
4/2016, Oxford

Independent Set Count & Independent Transversal Connectedness

Ross Kang

University of Amsterdam

2/2026, Oxford

Vraagstuk XXVIII.

K 13 a. Er zijn eenige punten gegeven waarvan geen vier in een zelfde vlak liggen. Hoeveel rechten kan men hoogstens tusschen die punten trekken zonder driehoeken te vormen? (W. Mantel.)

Mantel's theorem (1907)

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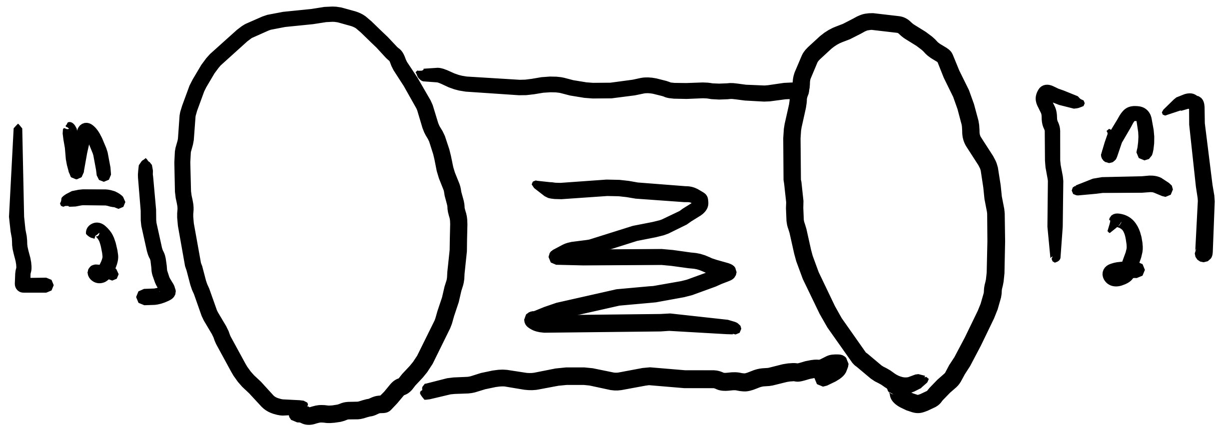

ON A PROBLEM OF FORMAL LOGIC

By F. P. Ramsey.

<u>Ramsey's theorem</u> (a special case)

There is some (smallest) integer $n = R(3,k)$ such that every graph on n vertices contains a triangle or its complement contains K_k

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Known(!): $\frac{k^2}{2\log k} \lesssim R(3,k) \lesssim \frac{k^2}{\log k}$

Once you have ONE

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You have MANY

Turán 1941:

Any graph on n vertices with more edges than the balanced complete $(r-1)$-partite graph $T_{r-1,n}$ on n vertices must contain a K_r

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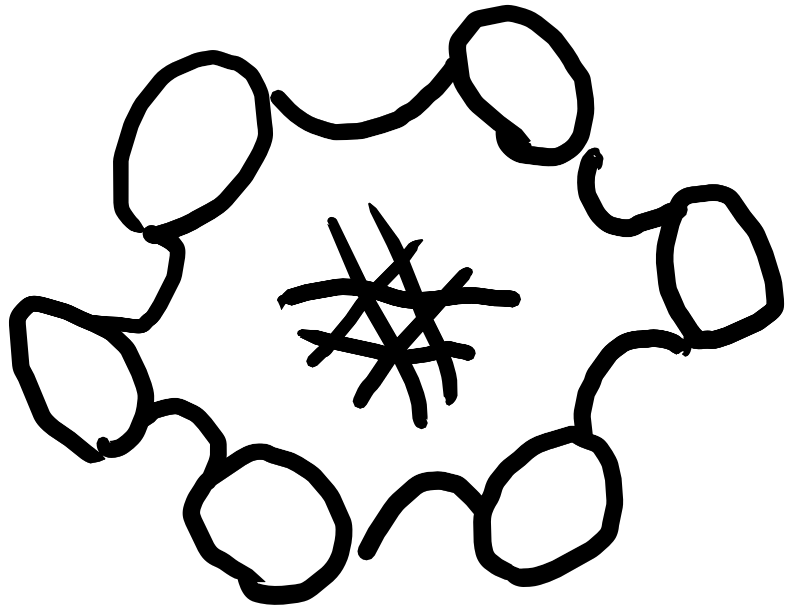

$T_{r-1,n}$ Turán graph

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Any graph on n vertices of average degree d must contain an independent set of size $\frac{n}{d+1}$

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NB: Complement of $T_{r-1,n}$ is balanced disjoint union of cliques

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Larger guarantee if we exclude (i.e. cliques)?

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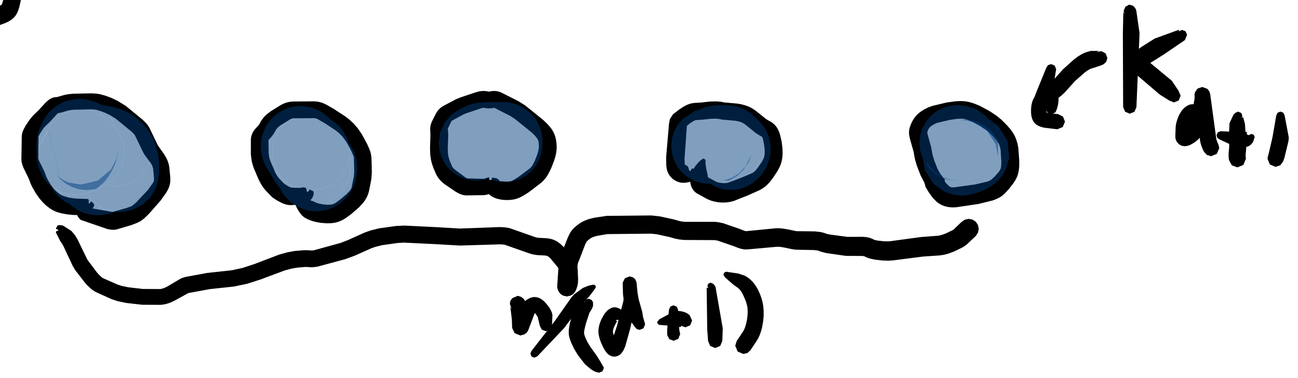

Does sparsity induce structure?

Shearer 1983, after Ajtai, Komlós, Szemerédi 1980/1:

Any triangle-free graph on n vertices of average degree d contains an independent set of size $(1+o(1))\frac{n}{d}\log d$

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NB: There are triangle-free graphs on n vertices of avg deg d with independent sets of size no larger than

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Hard-core model

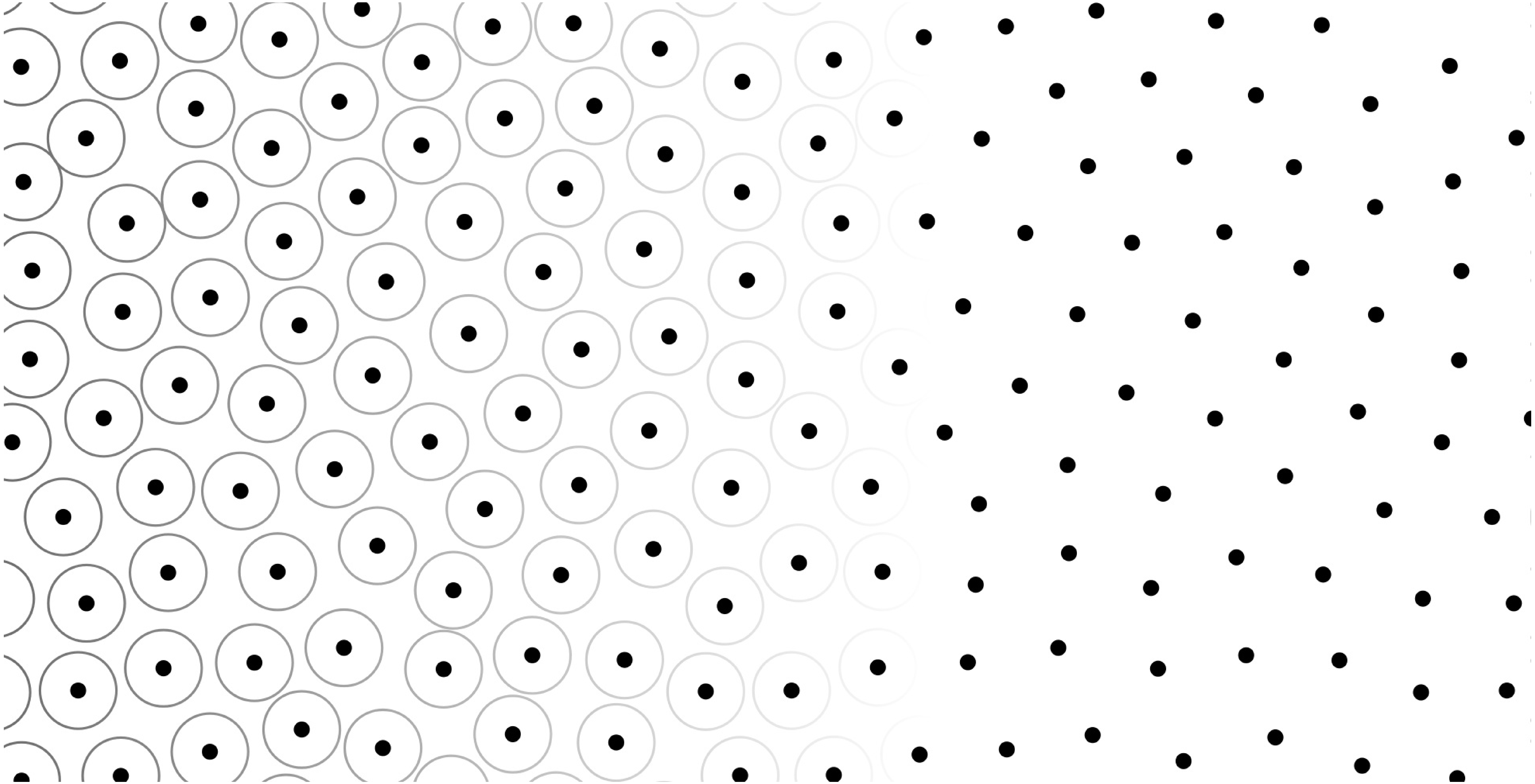

Image: Wiki/Grap-wh

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Any triangle-free graph on n vertices of maximum degree d has average independent set size at least $(1 + o(1)) \frac{n}{d} \log d$

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prob method

$\Longrightarrow R(3,k) \leq (1+o(1))\frac{k^2}{\log k}$ (stuck since 1983)

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$$\xRightarrow{\text{integration}} \quad \log Z_G(\lambda) \geq (1+o(1))\frac{n}{2d}(\log d)^2$$

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integration

$$\Longrightarrow \quad \log Z_G(\lambda) \geq (1+o(1))\frac{n}{2d}(\log d)^2$$

NB: There are triangle-free d-regular graphs on n vertices with log. independent set count $\log Z_G(\lambda)$ no larger than

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Buys, vd Heuvel, Kang 2025+

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Buys, vd Heuvel, Kang 2025+

Any triangle-free graph G on n vertices of average degree d has

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Shearer's induction for independence number α

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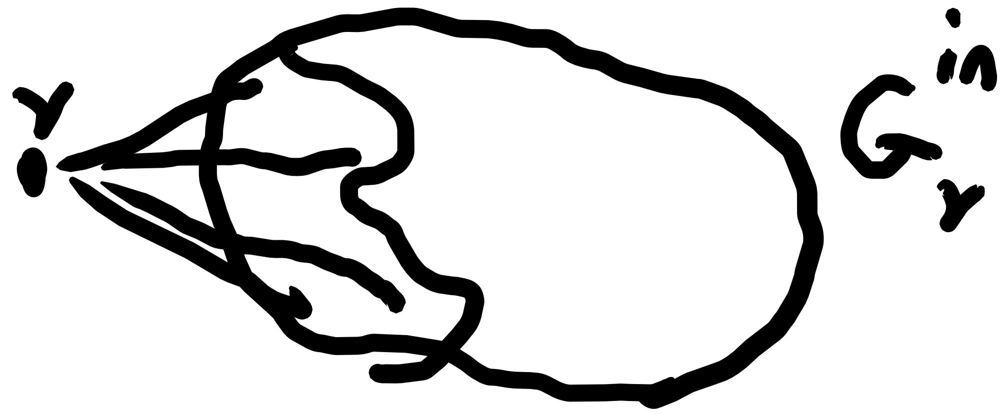
Observation: $\forall r \in V(G)$



