Guarantees of sparse or dense subgraphs

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Randomness and Graphs : Processes and Structures
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*The talk covers joint works with Eoin Long, Janos Pach, Viresh Patel and Guus Regts.

α and ω

A clique has all possible edges and a stable set has none.

The *clique number* ω is the size of a largest clique.

The *stability number* α is the size of a largest stable set.

α_c and ω_c

Consider sets "close" to cliques or stable sets, tuned by a parameter[†] $c \in [0, 1]$.

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A vertex subset with ℓ vertices
of minimum degree $\geq c(\ell - 1)$ is called a c-clique;
of maximum degree $\leq (1 - c)(\ell - 1)$ is called a c-stable set.

ω_c is size of a largest c-clique. α_c is size of a largest c-stable set.

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How does the behaviour change as we tune c between 0 and 1?

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Ramsey numbers‡

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$$R(k) = \min\{n : |V(G)| = n \Rightarrow \min\{\alpha(G), \omega(G)\} = k\}.$$

‡Picture borrowed from the cover of Soifer (2009).

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Theorem (Erdős 1947, Erdős and Szekeres 1935)

$$\sqrt{2}^{k+o(k)} \le R(k) \le 4^{k-o(k)} \text{ as } k \to \infty.$$

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Quasi-Ramsey numbers

Ramsey (1930) still implies, for any $c \in [0,1]$, the existence of

$$R_c^*(k) = \min\{n : |V(G)| = n \Rightarrow \min\{\alpha_c(G), \omega_c(G)\} = k\} \text{ and}$$
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Note that $R_c^*(k) \geq R_c(k)$ always, and both parameters are monotone in c.

Moreover, $R_0(k) = R_0^*(k) = k$ and $R_1(k) = R_1^*(k) = R(k) = \exp(\Theta(k))$.

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Proposition (Erdős and Pach 1983)

Fix $c \in [0, 1]$.

- *If $c < 1/2$, then $R_c^*(k) = \Theta(k)$.*
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Consider $c = 1/2 + \varepsilon$ where $\varepsilon = \varepsilon(\ell)$ is a real function tending to 0 as $\ell \to \infty$.

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The "variable" quasi-Ramsey numbers

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For some nonnegative real function $\nu = \nu(\ell)$, let $c = 1/2 + \nu\sqrt{\log \ell/\ell}$.

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Proof outline

Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of k vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

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Call a subset of ℓ vertices *excessive* if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

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3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.

Proof ingredient 1: graph discrepancy

Lemma (1)

For fixed $\nu \geq 0$, let $c = 1/2 + \nu/\sqrt{\ell - 1}$. Then $R_c(k) = O(k \log k)$.

$$\Longleftrightarrow$$

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of $\ell \geq k$ vertices inducing minimum degree $\ell/2 + \Omega(\sqrt{\ell})$ in the graph or complement.

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Theorem (Erdős and Spencer 1972)

For n *large any graph* $G = (V, E)$ *with* $|V| = n$ *has*

$$\max_{S \subseteq V} \left| e(S) - \frac{1}{2}\binom{|S|}{2} \right| = \Omega\left(n^{3/2}\right).$$

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Theorem (Erdős and Spencer 1974)
For n large and $\frac{1}{2} \log_2 n < t \leq n$ any graph $G = (V, E)$ with $|V| = n$ has

$$\max_{S \subseteq V, |S| \leq t} \left| e(S) - \frac{1}{2} \binom{|S|}{2} \right| = \Omega\left(t^{3/2} \sqrt{\log(n/t)}\right).$$

Proof ingredient 2: set system discrepancy

Lemma (2)

Suppose X is of size $\ell = Dk$, $D > 1$, inducing minimum degree $\geq \ell/2 + \nu\sqrt{\ell}$.
Some $X' \subseteq X$ of size k induces minimum degree $\geq k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}$.

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Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)

For $A_1, \ldots, A_n \subseteq [n]$ and $p \in [0,1]$, there exists $Y \subseteq [n]$ such that for all i $||A_i \cap Y| - p|A_i|| \leq 6\sqrt{n}$.

