Fractional colouring and the hard-core model *

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Graph Colouring Mini-Symposium
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*Based on joint work with Ewan Davies, Rémi de Joannis de Verclos, and François Pirot

Structure of triangle-free graphs

What large-scale structure forms if no 3 pairwise adjacent vertices, so if no neighbourhood induces any edge?

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What large-scale structure forms if no 3 pairwise adjacent vertices, so if no neighbourhood induces any edge?

i.e. "local to global graph structure"

Distinguished origins:

- Mantel (1907), Turán (1941)
- Ramsey (1930), Erdős & Szekeres (1935)
- Zykov (1949), Ungar & "Blanche Descartes" (1954)

$$\frac{n}{\alpha} \leq \chi_f \leq \chi \leq \chi_\ell \leq \Delta + 1$$

†

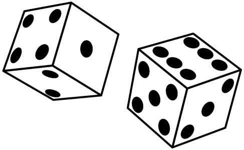
Independence and chromatic numbers

$$\frac{n}{\alpha} \le \chi_f \le \chi \le \chi_\ell \le \Delta + 1$$

In general, all can be strict [†]

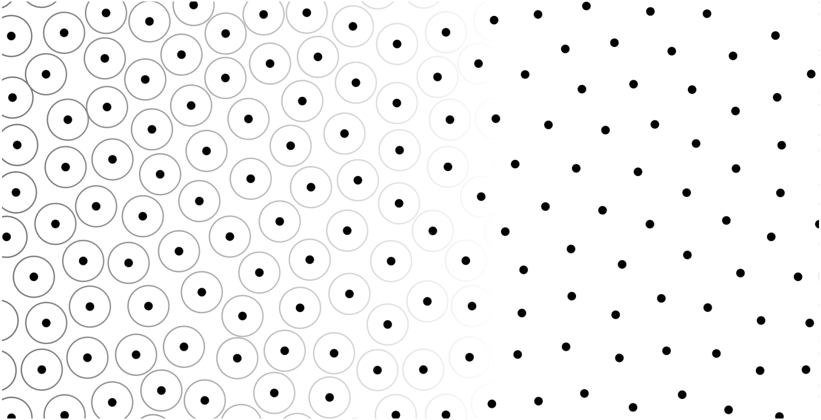
[†] On strictness of first, see Blumenthal, Lidický, Martin, Norin, Pfender, Volec (2018+), and Dvořák, Ossona de Mendez, Wu (2018+); nice open question in the triangle-free case

Probabilistic method



If a random object has desired property with positive probability, then there exists *at least one* object with that property

Hard-core model[‡]



[‡] More fully, the lattice gas with hard-core self-repulsion and nearest-neighbour exclusion. Picture credit: Wikipedia/Grap-wh

Off-diagonal Ramsey numbers[§]

§ Picture credit: Soifer 2009

Off-diagonal Ramsey numbers

$R(3,k)$: largest n such that there is red/blue-edge-coloured K_{n-1}
with no red triangle and no blue K_k

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Theorem (Bohman, Keevash 2013+, Fiz Pontiveros, Griffiths, Morris 2013+, cf. Kim 1995)

$$R(3,k) \gtrsim \frac{k^2}{4\log k}$$

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Does there exist a probability distribution on the set $\mathscr{I}$ of independent sets such that for a random $\mathbf{I}$

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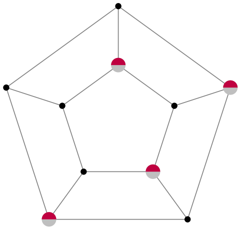
¶Yes, cf. Davies, Jenssen, Perkins, Roberts 2018...

Fractional chromatic number



$\alpha = 4$

$\chi_f \geq n/\alpha = 5/2$

fractional vertex-colouring : allow "fractions" of independent sets

fractional chromatic number χ_f : least "amount" needed

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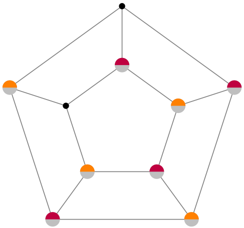

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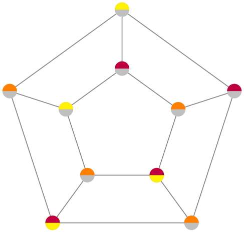

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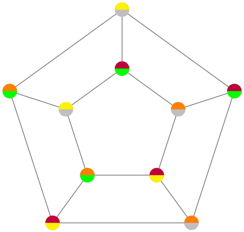

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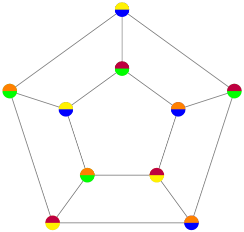

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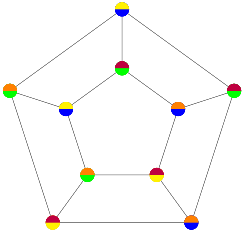

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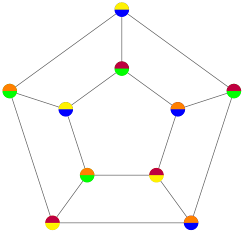

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$$\mathbb{P}(v \in \mathbf{I}) \geq 1/k \quad \text{for every vertex } v$$

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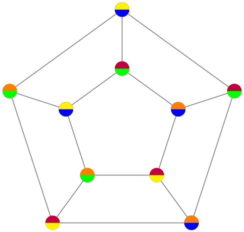

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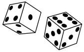
. . . 

