

Guarantees of sparse or dense subgraphs

Ross J. Kang*

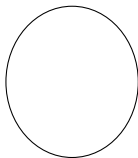
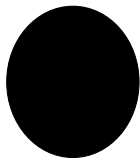


Radboud University Nijmegen

Randomness and Graphs : Processes and Structures
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*The talk covers joint works with Eoin Long, Janos Pach, Viresh Patel and Guus Regts.

α and ω



A clique has all possible edges and a stable set has none.

The *clique number* ω is the size of a largest clique.

The *stability number* α is the size of a largest stable set.

α_c and ω_c

Consider sets “close” to cliques or stable sets, tuned by a parameter[†] $c \in [0, 1]$.

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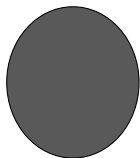
A vertex subset with ℓ vertices

of minimum degree $\geq c(\ell - 1)$

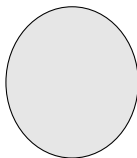
of maximum degree $\leq (1 - c)(\ell - 1)$

is called a c -clique;

is called a c -stable set.



ω_c is size of a largest c -clique.



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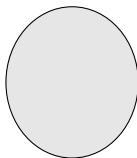
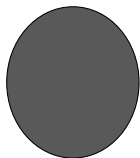
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How does the behaviour change as we tune c between 0 and 1?

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Ramsey numbers[‡]



Ramsey (1930) proved the existence of

$$R(k) = \min \{n : |V(G)| = n \Rightarrow \min\{\alpha(G), \omega(G)\} = k\}.$$

[‡]Picture borrowed from the cover of Soifer (2009).

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Theorem (Erdős 1947, Erdős and Szekeres 1935)

$$\sqrt{2}^{k+o(k)} \leq R(k) \leq 4^{k-o(k)} \text{ as } k \rightarrow \infty.$$

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[‡]Pictures borrowed from the cover of Soifer (2009) and homepages.

Quasi-Ramsey numbers



Ramsey (1930) still implies, for any $c \in [0, 1]$, the existence of

$$R_c^*(k) = \min \{n : |V(G)| = n \Rightarrow \min\{\alpha_c(G), \omega_c(G)\} = k\} \text{ and}$$

$$R_c(k) = \min \{n : |V(G)| = n \Rightarrow \min\{\alpha_c(G), \omega_c(G)\} \geq k\}.$$

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Note that $R_c^*(k) \geq R_c(k)$ always, and both parameters are monotone in c .

Moreover, $R_0(k) = R_0^*(k) = k$ and $R_1(k) = R_1^*(k) = R(k) = \exp(\Theta(k))$.

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Proposition (Erdős and Pach 1983)

Fix $c \in [0, 1]$.

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Consider $c = 1/2 + \varepsilon$ where $\varepsilon = \varepsilon(\ell)$ is a real function tending to 0 as $\ell \rightarrow \infty$.

Variable sharp threshold

The “variable” quasi-Ramsey numbers

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For some nonnegative real function $\nu = \nu(\ell)$, let $c = 1/2 + \nu\sqrt{\log \ell/\ell}$.

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- If $\nu = \Theta(1)$ as $\ell \rightarrow \infty$, then $R_c(k) = k^{\Theta(1)}$ as $k \rightarrow \infty$.
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Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of k vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

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Call a subset of ℓ vertices *excessive* if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

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3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.

Proof ingredient 1: graph discrepancy

Lemma (1)

For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.



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Theorem (Erdős and Spencer 1972)

For n large any graph $G = (V, E)$ with $|V| = n$ has

$$\max_{S \subseteq V} \left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| = \Omega(n^{3/2}).$$

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Theorem (Erdős and Spencer 1974)

For n large and $\frac{1}{2} \log_2 n < t \leq n$ any graph $G = (V, E)$ with $|V| = n$ has

$$\max_{S \subseteq V, |S| \leq t} \left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| = \Omega \left(t^{3/2} \sqrt{\log(n/t)} \right).$$

Proof ingredient 2: set system discrepancy

Lemma (2)

Suppose X is of size $\ell = Dk$, $D > 1$, inducing minimum degree $\geq \ell/2 + \nu\sqrt{\ell}$.
Some $X' \subseteq X$ of size k induces minimum degree $\geq k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}$.

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Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

*For $A_1, \dots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all i
 $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}$.*

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Apply Theorem to A_1, \dots, A_ℓ with $p = (k + 1 + 6\sqrt{\ell})/\ell$ (done if $p > 1$)
to produce $Y \subseteq [\ell]$ such that $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{\ell}$.

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$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu\sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}$.