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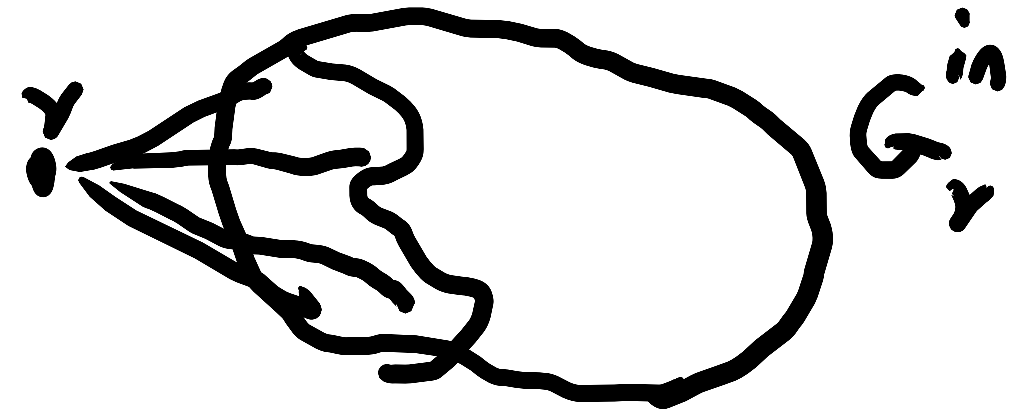
$$\alpha(G) \geq 1 + \alpha(G_v^{in})$$



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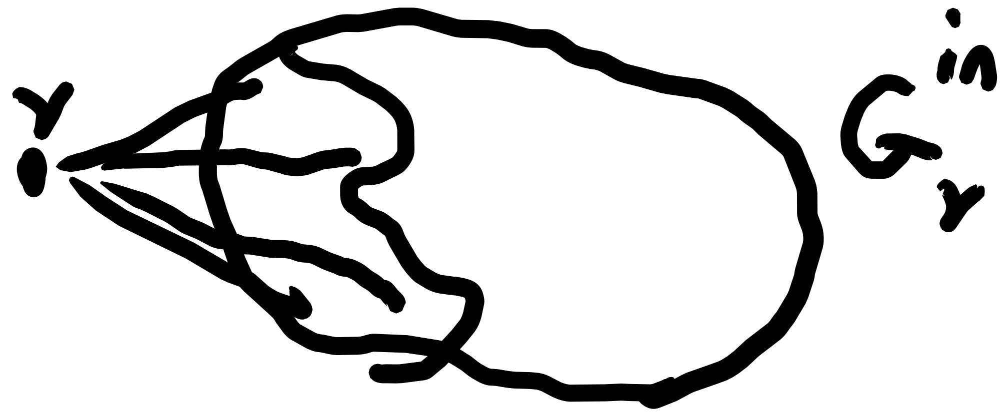

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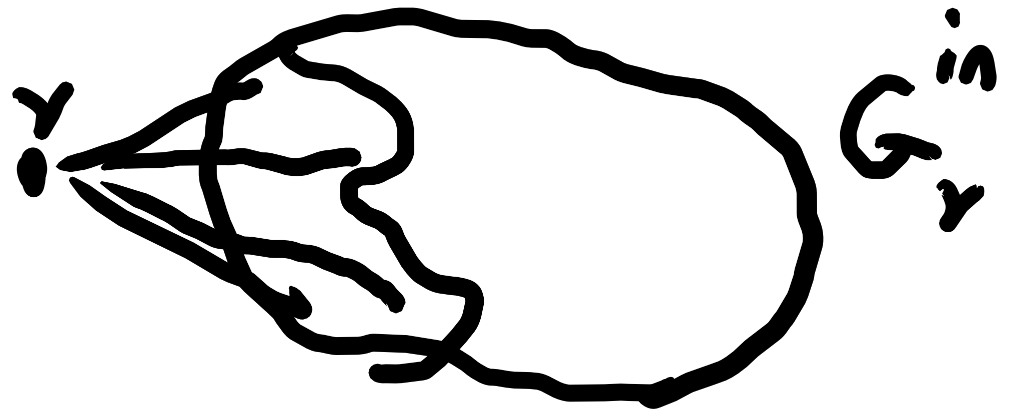

$$\begin{aligned}
\alpha(G) &\geq 1 + \alpha(G_v^{in}) \\
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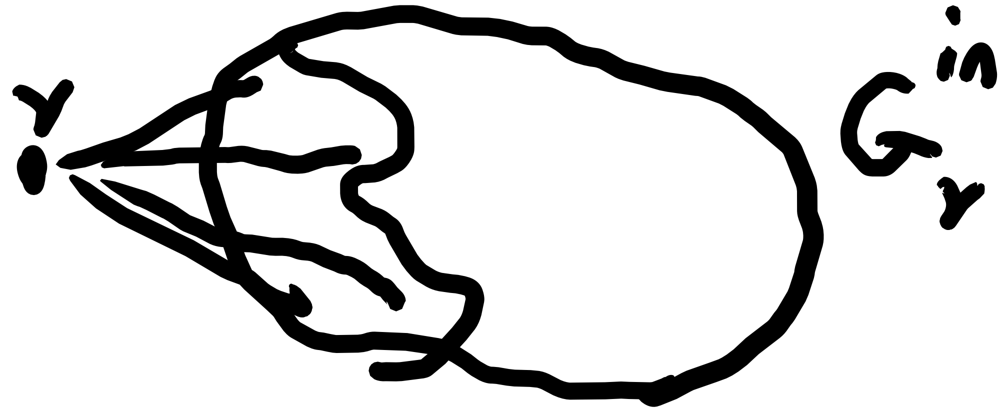

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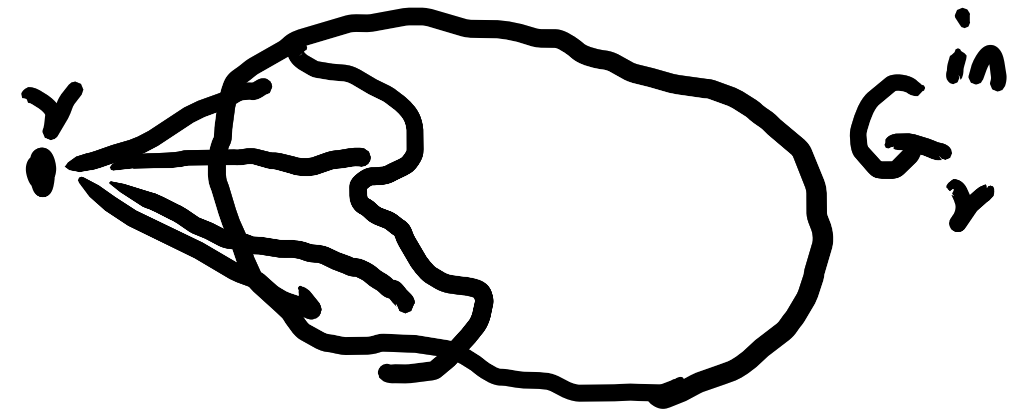

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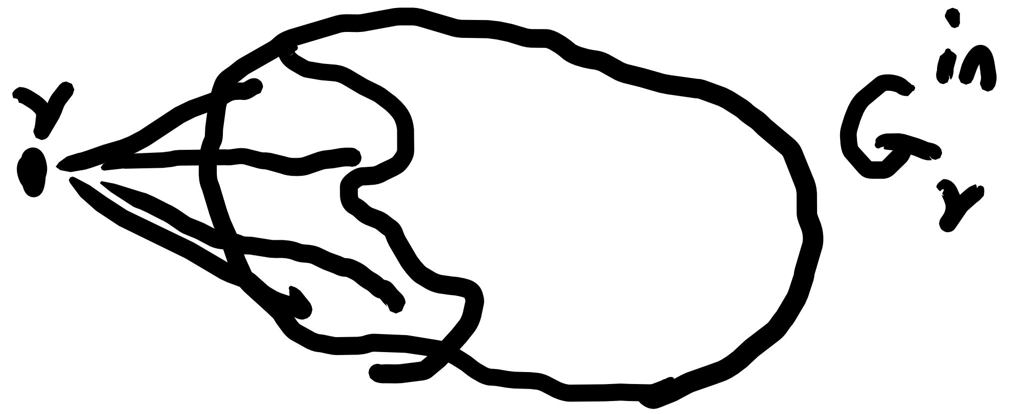

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reduces the problem to finding some $f : \mathbb{R}^+ \to \mathbb{R}$

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$$\Rightarrow \alpha \geq n f(d) \quad \text{where} \quad f(d) = \frac{d \log d - d + 1}{(d-1)^2}$$

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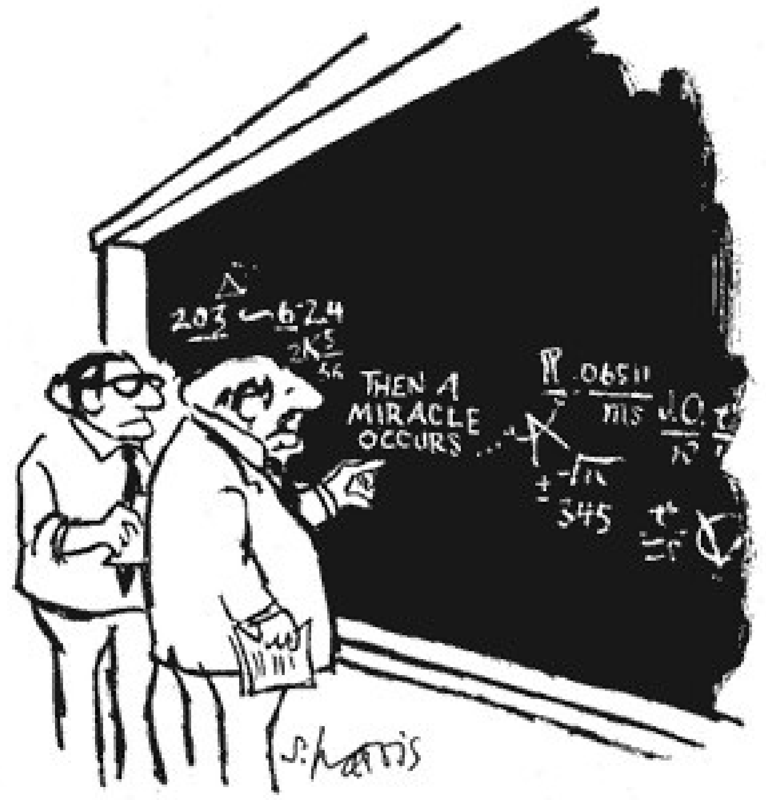

"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

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Buys, vd Heuvel, Kang 2025+

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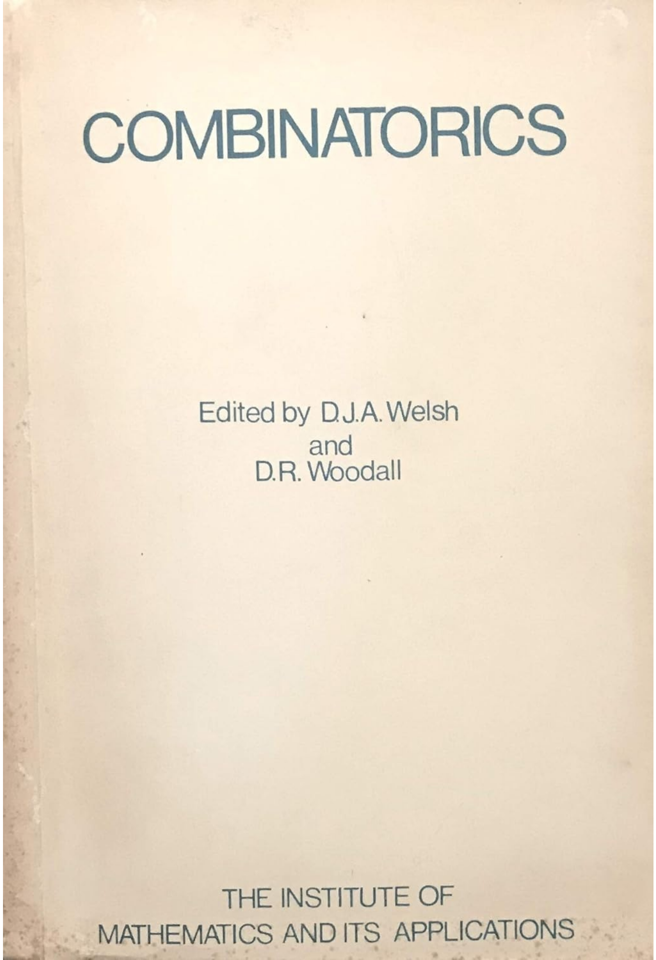
Andy Warhol



3rd BCC
in Oxford

Open problem session, 1972

Woodall: A Hall-type problem . . .

Open problem session, 1972

Woodall: A Hall-type problem . . . with Erdős . . .

Open problem session, 1972

Woodall: A Hall-type problem ... with Erdős ...

