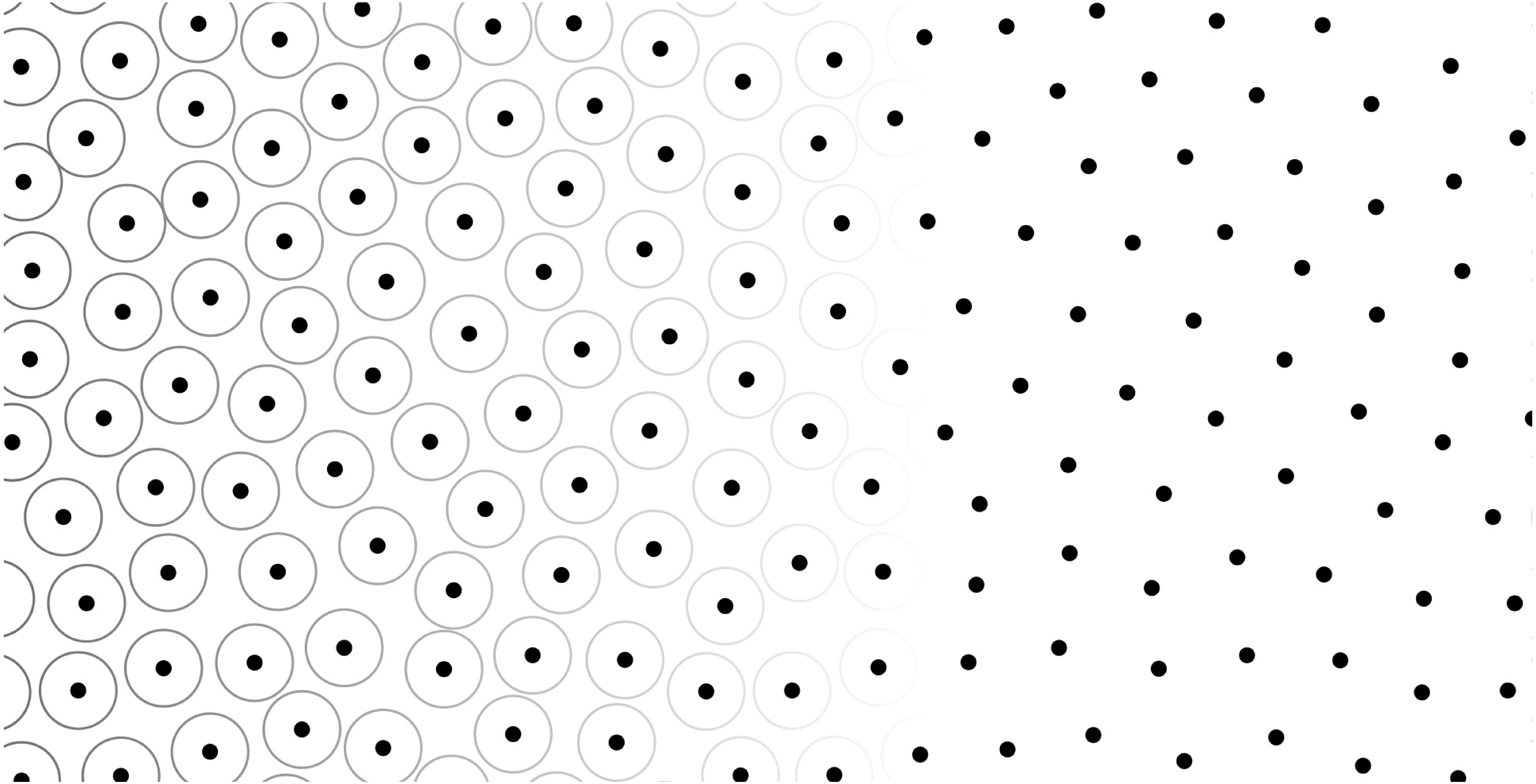

Image: Wiki/Grap-wh

Turán 1941:

Any graph on n vertices with more edges than the balanced complete $(r-1)$-partite graph $T_{r-1,n}$ on n vertices must contain a K_r

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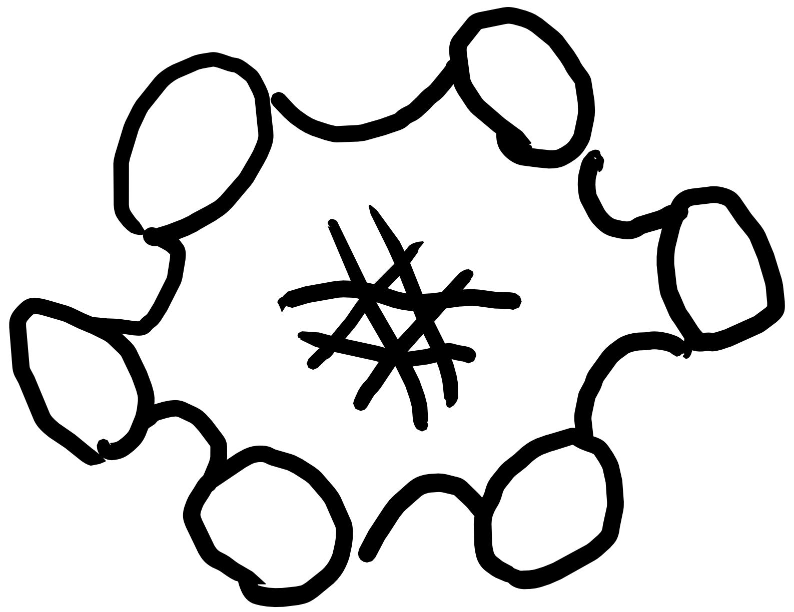

$T_{r-1,n}$ Turán graph

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NB: Complement of $T_{r-1,n}$ is balanced disjoint union of cliques

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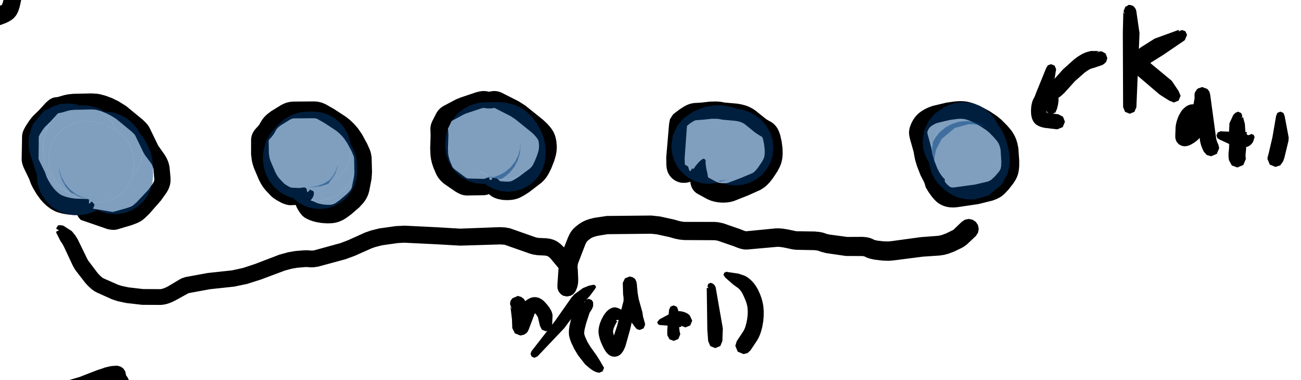

Larger guarantee if we exclude (i.e. cliques)?

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Does sparsity induce structure?

