Induced matchings in subcubic planar graphs

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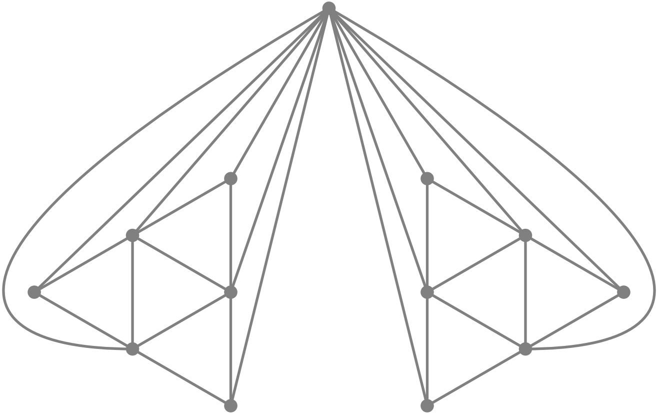
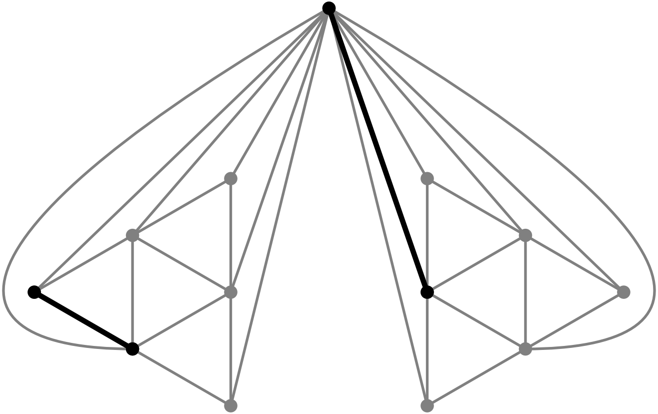
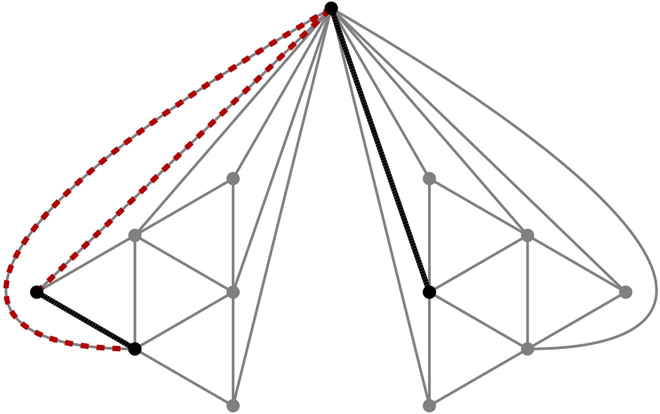
Induced matchings

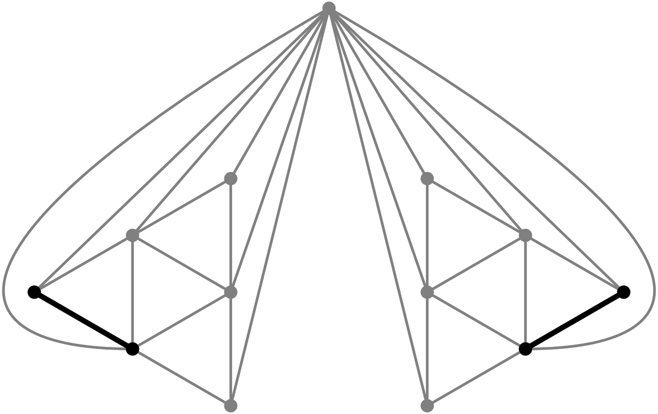
A set M of edges in a graph is an induced matching if the subgraph induced by M forms a set of disjoint edges.

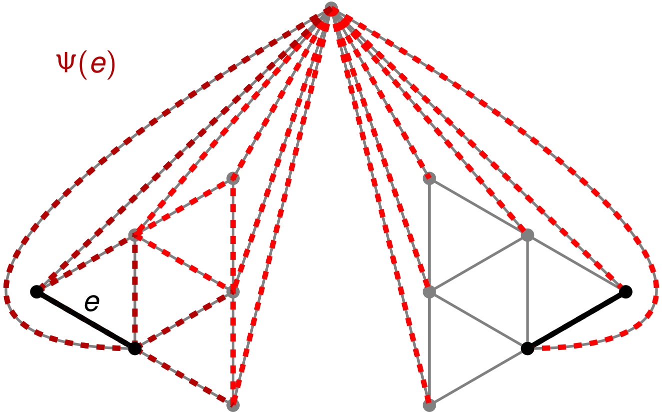
We seek efficient ways to find large induced matchings.