Problem (see Bollobás, Erdős, Szemerédi 1975)

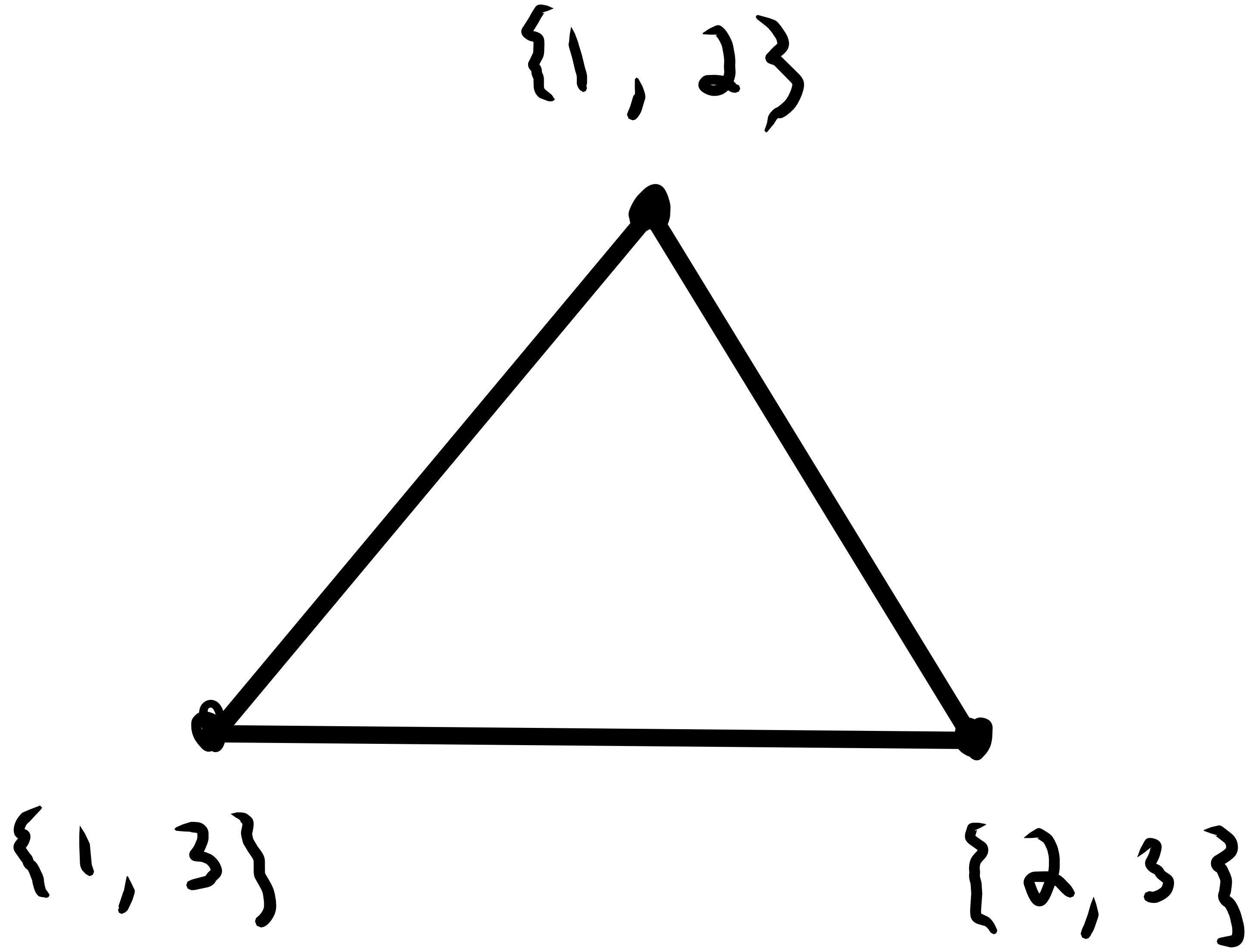
Given Δ, what is least $t = t(\Delta)$?

For any G of maximum degree Δ,

vertex-partitioned with each part $\geq t$ vertices,

there is guaranteed an indep set transversal to partition

Example: **list colouring**



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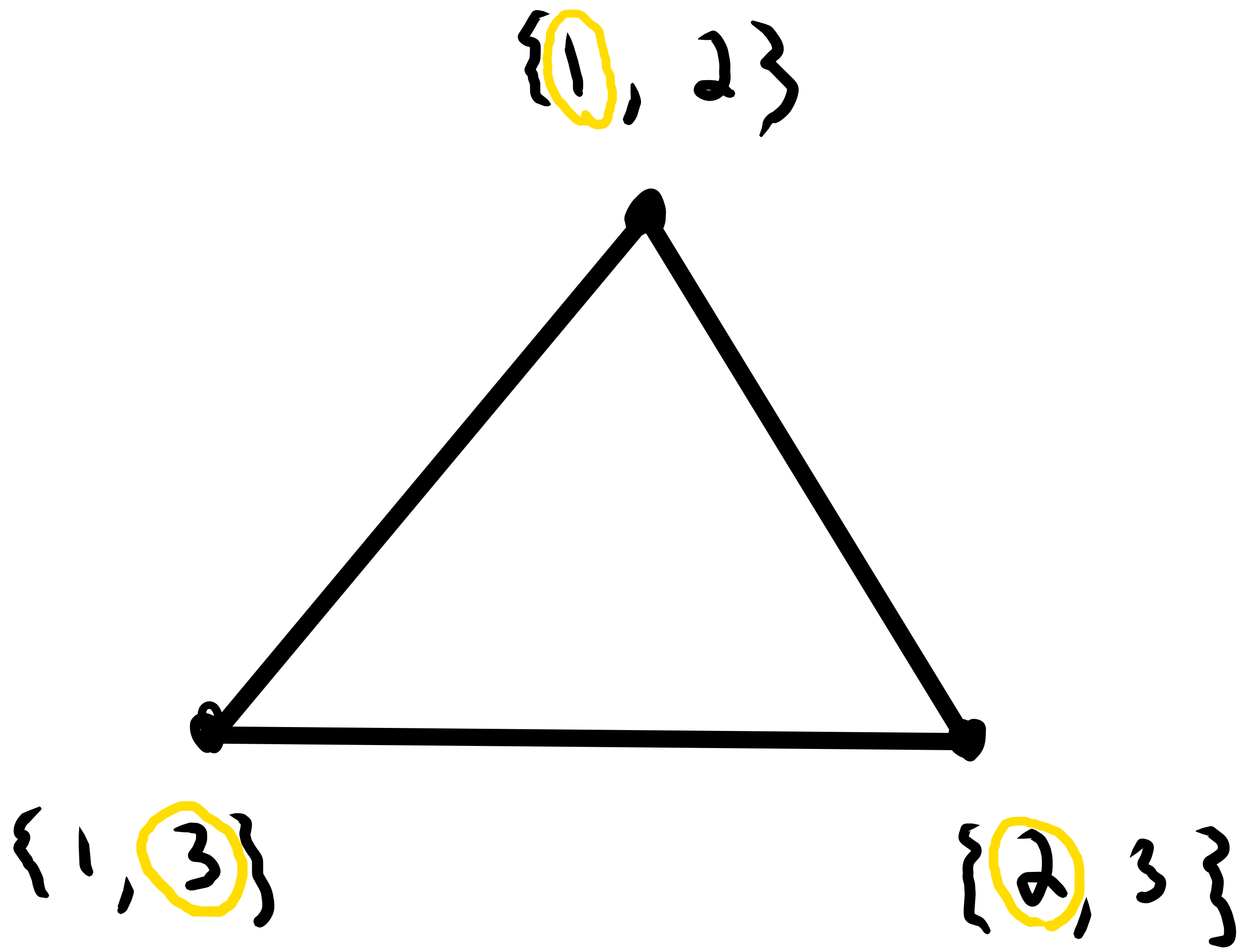

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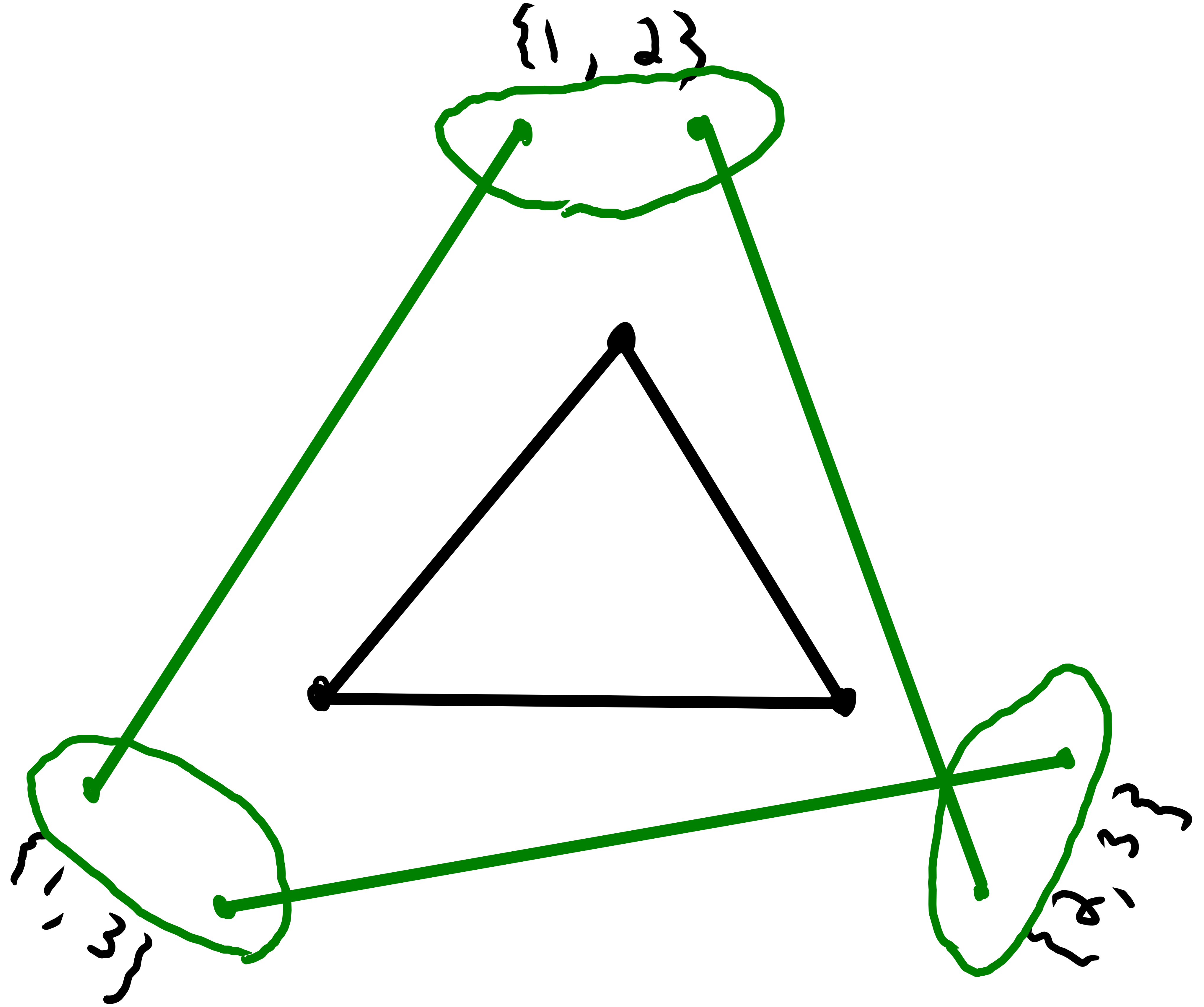

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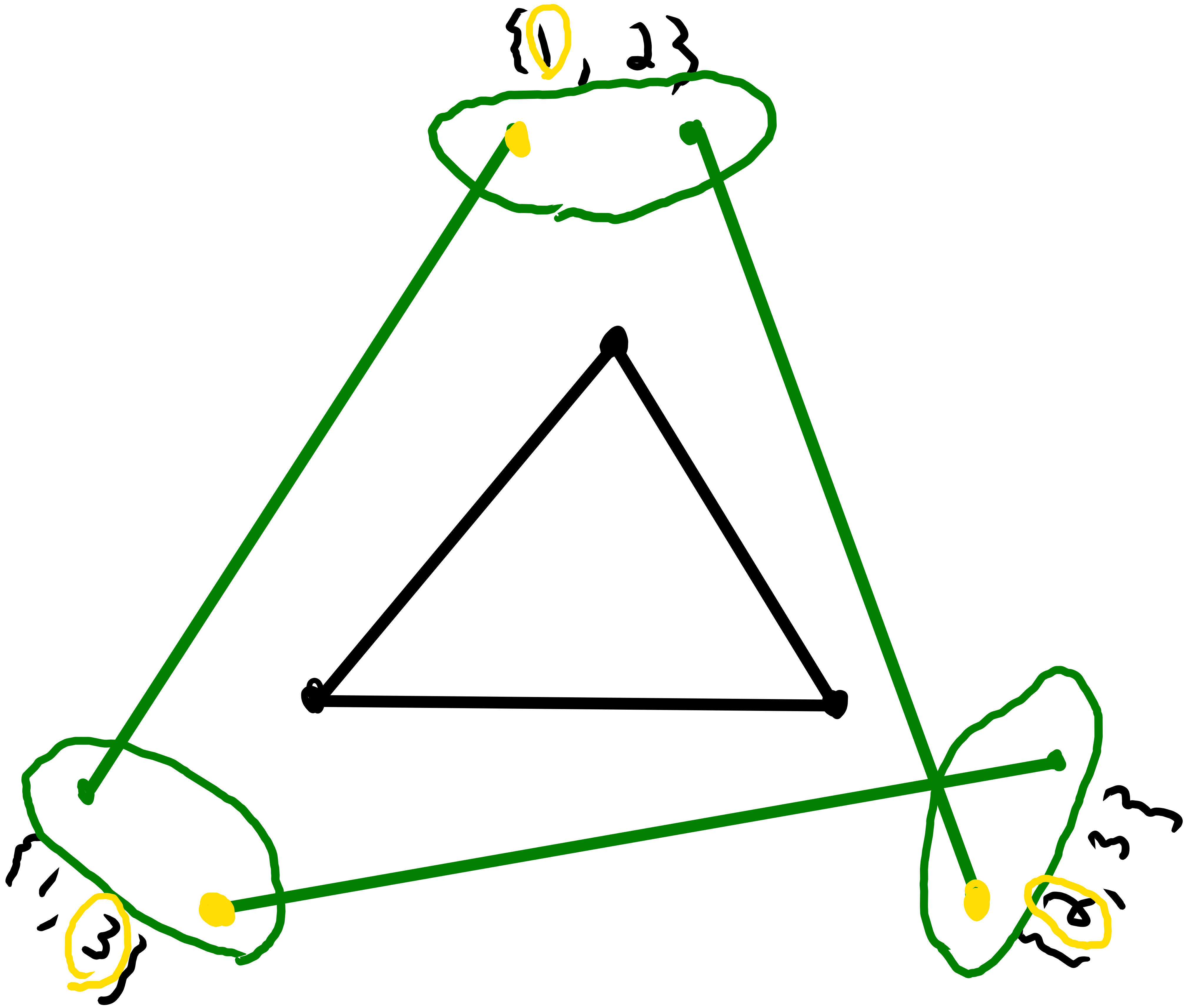