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Any triangle-free graph on n vertices of average degree d contains an independent set of size $(1+o(1))\,\frac{n}{d}\log d$

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$$\implies R(3,k) \leq (1+o(1))\frac{k^2}{\log k} \quad \text{(stuck since 1983)}$$

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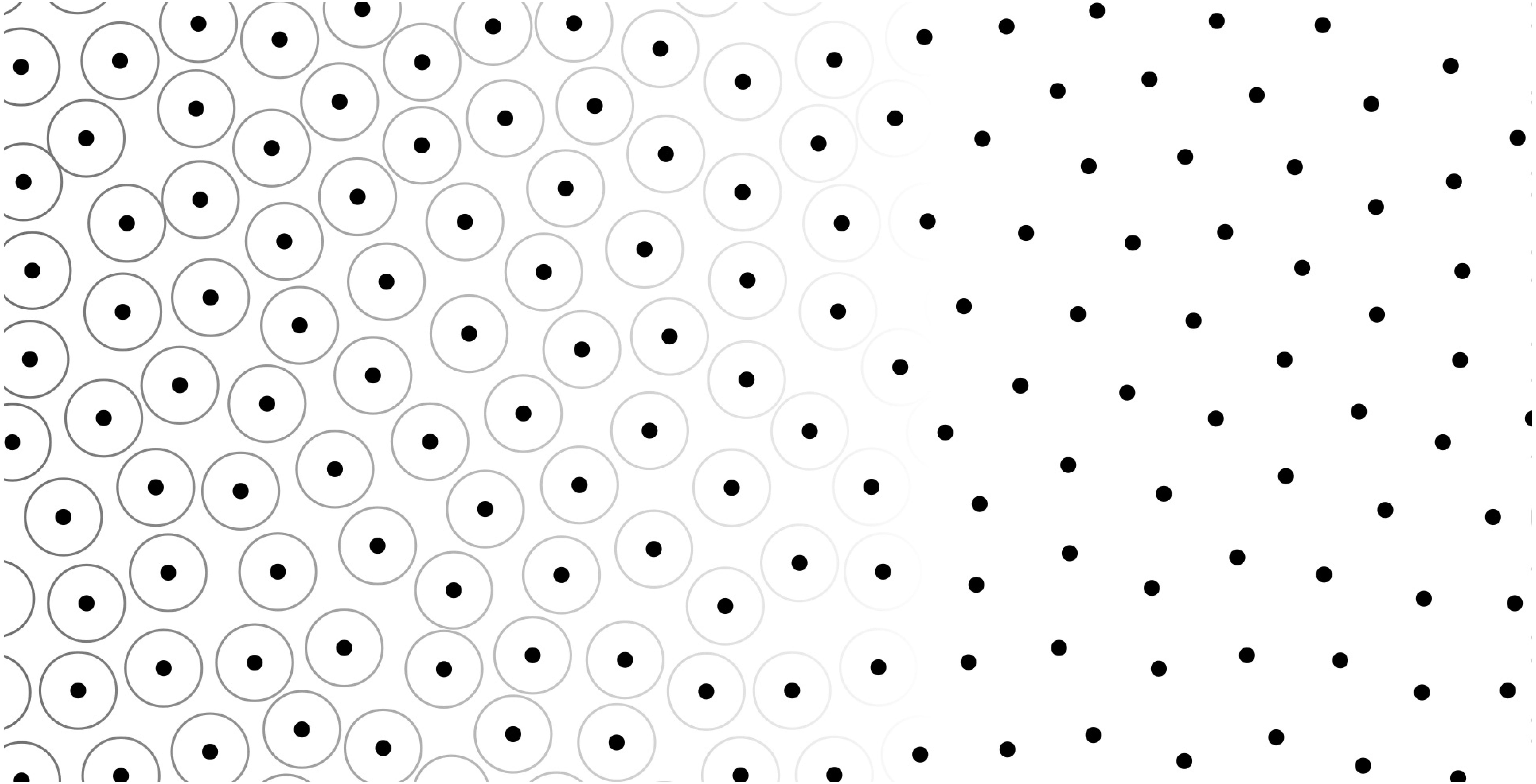
$$\Rightarrow R(3,k) \leq (1+o(1))\frac{k^2}{\log k} \quad \text{(stuck since 1983)}$$