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Induced matchings

Introduced by Stockmeyer and Vazirani (1982) as 'risk-free marriage'.

More recent work associates the size of a largest induced matching to maximum capacity in ad hoc wireless networks.

Induced matchings

Computing the size of a largest induced matching is NP-hard

- for bipartite planar graphs of maximum degree 4 (Stockmeyer & Vazirani, 1982);
- for subcubic planar graphs (Lozin, 2002).

Polynomial-time for trees, chordal graphs, cocomparability graphs, asteroidal-triple free graphs, graphs of bounded cliquewidth.

Duckworth, Manlove & Zito (2005) give a survey of complexity and approximation status for various classes.

Strong chromatic index

A strong edge colouring is a partition of the edge set so that each colour is an induced matching. The strong chromatic index is the least number of colours needed in such a colouring.

Erdős and Nešetřil in 1985 wondered about bounds for graph of maximum degree d, asking if $(2-\varepsilon)d^2$ could be achieved. The "bounty" was collected by Molloy and Reed (1997), with $\varepsilon \geq 0.002$. (Trivially, $0 \leq \varepsilon \leq 0.75$.)

Strong chromatic index for planar graphs

Conjecture 1 (Faudree, Schelp, Gyárfás, Tuza, 1989)

Every planar graph of maximum degree 3 *has strong chromatic index at most* 9*.*

Conjecture 2 (Wegner, 1977)

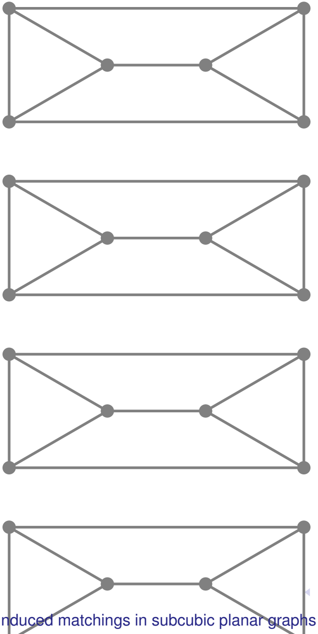
The square of any planar graph of maximum degree 4 *has chromatic number at most* 9*.*

The main theorem

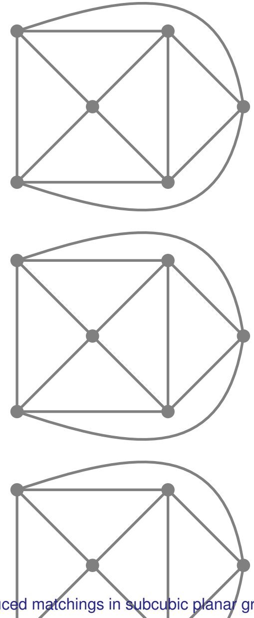
Any planar graph of maximum degree 3 *with* m *edges contains an induced matching of at least* $m/9$ *edges.*

Furthermore, such a matching can be found in $O(m)$ *time.*

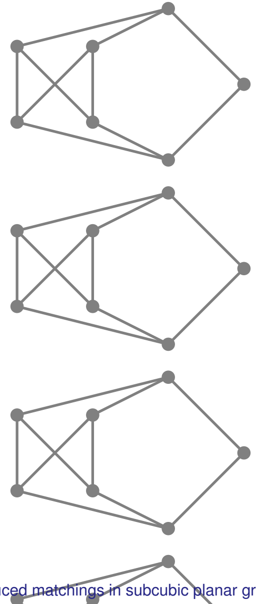
A tight bound



A tight bound



A tight bound



A structural result

A set E' of edges with between one and five edges is good if it is an induced matching and has $|\Psi(E')| \leq 9 \cdot |E'|$.

Lemma 1

Given a subcubic plane graph, if it contains a "special" structure, then it contains a good set of edges.

Lemma 2

Every subcubic plane graph contains a "special" structure.

Special structures

-
  - a 1-vertex;
  - a 2-vertex incident to an ($\leqslant$6)-cycle or 7-face;
  - a 2-vertex at distance at most 2 from a 2-vertex;
  - a 2-vertex at distance at most 2 from an ($\leqslant$5)-cycle;
  - a 3-cycle adjacent to an ($\leqslant$7)-cycle;
  - a 4- or 5-cycle in sequence with a 5- or 6-cycle;
  - a 3-cycle at distance 1 from an ($\leqslant$5)-cycle;
  - a double 4-face adjacent to an ($\leqslant$7)-cycle;
  - a 4-cycle, ($\leqslant$8)-cycle and 4-cycle in sequence;
  - a 4-cycle, 7-cycle and 5-cycle in sequence;
  - a 3-cycle or double 4-face at distance at most 2 from a 3-cycle or double 4-face; and
  - a double 4-face at distance 1 from a 5-cycle.