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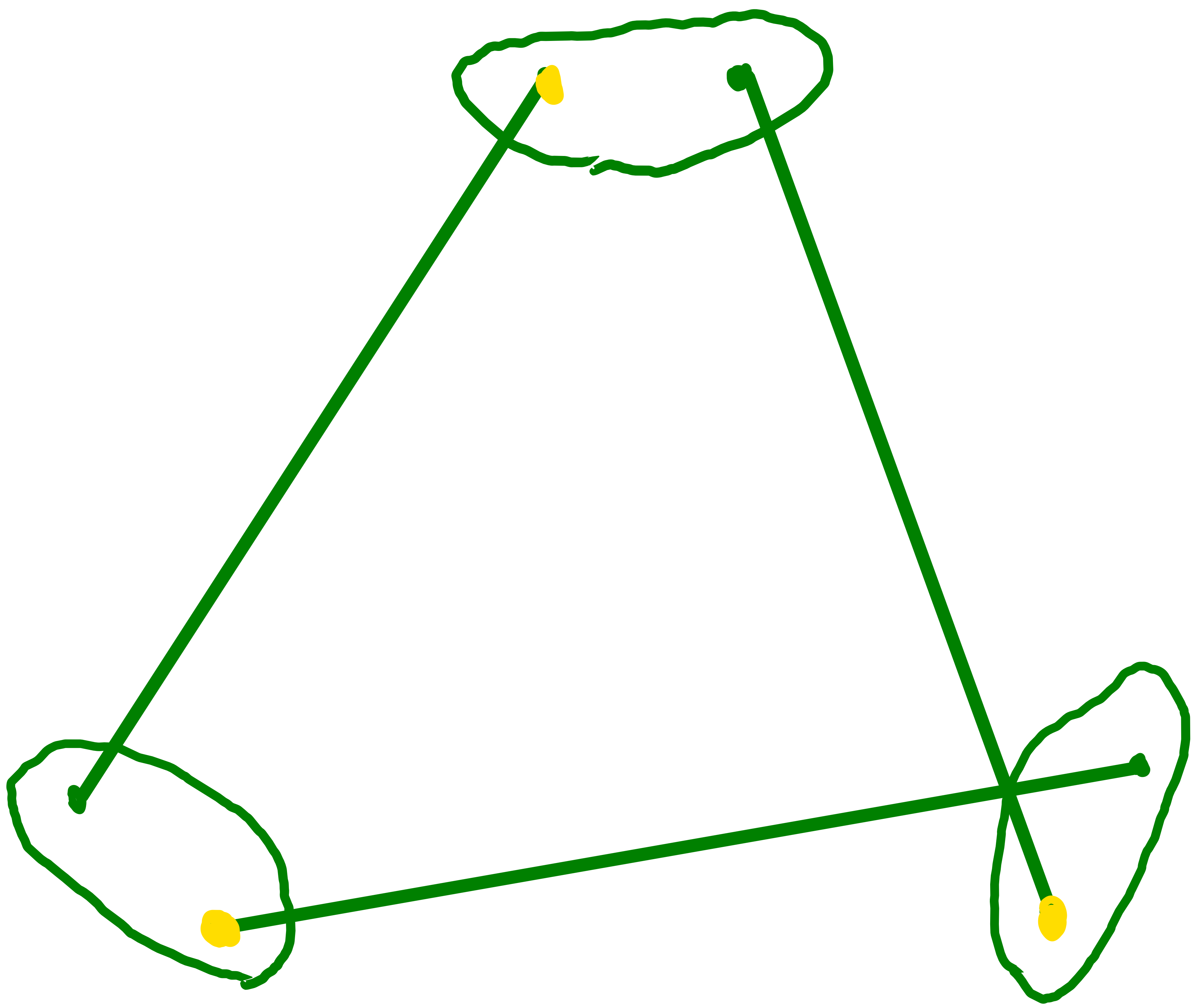


Example: list colouring

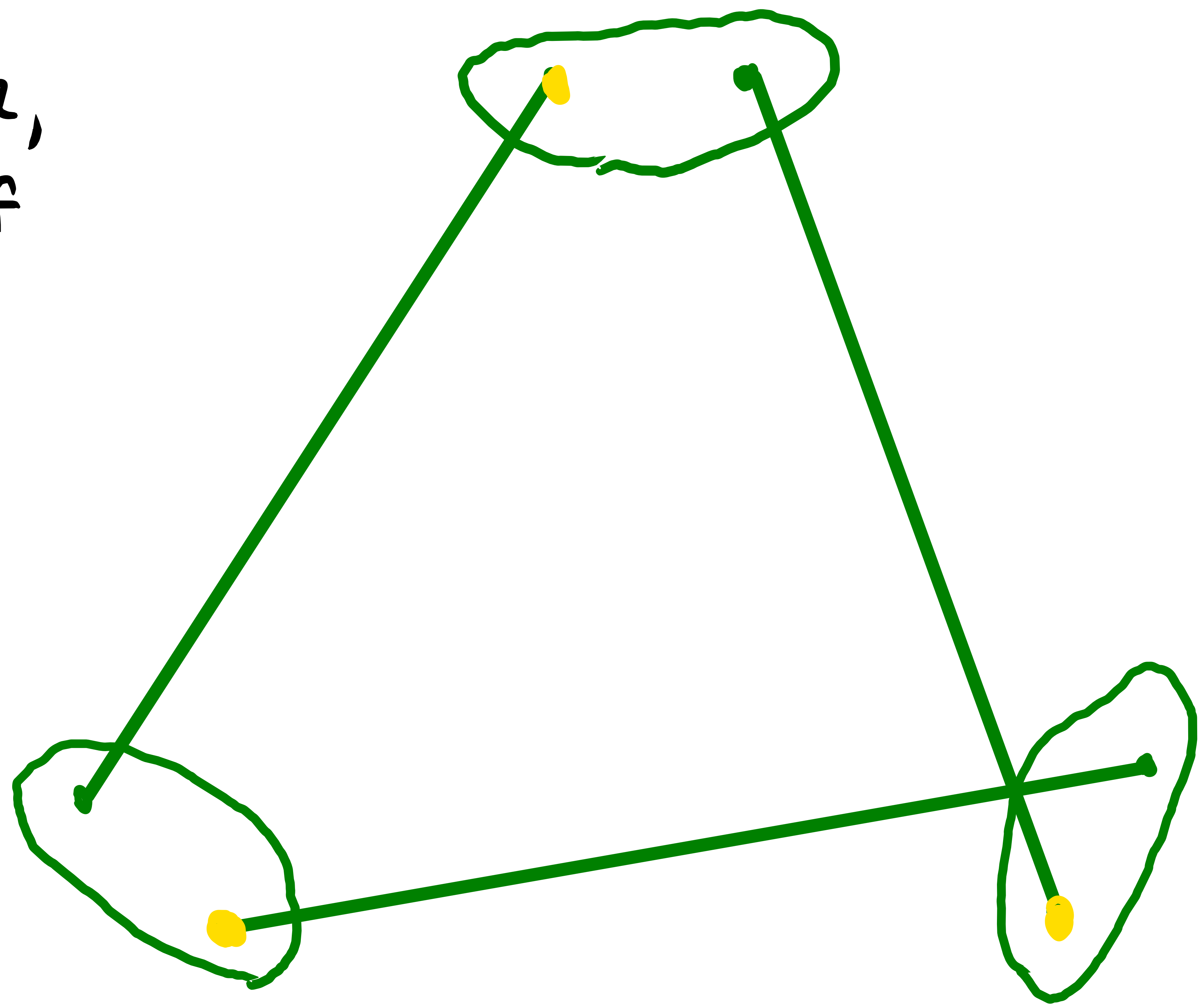
In this special case,

$\Delta \leq$ max degree of underlying graph

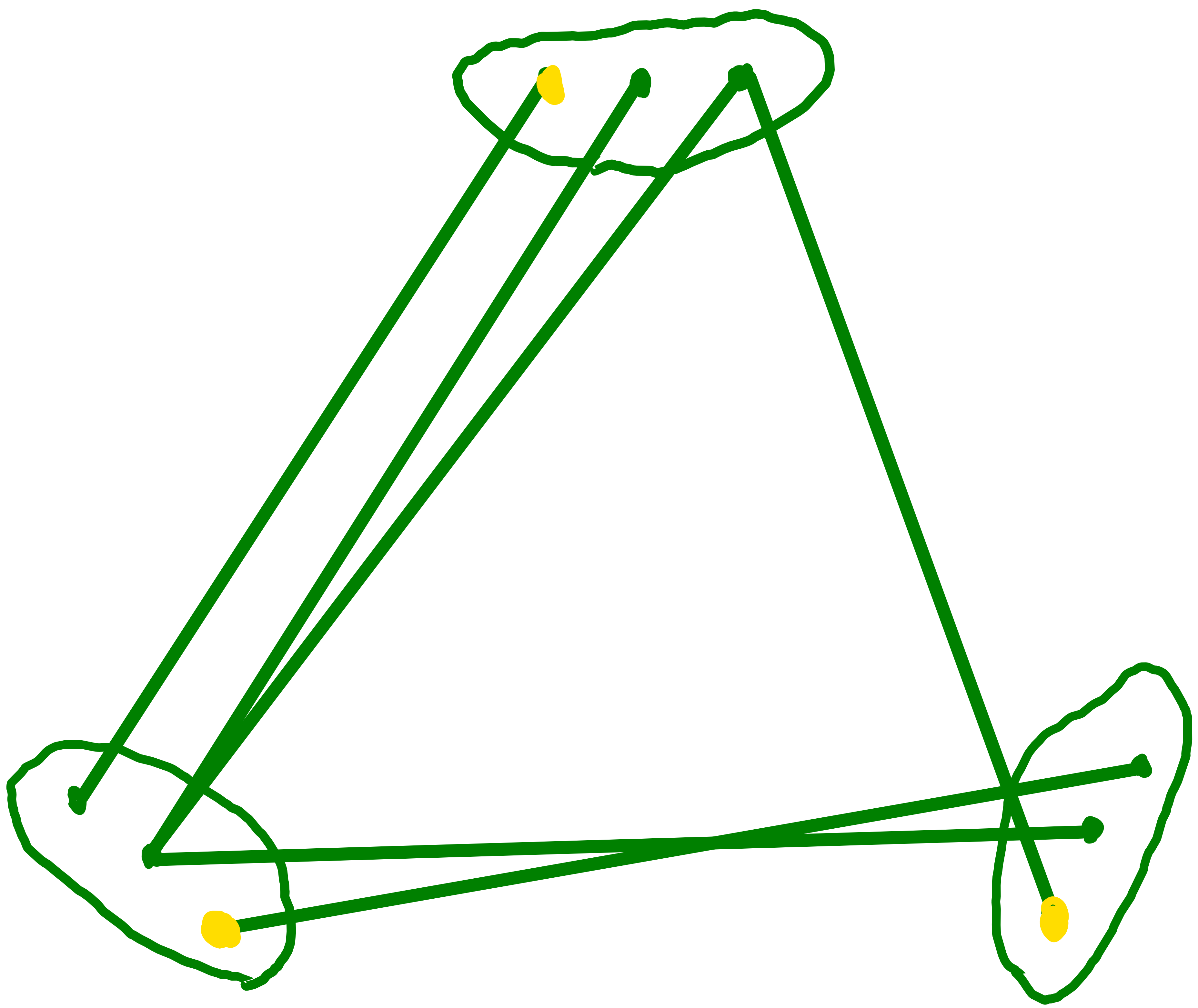
$t = \Delta + o(\Delta)$

suffices

(Reed - Sudakov '02)



But what if parts didn't derive from lists?