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$$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu\sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}.$$

$$i = \ell \implies k + 1 = p\ell - 6\sqrt{\ell} \leq |Y| \leq p\ell + 6\sqrt{\ell} = k + 1 + 12\sqrt{\ell}.$$

Proof ingredient 2: set system discrepancy

Lemma (2)

Suppose X is of size $\ell = Dk$, $D > 1$, inducing minimum degree $\geq \ell/2 + \nu\sqrt{\ell}$.
Some $X' \subseteq X$ of size k induces minimum degree $\geq k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}$.

Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

For $A_1, \dots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all i
 $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}$.

"Six standard deviations suffice."

Proof of Lemma (2).

Writing $X = [\ell]$, let $A_i \subseteq X$ be neighbourhood $N(i)$ of $i \in [\ell - 1]$, and $A_\ell = X$.
Apply Theorem to A_1, \dots, A_ℓ with $p = (k + 1 + 6\sqrt{\ell})/\ell$ (done if $p > 1$)
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$$i = \ell \implies k + 1 = p\ell - 6\sqrt{\ell} \leq |Y| \leq p\ell + 6\sqrt{\ell} = k + 1 + 12\sqrt{\ell}.$$

Take $X' \subseteq [\ell - 1]$ arbitrary with $|X'| = k$. By the above, for all $i \in X'$

$$|N(i) \cap X'| \geq k/2 + \nu k/\sqrt{\ell} - 15\sqrt{\ell} = k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}. \quad \square$$

Proof ingredient 3: greedy swaps

Lemma (3)

Suppose X is of size $\ell = \ell_1 + \ell_2$ inducing minimum degree $\geq \delta = \delta_1 + \delta_2$.

Then there exists $X_1, X_2 \subseteq X$ with $|X_1| = \ell_1$ and $|X_2| = \ell_2$ such that either X_1 induces minimum degree $\geq \delta_1$ or X_2 induces minimum degree $\geq \delta_2$

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Proof.

Start with X_1, X_2 an arbitrary partition of X with $|X_1| = \ell_1$ and $|X_2| = \ell_2$.

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The number of edges in X_1 increases by at least

$$\deg_{X_1}(b) - \deg_{X_1}(a) - 1 \geq \delta - \deg_{X_2}(b) - \deg_{X_1}(a) - 1 \geq \delta - \delta_2 - \delta_1 + 1 = 1$$

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(where the -1 accounts for the possibility of the edge ab).

At some point we cannot find two vertices to swap, but then we are done. \square

Proof ingredients

Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of k vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

Call a subset of ℓ vertices *excessive* if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

1. A “variable” quasi-Ramsey bound to produce a set of $\ell \geq 2k$ vertices that is $\Omega(\sqrt{\ell})$ excessive either in the graph or complement.
2. Reduction from a $\Omega(\sqrt{\ell})$ excessive set of Dk vertices, $D > 1$ fixed, to an excessive set of exactly k vertices.
3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.

First apply 1.

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First apply 1. If $\ell \not\equiv 0 \pmod{k}$, then apply 3 to lop off a possibly excessive piece of size Dk , with $Dk \equiv \ell \pmod{k}$ and $D > 1$ fixed, then possibly apply 2. Otherwise apply 3 repeatedly to partition an excessive set of size $\equiv 0 \pmod{k}$ into roughly equal parts of size $\equiv 0 \pmod{k}$, one excessive.

Summary and open questions

For the quasi-Ramsey numbers

$$R_c^*(k) = \min \{n : |V(G)| = n \Rightarrow \min\{\alpha_c(G), \omega_c(G)\} = k\} \text{ and}$$

$$R_c(k) = \min \{n : |V(G)| = n \Rightarrow \min\{\alpha_c(G), \omega_c(G)\} \geq k\},$$

- identified a sharp transition for $R_c(k)$ at $c = 1/2 + \Theta(\sqrt{\log \ell / \ell})$, and
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- Hypergraphs? (See next page.)

Thank you!

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Heterogeneously weighted random graph

Theorem (Erdős and Pach 1983, cf. Kang, Pach, Patel and Regts 2015)

$$R_{1/2}(k) = \Omega(k \log k / \log \log k).$$

i.e. there is some graph on $Ck \log k / \log \log k$ vertices such that any set of $\ell \geq k$ vertices is excessive in neither the graph nor complement.

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Let $z = \frac{\zeta \log k}{\log \log k}$ for some suitably chosen fixed $\zeta > 0$.

Let $V = V_1 \cup \dots \cup V_z$ where $|V_1| = \dots = |V_z| = \left(1 - \frac{1}{2z}\right) k$.