NB: There are triangle-free graphs on n vertices of avg deg d with independent sets of size no larger than

$$(2+o(1))\frac{n}{d}\log d$$

Hard-core model



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Davies, Jenssen, Perkins, Roberts 2018:

Any triangle-free graph on n vertices of maximum degree d has average independent set size at least $(1 + o(1)) \frac{n}{d} \log d$

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prob method

$\Longrightarrow$ $R(3,k) \leq (1+o(1)) \frac{k^2}{\log k}$ (stuck since 1983)

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integration

$$\Longrightarrow \quad \log Z_G(\lambda) \geq (1+o(1))\frac{n}{2d}(\log d)^2$$

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$$\overset{\text{integration}}{\Longrightarrow} \qquad \log Z_G(\lambda) \geq (1+o(1)) \frac{n}{2d} (\log d)^2$$

NB: There are triangle-free d-regular graphs on n vertices with log. independent set count $\log Z_G(\lambda)$ no larger than

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Buys, vd Heuvel, Kang 2025+

Any triangle-free graph G on n vertices of average degree d has

$$\log Z_G(\lambda) \geq$$

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Buys, vd Heuvel, Kang 2025+

Any triangle-free graph G on n vertices of average degree d has

$$\log Z_G(\lambda) \geq \frac{n}{2(d-2)} \left(W(\lambda d)^2 + 2W(\lambda d) - O(1) \right)$$

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Buys, vd Heuvel, Kang 2025+

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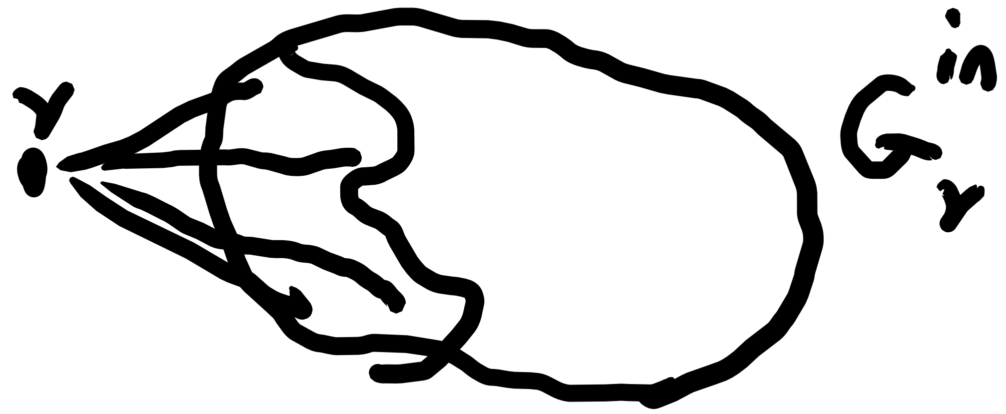
Observation: $\forall r \in V(G)$



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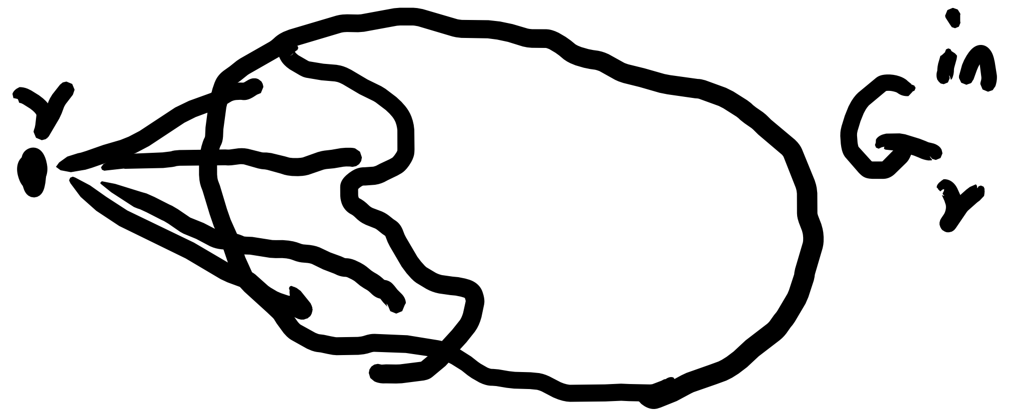