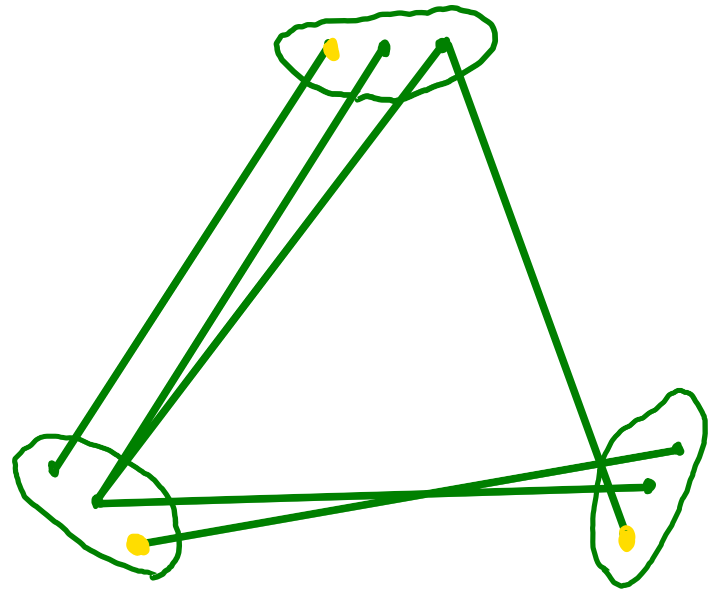
Problem (see Bollobás, Erdős, Szemerédi 1975)

Given Δ, what is least $t = t(\Delta)$?

For any G of maximum degree Δ,

vertex-partitioned with each part $\geq t$ vertices,

there is guaranteed an indep set transversal to partition



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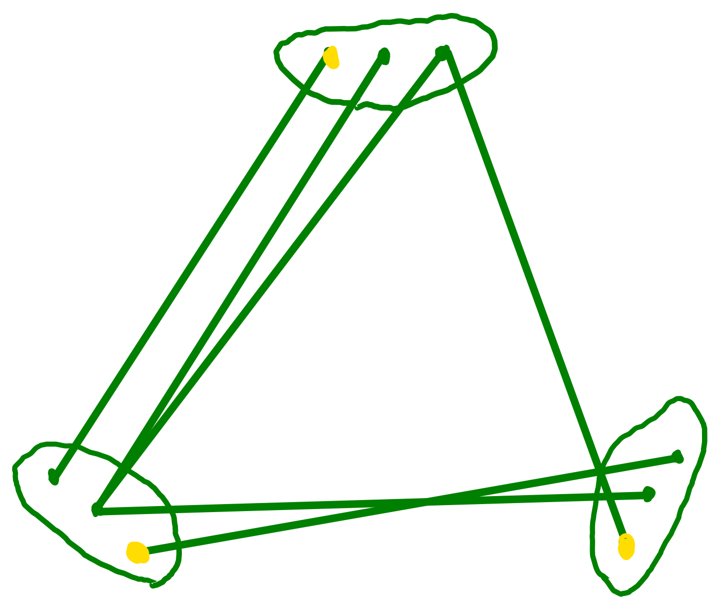

LLL $\Longrightarrow$ $t = O(\Delta)$ always suffices (Alon?)

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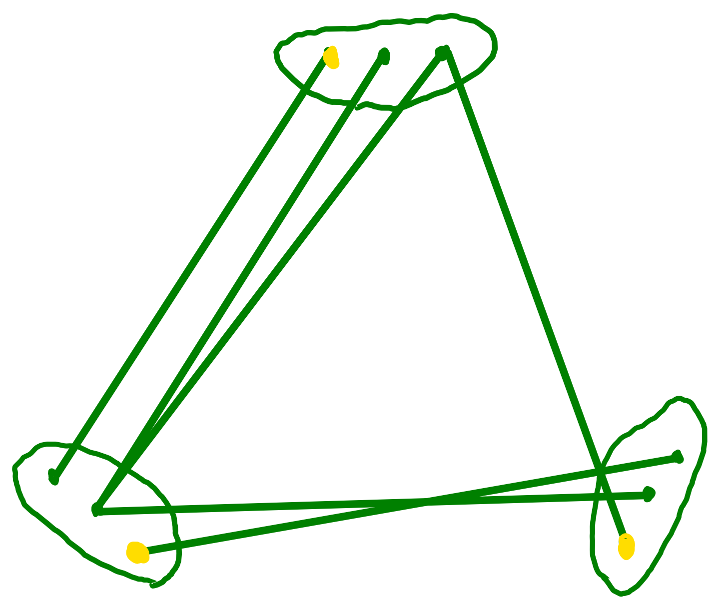

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<u>Haxell 1995/2001</u>: $t = 2\Delta$ always suffices

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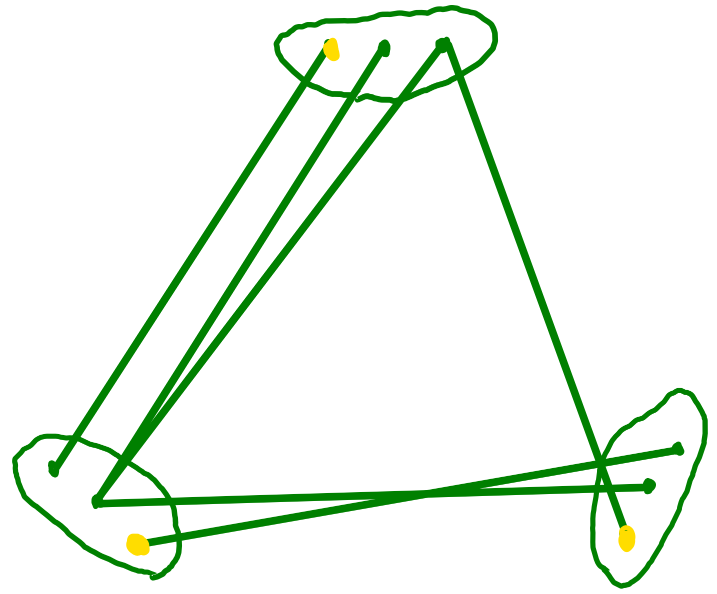

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Szabó-Tardos 2006: $t = 2\Delta$ always necessary

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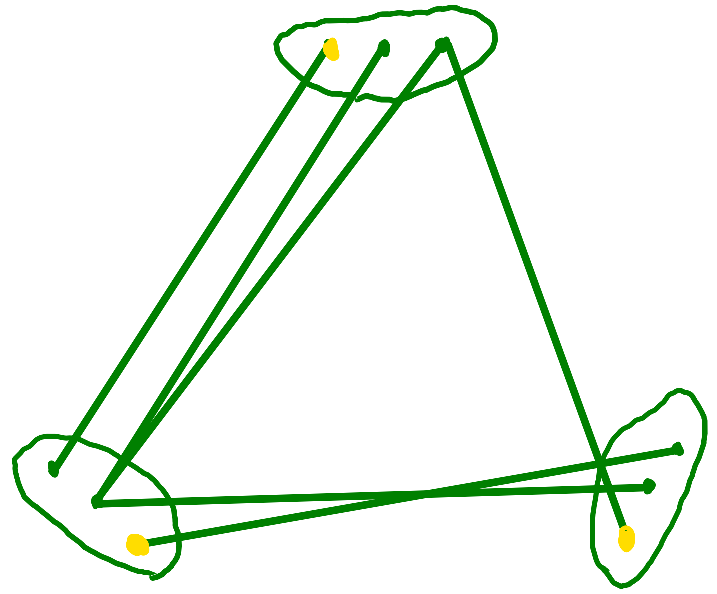

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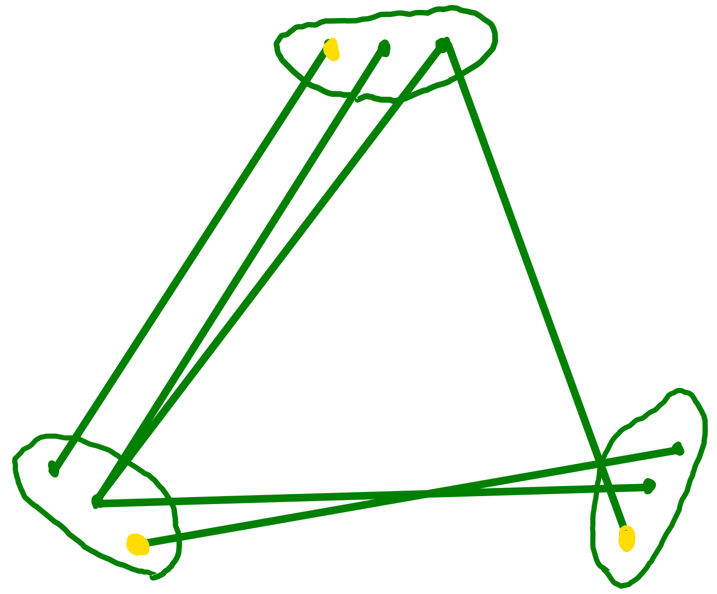

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For any G of maximum degree Δ,

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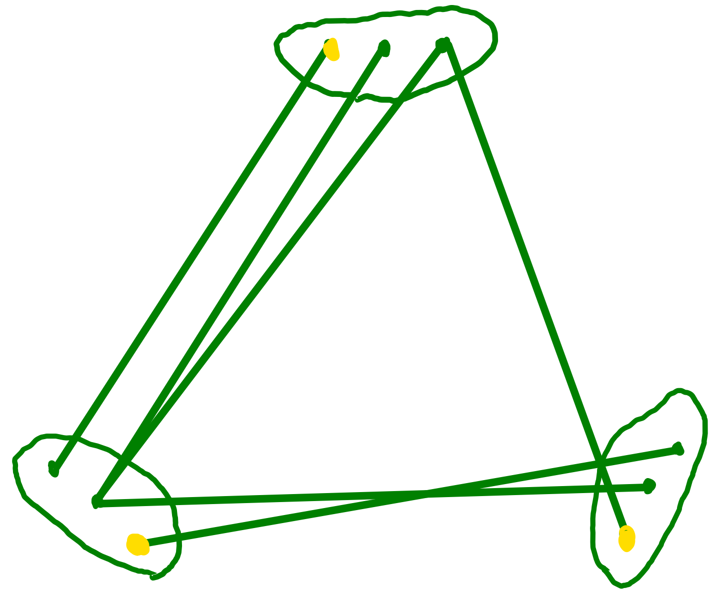

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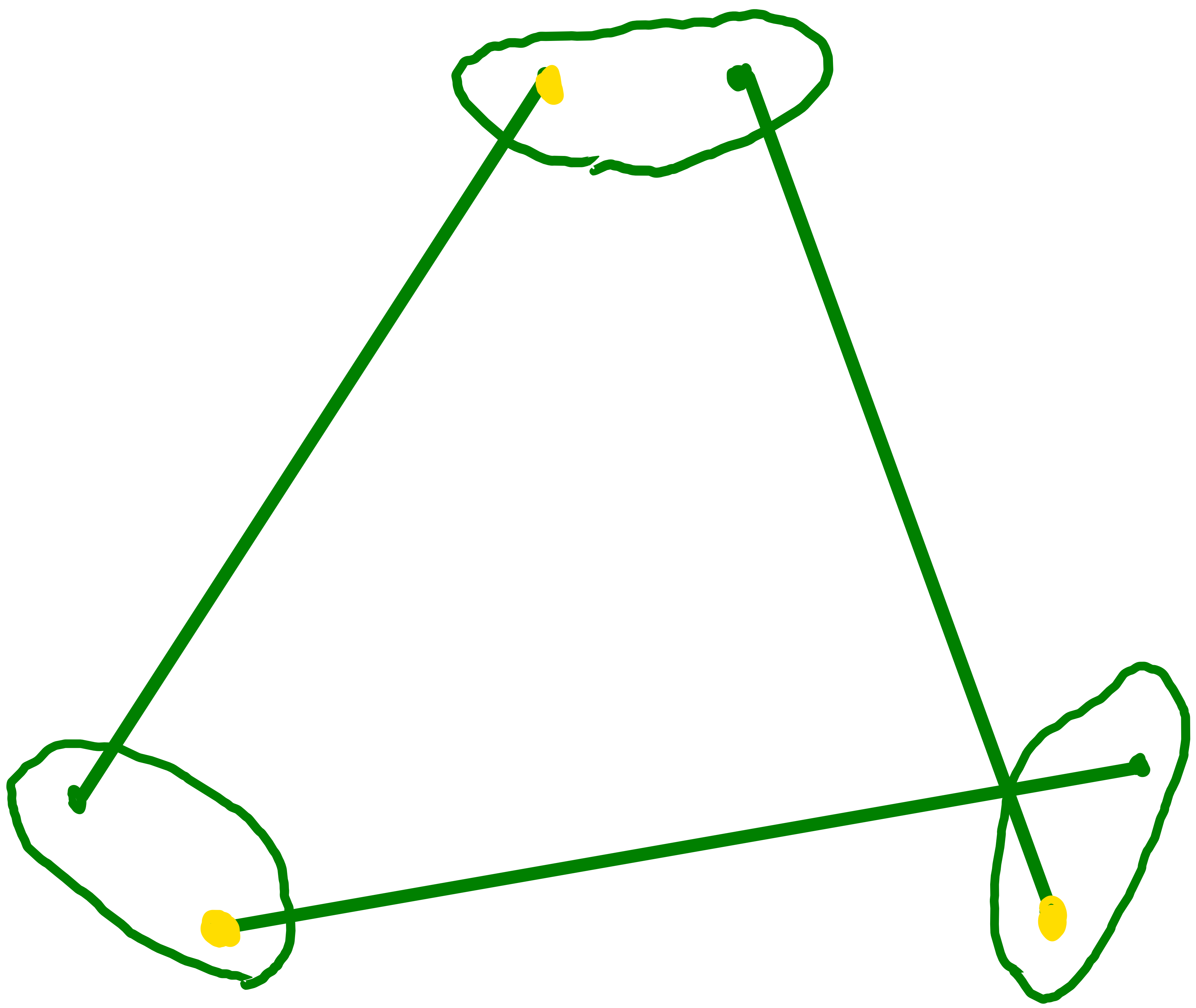