Generate E randomly for any $v_i \in V_i$ and $v_j \in V_j$ by

$$\mathbb{P}(v_i v_j \in E) = \begin{cases} \frac{1}{2} - (2z)^{-4(i+j)-1} & \text{if } i \neq j; \\ \frac{1}{2} + (2z)^{-8i} & \text{if } i = j. \end{cases}$$

There is a chance the graph $G = (V, E)$ has the desired properties.

Proof ingredient 2: set system discrepancy

Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

For $A_1, \dots, A_n \subseteq [n]$ and $p \in [0, 1]$, there exists $Y \subseteq [n]$ such that for all i

$$\left| |A_i \cap Y| - p|A_i| \right| \leq 6\sqrt{n}.$$

For $\mathcal{H} = \{A_1, \dots, A_n\} \subseteq 2^{[n]}$, define the *discrepancy* of \mathcal{H} as

$$\text{disc}(\mathcal{H}) := \min_{\chi \in \{-1, 1\}^V} \max_{S \in \mathcal{H}} \sum_{i \in S} \chi(i).$$

Spencer showed that $\text{disc}(\mathcal{H}) \leq 6\sqrt{n}$ for any such \mathcal{H} .

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If A is the incidence matrix of \mathcal{H} , i.e. A is the $n \times n$ matrix given by

$$A_{ij} = \begin{cases} 1 & \text{if } j \in A_i, \\ 0 & \text{otherwise.} \end{cases},$$

then the *linear discrepancy* is

$$\text{lindisc}(\mathcal{H}) := \max_{c \in [0, 1]^V} \min_{x \in \{0, 1\}^V} \|A(x - c)\|_\infty.$$

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Apply this result with c the all p vector.

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Lemma (1)

For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.

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Outline proof of Lemma (1).

Let $G = (V, E)$ be a graph on $Ck \log k$ vertices and define for any $X \subseteq V$

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Pigeonhole guarantees some $|X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k$ if $C \geq \frac{25}{\log(6/5)}$.

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For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.

Outline proof of Lemma (1).

Let $G = (V, E)$ be a graph on $Ck \log k$ vertices and define for any $X \subseteq V$

$$D_\nu(X) := |D(X)| - \nu|X|^{3/2}.$$

Let $V_0 = V$. Form V_{i+1} in step $i + 1$ by letting $X_i \subseteq V_i$ maximise $D_\nu(X_i)$ and $V_{i+1} = V_i \setminus X_i$. Stop after step $t + 1$ if $|V_{t+1}| < \frac{1}{2}|V|$.

Let $\{i_1, \dots, i_m\} \subseteq [t]$ be those i with $D(X_i) > 0$. $\forall \log \sum_{j \in [m]} |X_{i_j}| \geq \frac{1}{4}|V|$.

Claim 1 For any $j \in [m]$, H_{i_j} has minimum degree $\geq \frac{1}{2}(|X_{i_j}| - 1) + \nu\sqrt{|X_{i_j}|} - 1$.

Claim 2 For any $\ell \in [m - 3]$, $D(X_{i_{\ell+3}}) \leq \frac{5}{6}D(X_{i_\ell})$.

Then $(\frac{5}{6})^{(m-1)/3} D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log_{6/5} D(X_{i_1}) \leq 6 \log_{6/5} k$.

Pigeonhole guarantees some $|X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k$ if $C \geq \frac{25}{\log(6/5)}$.

Claim 1 implies that X_{i_j} is the desired subset. \square

Claim 1 For any $j \in [m]$, H_j has minimum degree $\geq \frac{1}{2}(|X_j| - 1) + \nu\sqrt{|X_j| - 1}$.

Proof of Claim 1.

If not there exists $x \in X_j$ with $\deg_{H_j}(x) < \frac{1}{2}(|X_j| - 1) + \nu\sqrt{|X_j| - 1}$.

Let $X'_j = X_j \setminus \{x\}$. Since $D(X_j) > 0$,

$$\begin{aligned} D_\nu(X'_j) &= e(X'_j) - \frac{1}{2} \binom{|X'_j| - 1}{2} - \nu(|X'_j| - 1)^{3/2} \\ &> e(X_j) - \frac{1}{2} \binom{|X_j|}{2} - \nu\sqrt{|X_j| - 1} - \nu(|X_j| - 1)^{3/2} \\ &> e(X_j) - \frac{1}{2} \binom{|X_j|}{2} - \nu|X_j|^{3/2} = D_\nu(X_j) \end{aligned}$$

(since $n^{3/2} > \sqrt{n-1} + (n-1)^{3/2}$), contradicting maximality of $D_\nu(X_j)$. \square