$$\alpha(G) \geq 1 + \alpha(G_v^{in})$$

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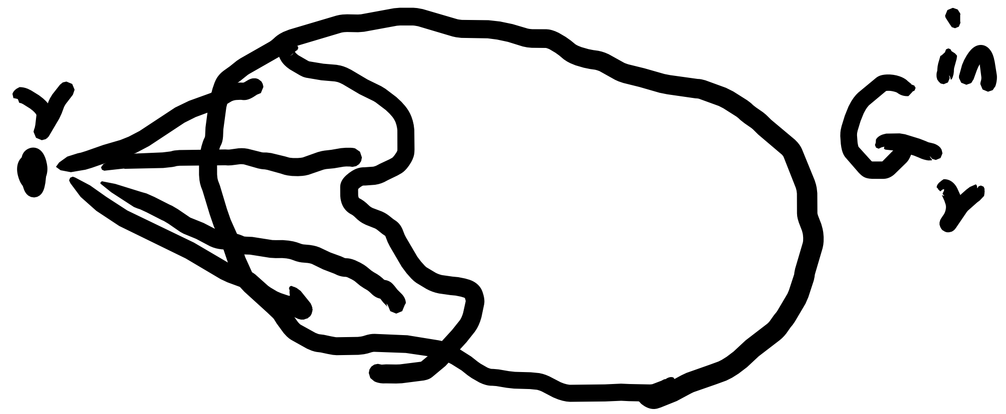

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$$\overset{ind}{\geq} 1 + f(d(G_v^{in})) \cdot n(G_v^{in})$$

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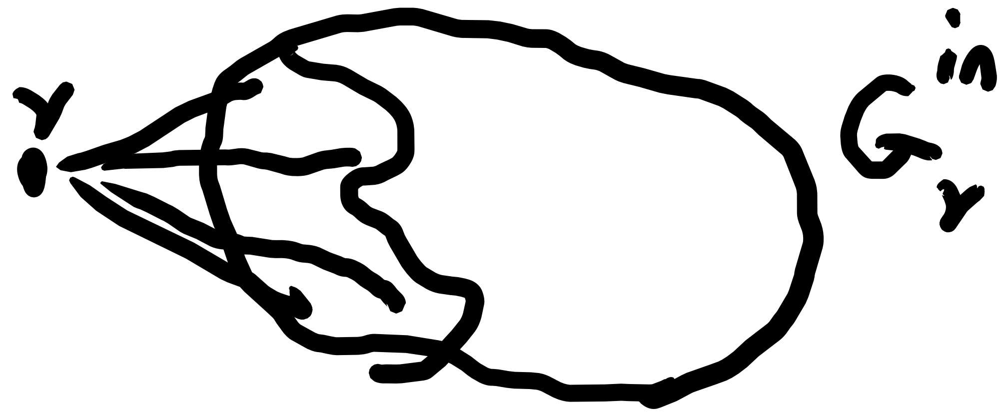