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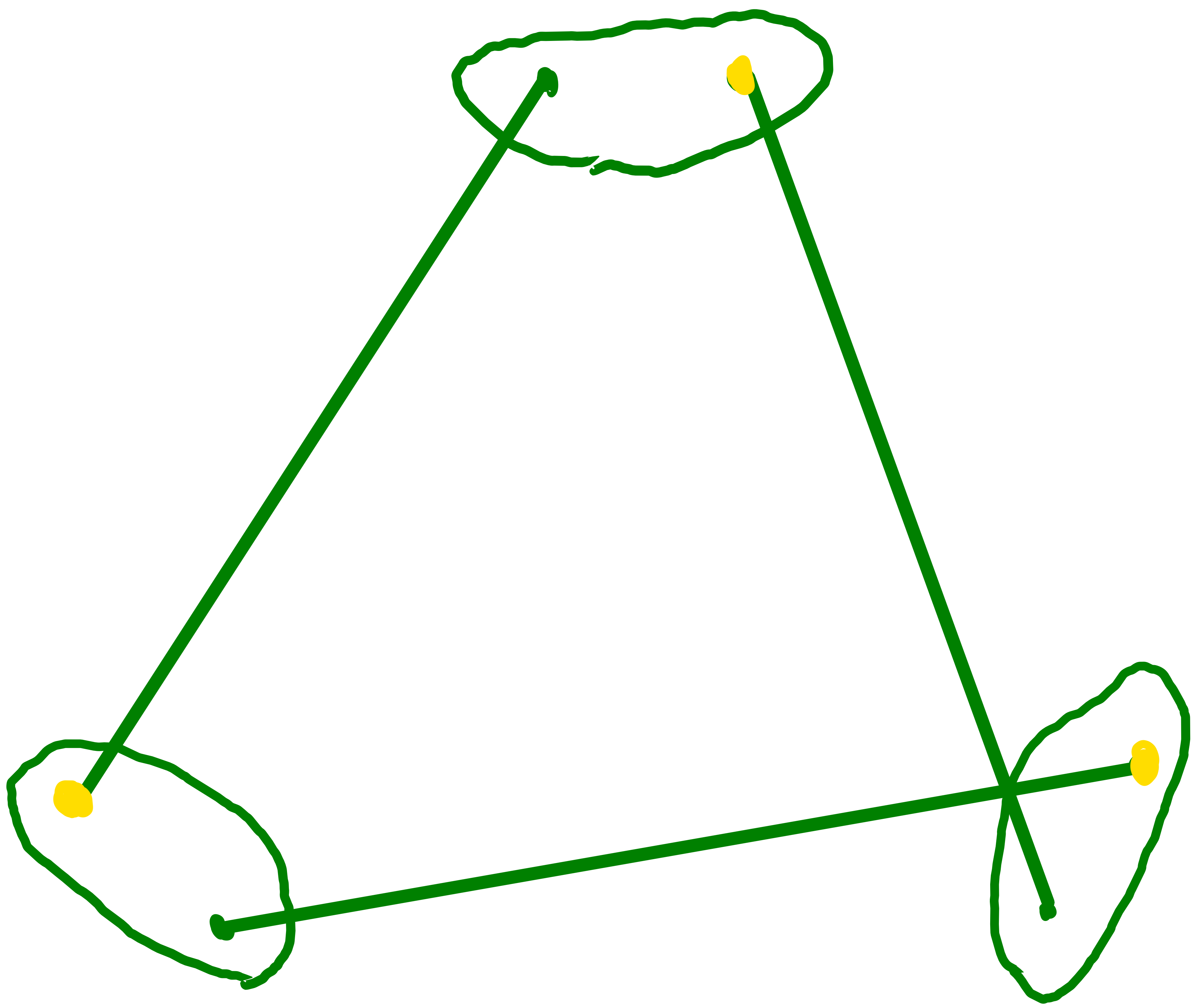

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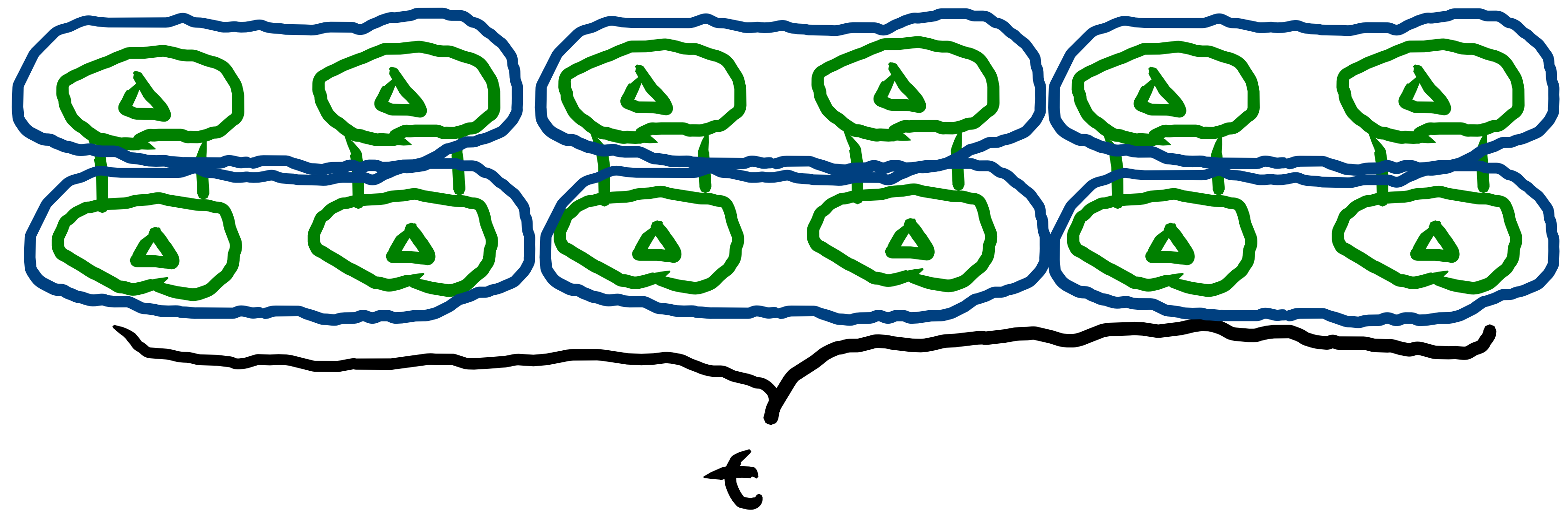

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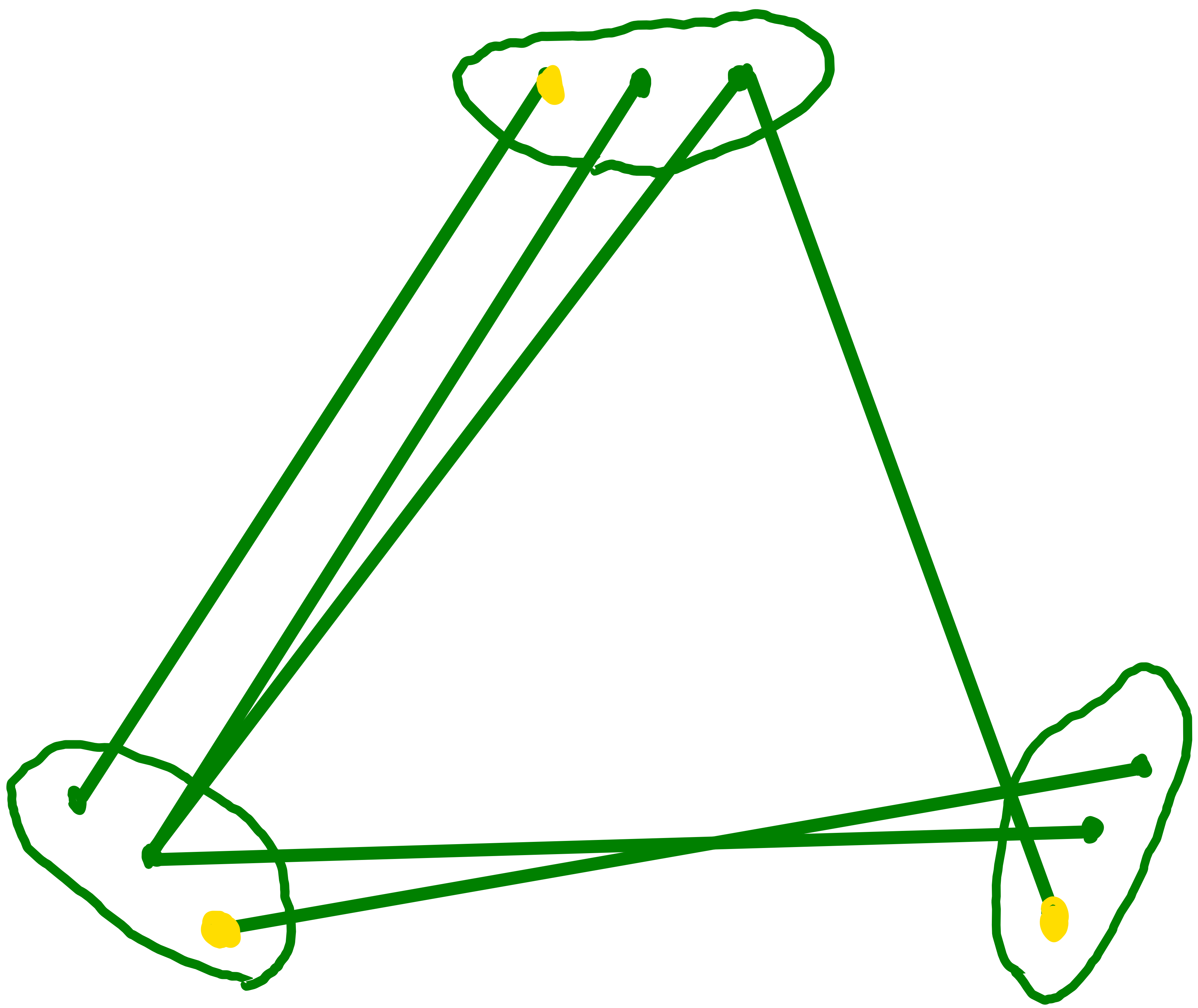