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Chromatic number of triangle-free graphs

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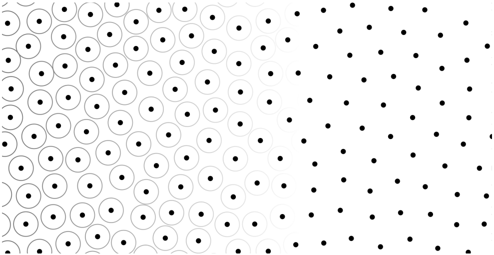
Why? Simpler, conceptual, more versatile, e.g. local colouring[‖], local sparsity

[‖] As in next talk

A probability distribution on $\mathscr{I}$ the set of independent sets



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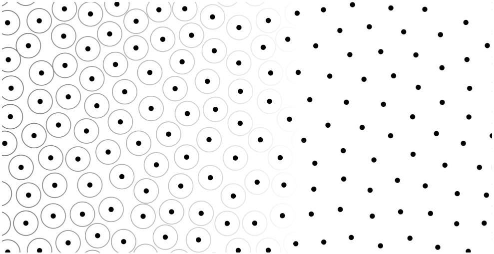
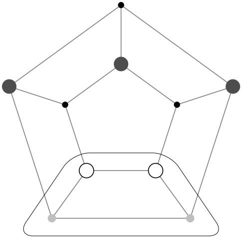


The hard-core model at fugacity $\lambda > 0$ is the probability distribution on $\mathscr{I}$ such that a random $\mathbf{I}$ satisfies for all $S \in \mathscr{I}$

$$\mathbb{P}(\mathbf{I} = S) = \frac{\lambda^{|S|}}{Z(\lambda)}, \quad \text{where } Z(\lambda) = \sum_{S \in \mathscr{I}} \lambda^{|S|}$$

Spatial Markov property

For any $S \in \mathscr{I}$, call u occupied if $u \in S$ and call u uncovered if $N(u) \cap S = \emptyset$



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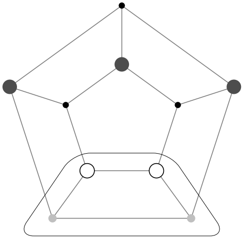

Take $\mathbf{I}$ from the hard-core model on G at fugacity λ and let $X \subseteq V(G)$

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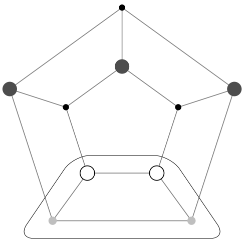

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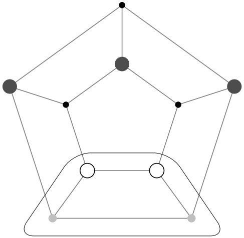

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Then $\mathbf{I} \cap X$ is hard-core on $G[\mathbf{U}_X]$ at fugacity λ

Local occupancy method

A distribution $\mathbf{I}$ on $\mathscr{I}$ has local (a, b)-occupancy $(a, b > 0)$ if for every vertex v

$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq 1$$

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$\rightsquigarrow$ analysis exercise to choose a, b, λ $\rightsquigarrow$ $\chi_f \lesssim \dfrac{\Delta}{\log \Delta}$ for triangle-free 

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(needs triangle-free!)

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Fact 2 $\mathbb{P}(v \text{ uncovered} \mid v \text{ has } j \text{ uncovered neighbours}) = \dfrac{1}{(1+\lambda)^j}$

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A distribution $\mathbf{I}$ on $\mathscr{I}$ has local (a, b)-occupancy $(a, b > 0)$ if for every vertex v

$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq 1$$

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Summing over v,

$$a \cdot \mathbb{E}|\mathbf{I}| + b \cdot \Delta \cdot \mathbb{E}|\mathbf{I}| \geq n \iff \mathbb{E}|\mathbf{I}| \geq \frac{n}{a + b \cdot \Delta},$$

as each u appears $\leq \Delta$ times in

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 [¶]

[¶] Yes, cf. Davies, Jenssen, Perkins, Roberts 2018...

Graphs with sparse neighbourhoods

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$\chi = O\left(\frac{\Delta}{\log \Delta - \log\sqrt{T+1}}\right)$ *for graphs of maximum degree* Δ *with each vertex in* $\leq T$ *triangles,* $0 \leq T = o(\Delta^2)$

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Theorem (Davies, de Joannis de Verclos, Kang, Pirot 2018+)
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Central, classic topic, yet basic discoveries potentially still to be made:

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**See François's talk on Friday