$$\alpha(G) \geq 1 + \alpha(G_r^{in})$$

$$\overset{\text{ind}}{\geq} 1 + f(d(G_r^{in})) \cdot n(G_r^{in})$$

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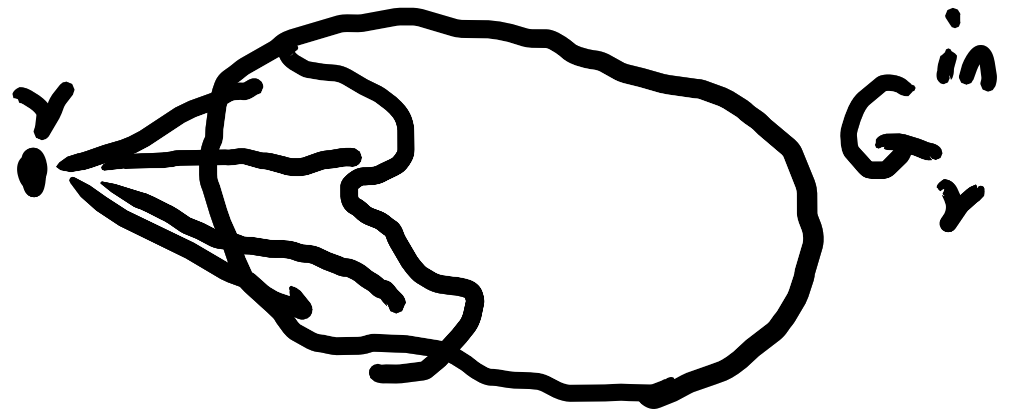

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Uniform random v, writing $d = d(G)$,

$$\mathbb{E}(\text{RHS}) \overset{\Delta\text{-free}}{\geq}$$

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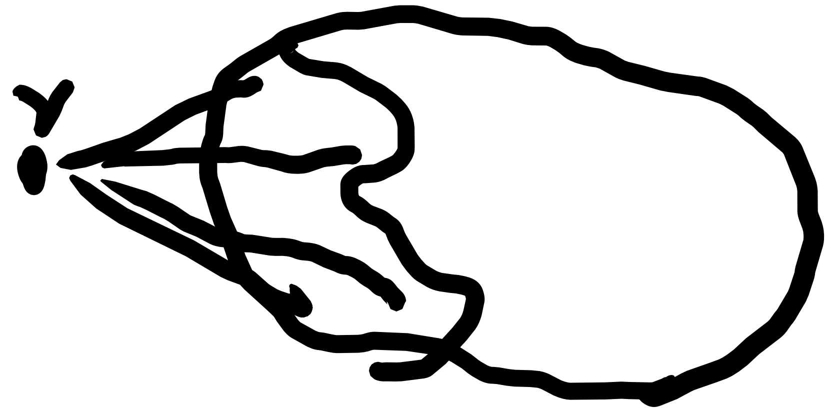

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Uniform random v, writing $d = d(G)$,

$$\mathbb{E}(RHS) \overset{\Delta\text{-free}}{\geq} f(d)n + 1 + (d - d^2) f'(d) - (1+d) f(d)$$

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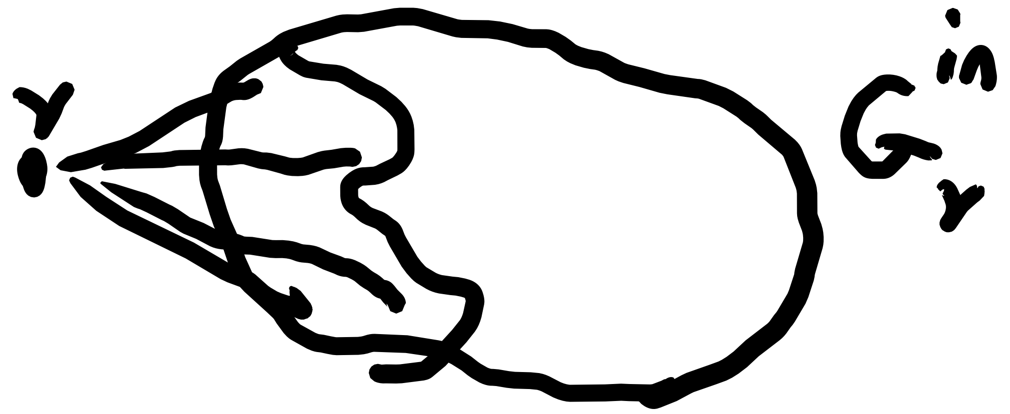