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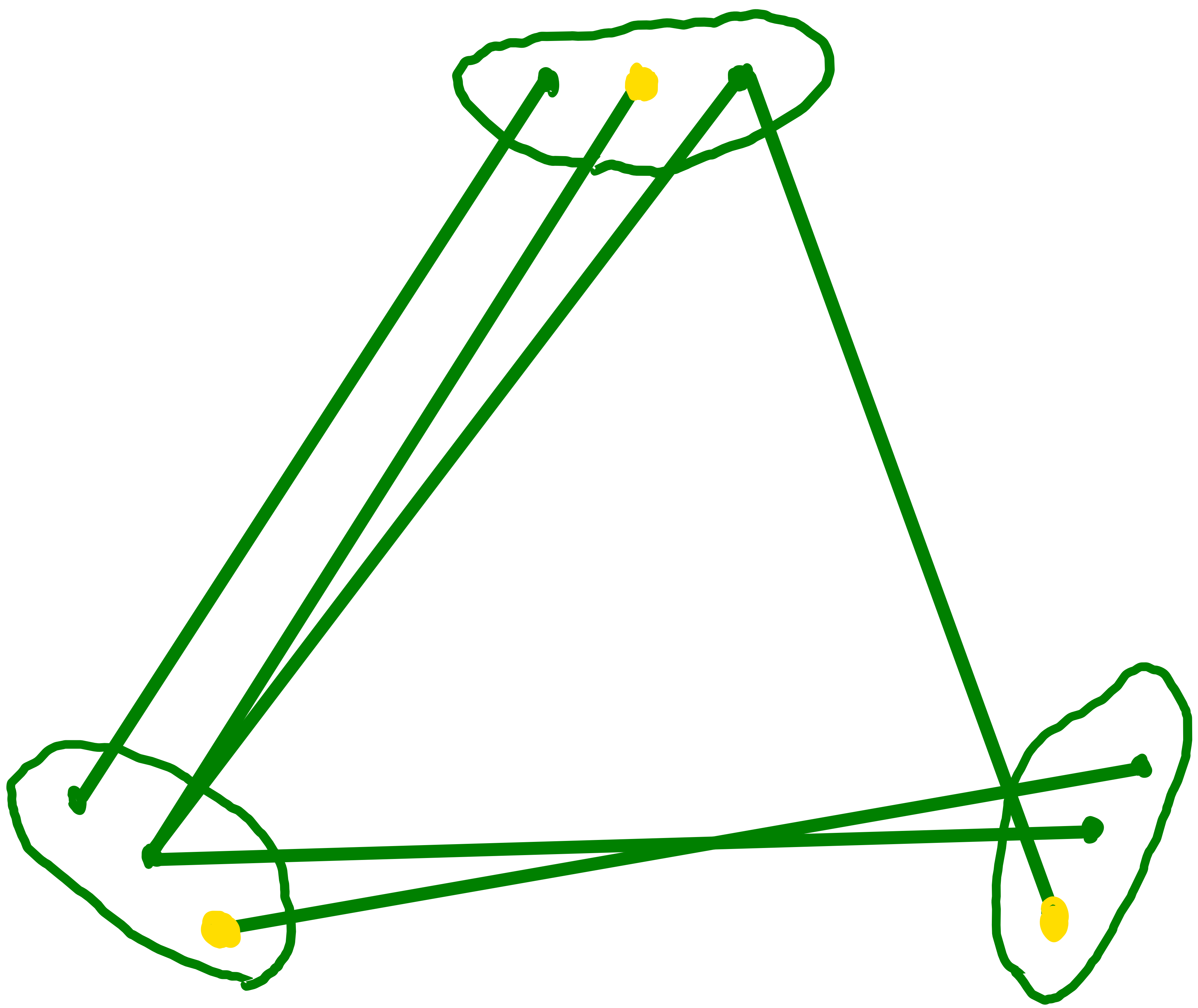

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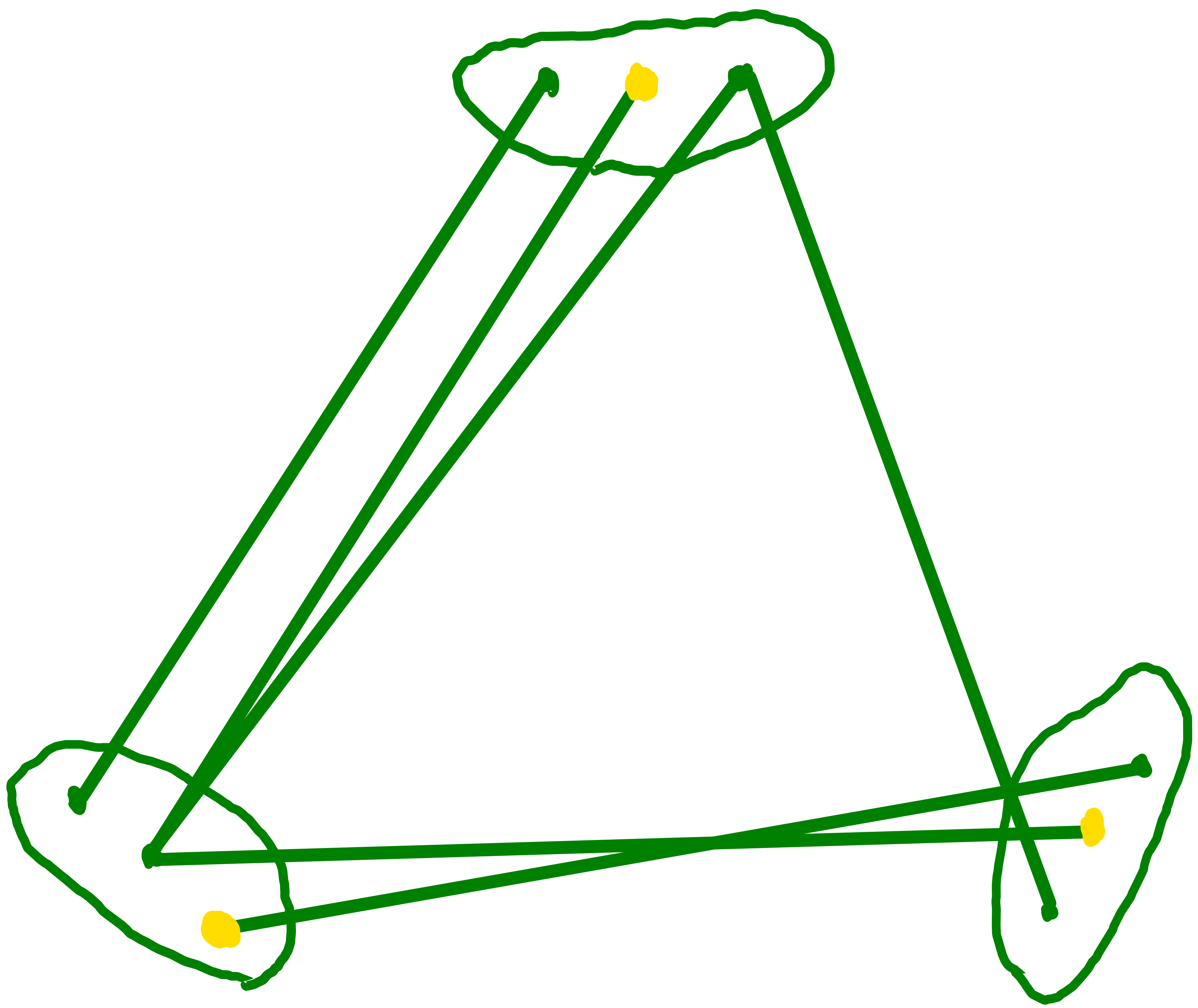

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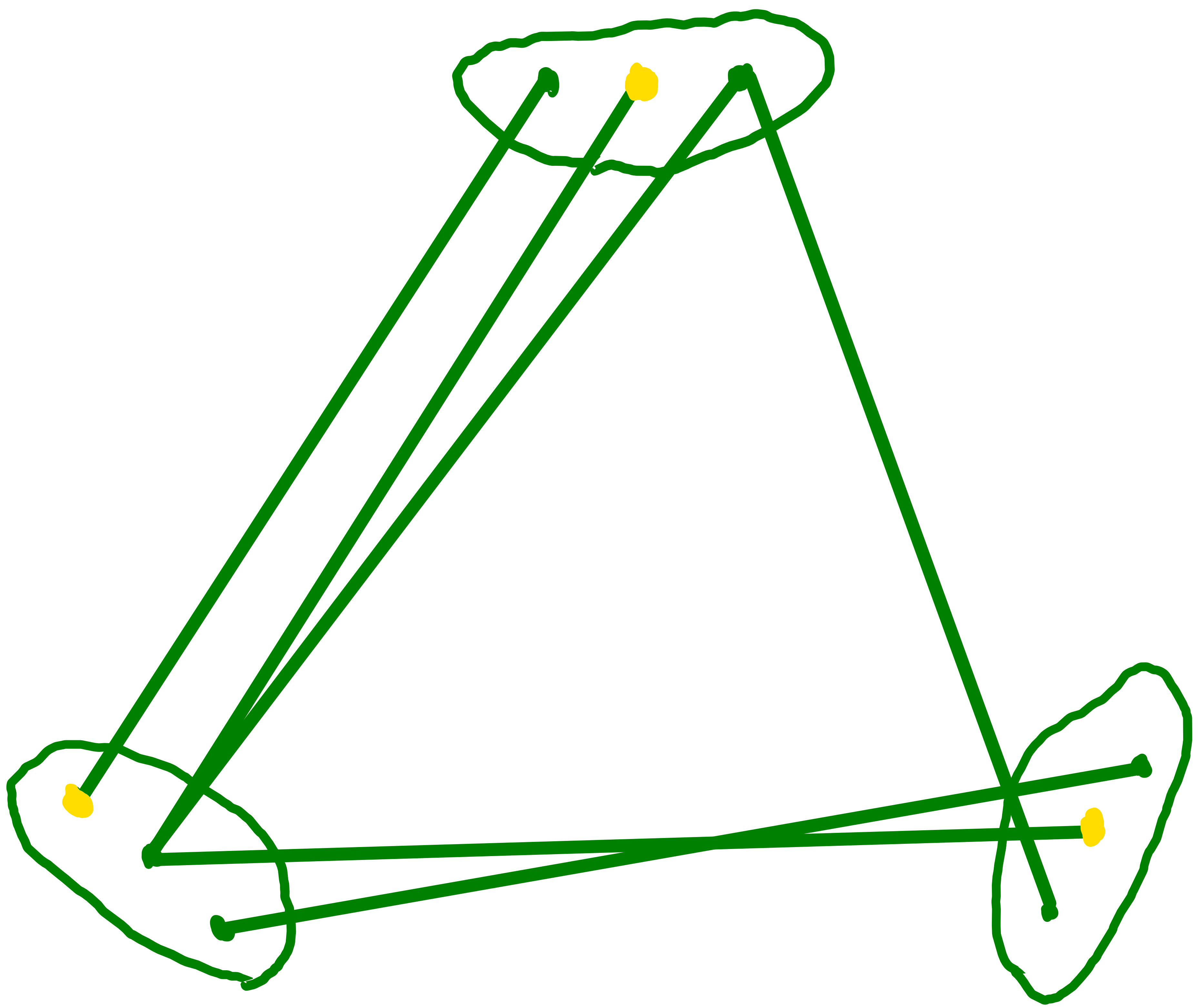

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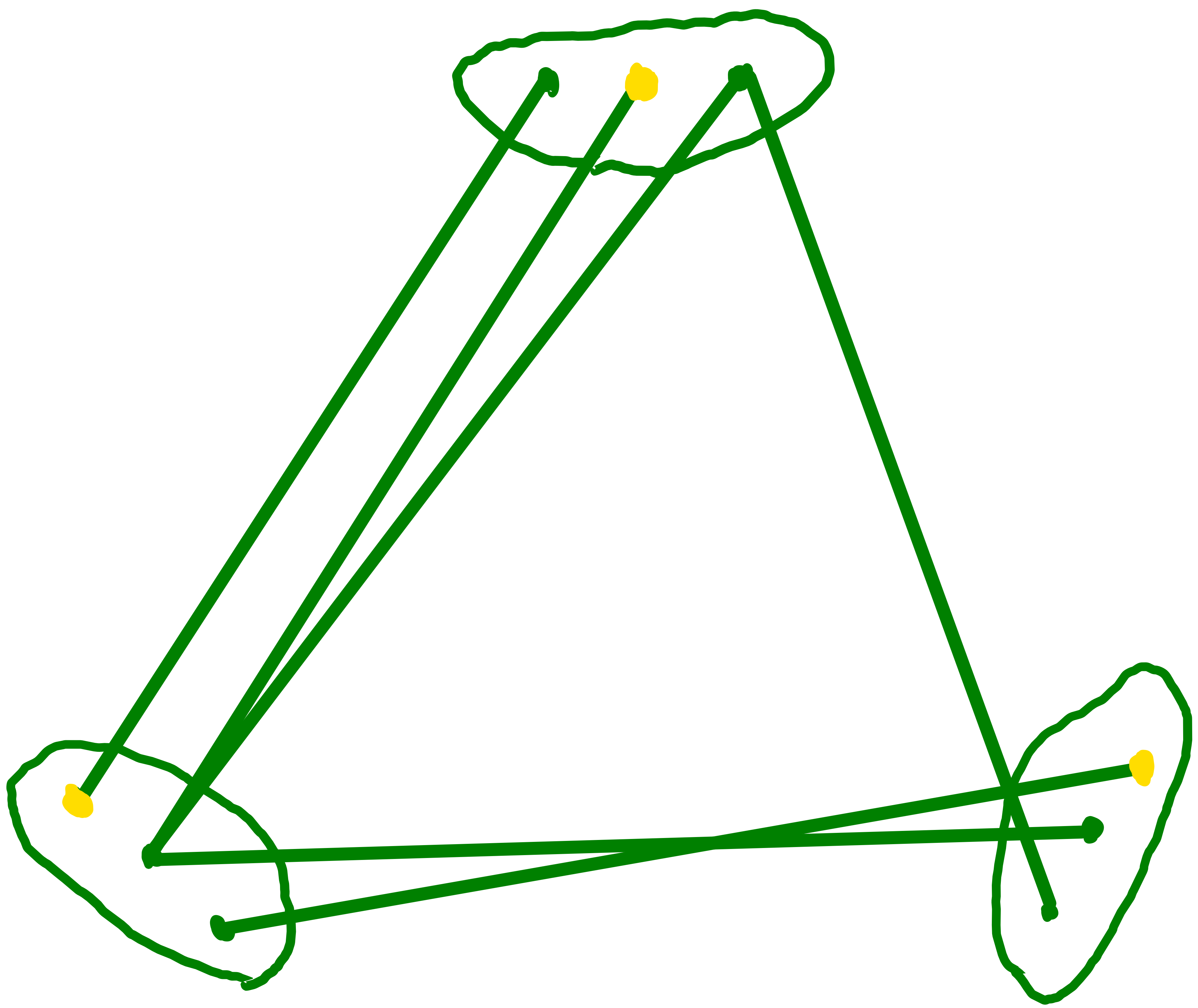