The algorithm

Start with $M := \emptyset$, $H := G$ and $Q := V$. Let v denote the first element of Q. Repeat the following while Q is non-empty.

1. If v isolated, then remove v from Q.
2. Otherwise, check for a minimally good set E' in H involving v, and
   - 2a. if such an E' exists, set $M := M \cup E'$, $H := H \setminus \Psi(E')$, and put the vertices of $N^{20}(E')$ at the beginning of Q in an arbitrary order,
   - 2b. otherwise, move v to the end of Q.

Proof of Lemma 2 by discharging

Let $G = (V, E, F)$ be a counter-example, minimal w.r.t. $|E|$.

We obtain a contradiction by using the discharging method:

- each vertex and face of G is given an initial charge (chosen so that the total sum of initial charges is negative by Euler's formula);
- charge is redistributed according to specific rules;
- by absence of special structures, it follows that the total sum of redistributed charge is non-negative.

Proof of Lemma 2 by discharging

For $v \in V$ and $f \in F$, set initial charges $\mathrm{ch}(v) := 2\deg(v) - 6$ and $\mathrm{ch}(f) := \deg(f) - 6$.

Each ($\geqslant$7)-face f sends

- 1 to each incident 2-vertex,
- 1 to each adjacent 3-face,
- 1 to each adjacent 4-face in a double 4-face if f and the two 4-faces are in sequence,
- $1/2$ to each other adjacent 4-face, and
- $1/5$ to each adjacent 5-face.

Proof of Lemma 1 by case analysis

-
  - a 1-vertex;
  - a 2-vertex incident to an ($\leqslant$6)-cycle or 7-face;
  - a 2-vertex at distance at most 2 from a 2-vertex;
  - a 2-vertex at distance at most 2 from an ($\leqslant$5)-cycle;
  - a 3-cycle adjacent to an ($\leqslant$7)-cycle;
  - a 4- or 5-cycle in sequence with a 5- or 6-cycle;
  - a 3-cycle at distance 1 from an ($\leqslant$5)-cycle;
  - a double 4-face adjacent to an ($\leqslant$7)-cycle;
  - a 4-cycle, ($\leqslant$8)-cycle and 4-cycle in sequence;
  - a 4-cycle, 7-cycle and 5-cycle in sequence;
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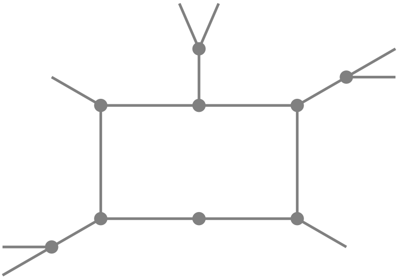
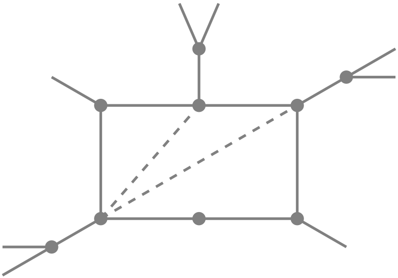
Proof of Lemma 1 by case analysis

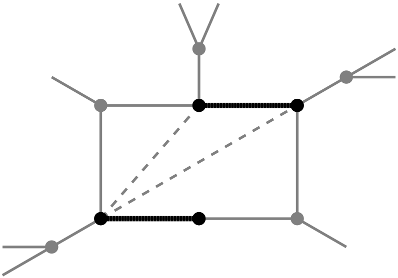


Proof of Lemma 1 by case analysis



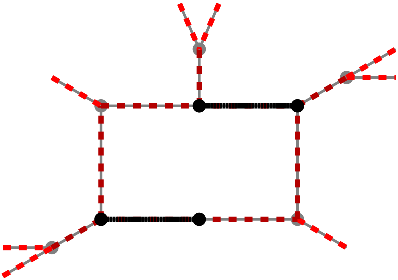
Proof of Lemma 1 by case analysis

E' is induced matching, $|E'| = 2$



Proof of Lemma 1 by case analysis

$$\Psi(E') \leq 17 < 9 \cdot |E'|$$



Some open problems

- Proof not using (too much) discharging?
- Induced matchings in planar graphs of maximum degree d?
- Independent sets in squares of planar graphs of maximum degree d.
- Conjectures 1 and 2 (and their maximum degree d analogues).

(Note: Maximum degree d analogue of Conjecture 2 settled asymptotically via “hard-core” colouring result by Havet, van den Heuvel, McDiarmid and Reed (2010+). The $d = 3$ case is solved in a preprint of Thomassen (2010+).)