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reduces the problem to finding some $f : \mathbb{R}^+ \to \mathbb{R}$

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$$\Rightarrow \alpha \geq n f(d) \quad \text{where} \quad f(d) = \frac{d \log d - d + 1}{(d-1)^2}$$

Shearer-style induction for indep set count

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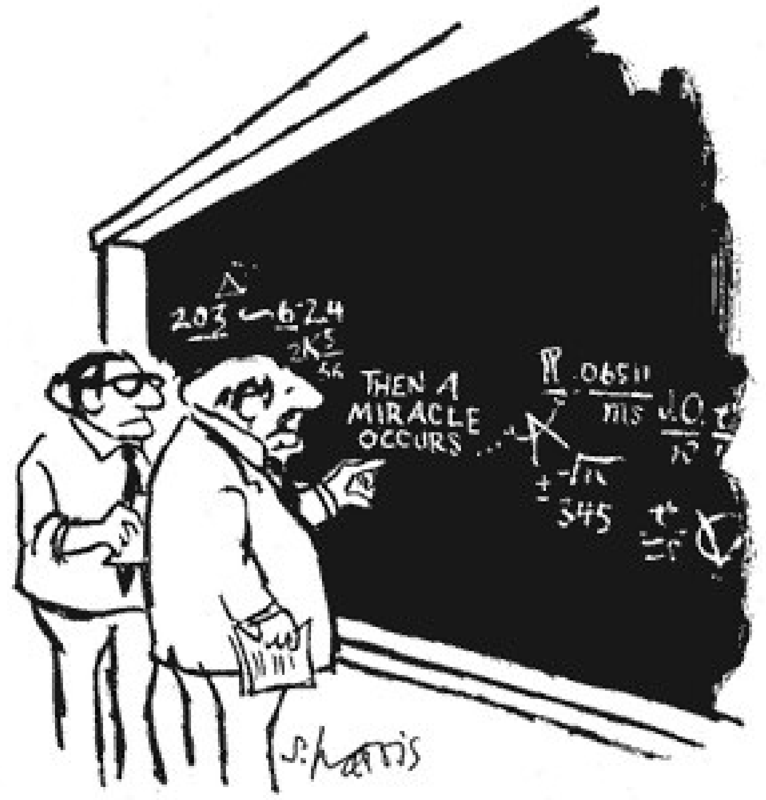
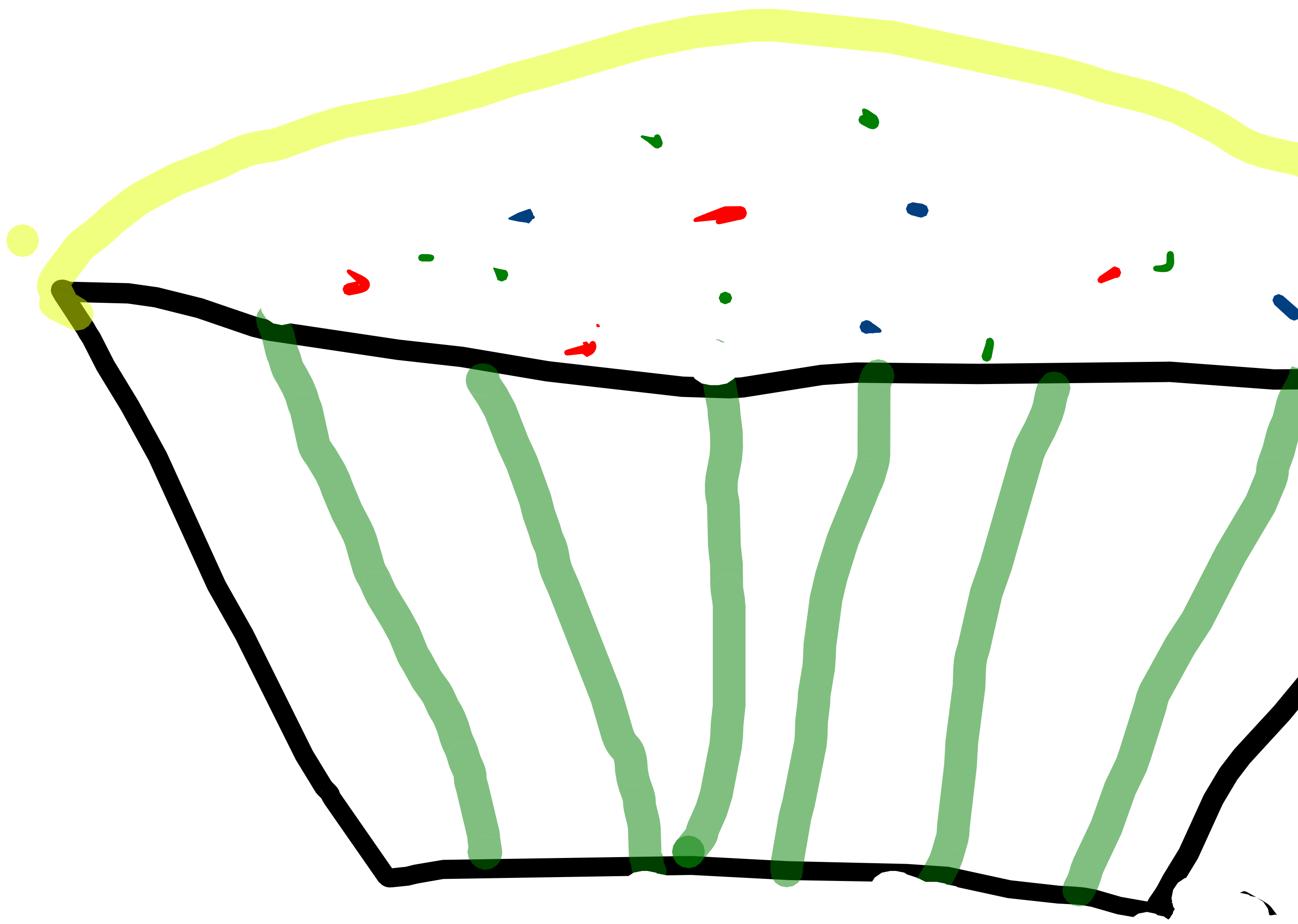
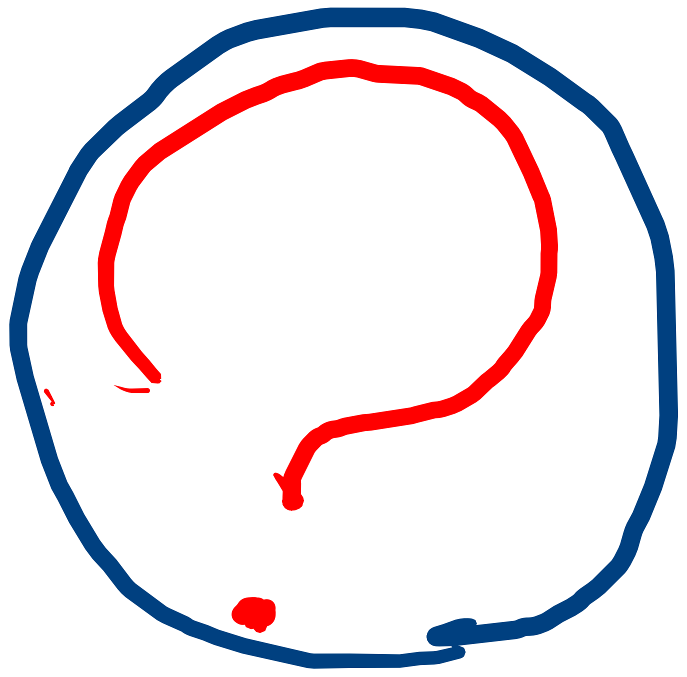


"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Thanks for your attention!





Questions?