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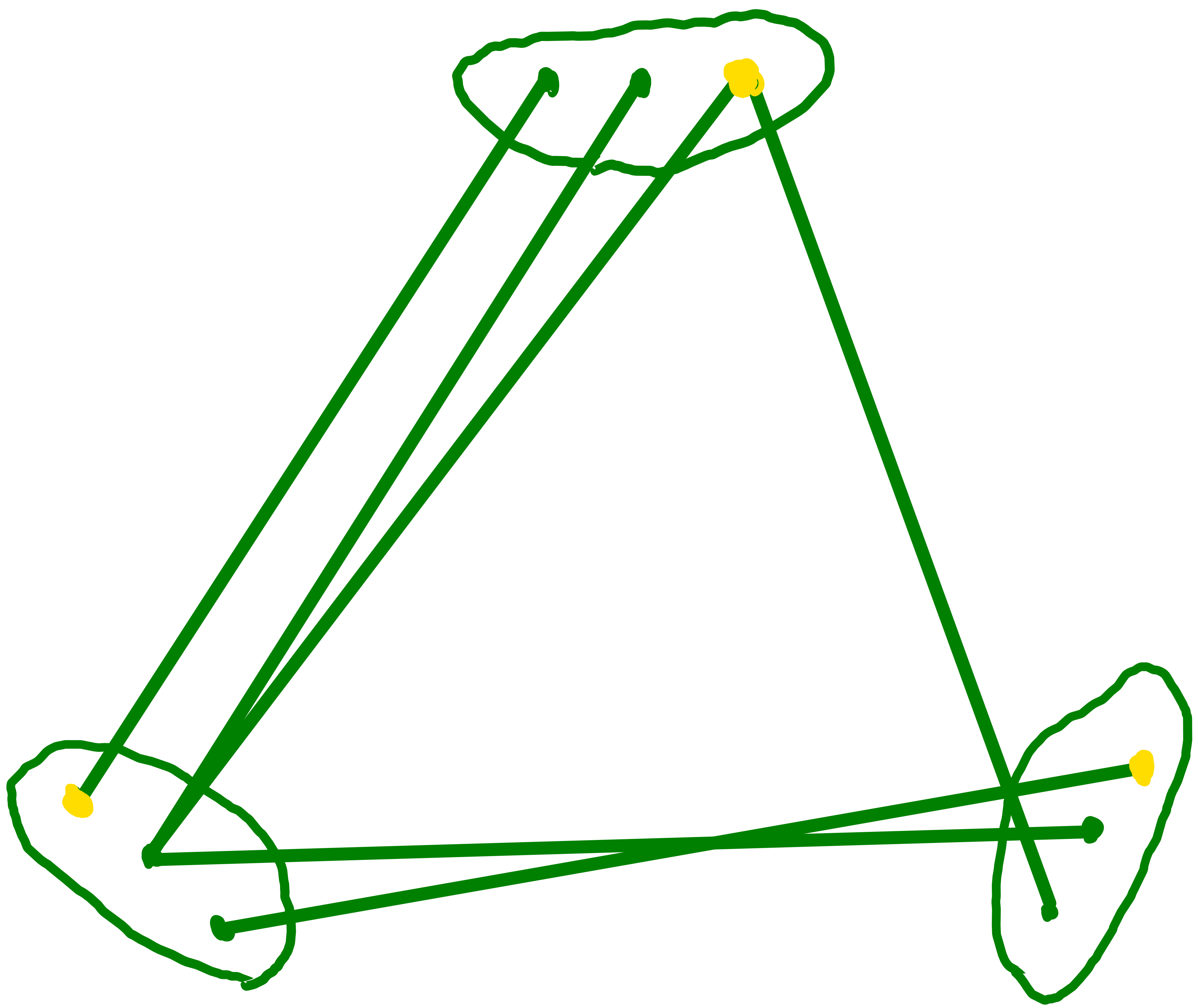

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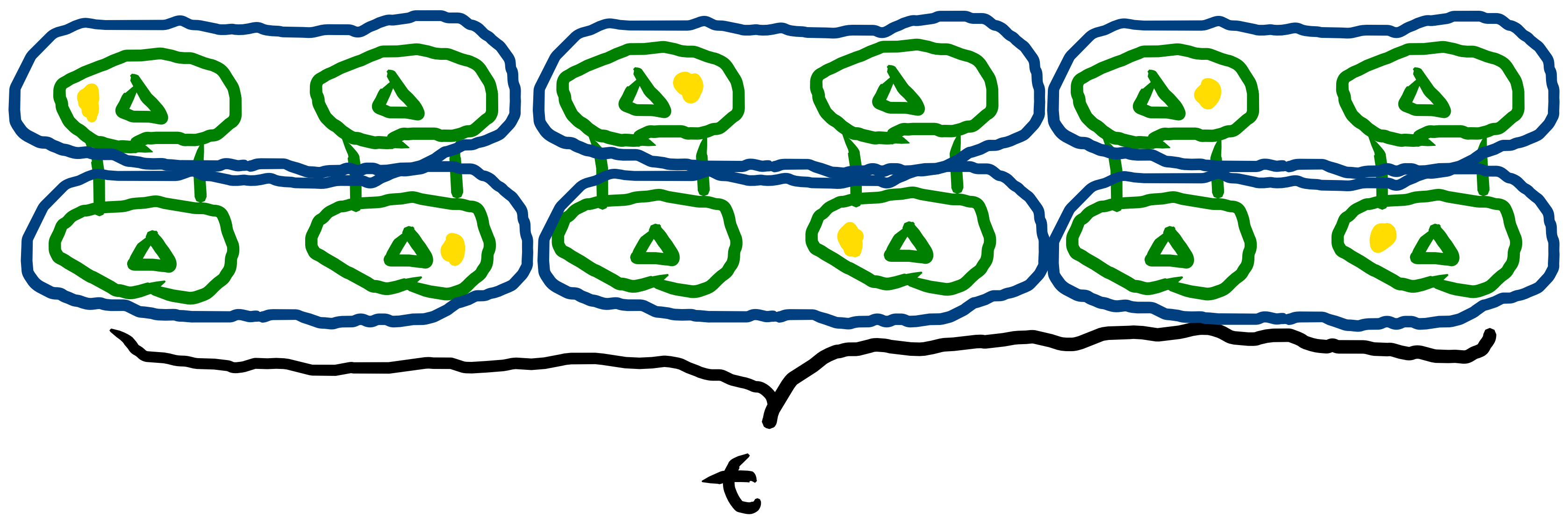
For $t = 2\Delta$, consider disjoint union of $K_{\Delta,\Delta}$'s:



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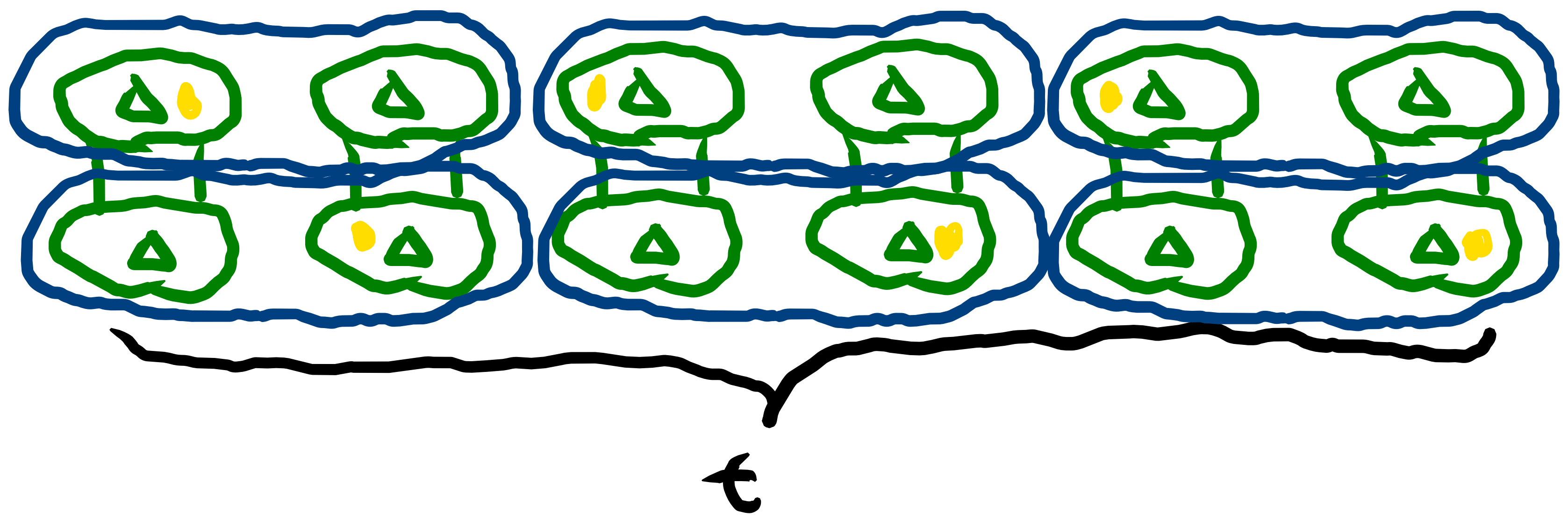

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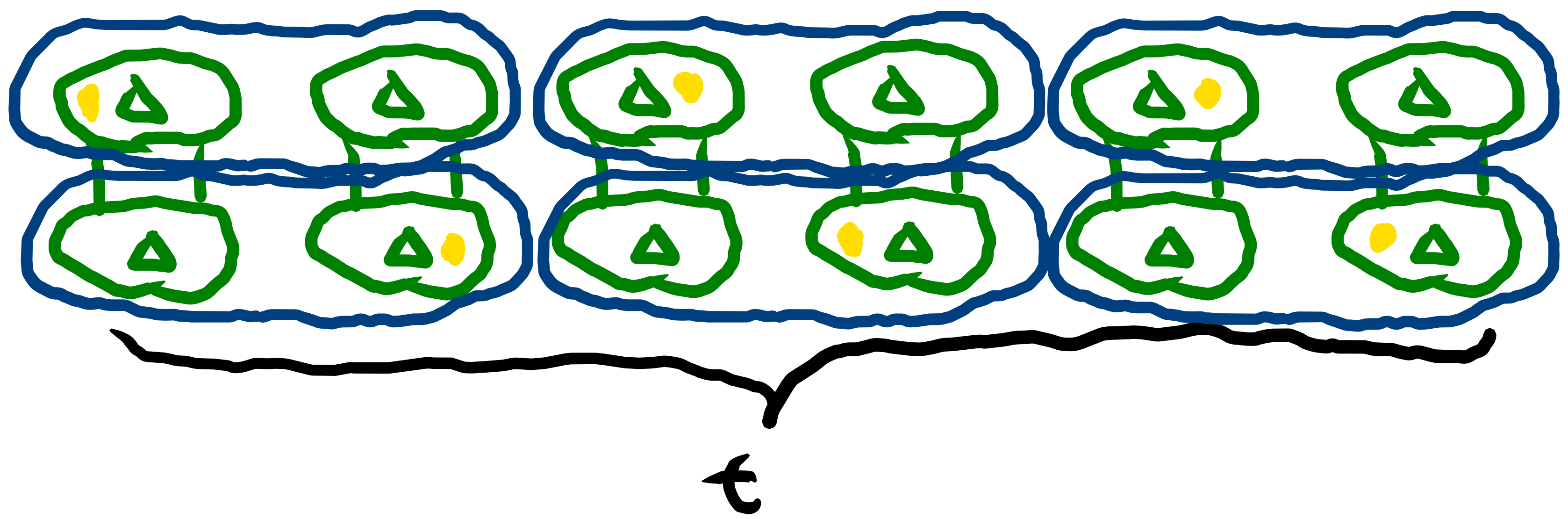

Their ITs are 'frozen'

They form disconnected components in reconfiguration graph

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Wdowinski 2025+: topological pf via Sperner-type lemma

Haxell 2026+: algorithm for finding sequences

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Any triangle-free graph G on n vertices of average degree d has average independent set size $\dfrac{\lambda Z_G'(\lambda)}{Z_G(\lambda)} \geq (1+o(1))\dfrac{n}{d}\log d$

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