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$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu\sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}$.

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$i = \ell \implies k + 1 = p\ell - 6\sqrt{\ell} \leq |Y| \leq p\ell + 6\sqrt{\ell} = k + 1 + 12\sqrt{\ell}$.

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$i \in [\ell - 1] \implies |N(i) \cap Y| \geq p(\ell/2 + \nu\sqrt{\ell}) - 6\sqrt{\ell} \geq k/2 + \nu k/\sqrt{\ell} + 1 - 3\sqrt{\ell}$.

$i = \ell \implies k + 1 = p\ell - 6\sqrt{\ell} \leq |Y| \leq p\ell + 6\sqrt{\ell} = k + 1 + 12\sqrt{\ell}$.

Take $X' \subseteq [\ell - 1]$ arbitrary with $|X'| = k$. By the above, for all $i \in X'$

$$|N(i) \cap X'| \geq k/2 + \nu k/\sqrt{\ell} - 15\sqrt{\ell} = k/2 + (\nu/\sqrt{D} - 15\sqrt{D})\sqrt{k}. \qquad \square$$

Proof ingredient 3: greedy swaps

Lemma (3)

Suppose X *is of size* $\ell = \ell_1 + \ell_2$ *inducing minimum degree* $\geq \delta = \delta_1 + \delta_2$*. Then there exists* $X_1, X_2 \subseteq X$ *with* $|X_1| = \ell_1$ *and* $|X_2| = \ell_2$ *such that either* X_1 *induces minimum degree* $\geq \delta_1$ *or* X_2 *induces minimum degree* $\geq \delta_2$

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Proof.

Start with X_1, X_2 an arbitrary partition of X with $|X_1| = \ell_1$ and $|X_2| = \ell_2$.

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Start with X_1, X_2 an arbitrary partition of X with $|X_1| = \ell_1$ and $|X_2| = \ell_2$.
If $a \in X_1$ has $\deg_{X_1}(a) \leq \delta_1 - 1$ and $b \in X_2$ has $\deg_{X_2}(b) \leq \delta_2 - 1$, swap them.

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The number of edges in X_1 increases by at least

$$\deg_{X_1}(b) - \deg_{X_1}(a) - 1 \geq \delta - \deg_{X_2}(b) - \deg_{X_1}(a) - 1 \geq \delta - \delta_2 - \delta_1 + 1 = 1$$

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(where the -1 accounts for the possibility of the edge ab).
At some point we cannot find two vertices to swap, but then we are done. $\square$

Proof ingredients

Theorem (Kang, Long, Patel and Regts 2016+)

For some $C > 0$, in any graph on $Ck \log k$ vertices there is a set of k vertices inducing minimum degree $k/2 + \Omega(\sqrt{k/\log k})$ in the graph or complement.

Call a subset of ℓ vertices *excessive* if it induces minimum degree $\geq \frac{1}{2}(\ell - 1) + \zeta$ for some excess $\zeta \geq 0$.

Beginning with $Ck \log k$ vertices, here are the rough proof ingredients.

1. A "variable" quasi-Ramsey bound to produce a set of $\ell \geq 2k$ vertices that is $\Omega(\sqrt{\ell})$ excessive either in the graph or complement.
2. Reduction from a $\Omega(\sqrt{\ell})$ excessive set of Dk vertices, $D > 1$ fixed, to an excessive set of exactly k vertices.
3. Partition of an excessive set into two parts of prescribed size at least one of which is excessive.

First apply 1.

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First apply 1. If $\ell \not\equiv 0 \pmod{k}$, then apply 3 to lop off a possibly excessive piece of size Dk, with $Dk \equiv \ell \pmod{k}$ and $D > 1$ fixed, then possibly apply 2. Otherwise apply 3 repeatedly to partition an excessive set of size $\equiv 0 \pmod{k}$ into roughly equal parts of size $\equiv 0 \pmod{k}$, one excessive.

Summary and open questions

For the quasi-Ramsey numbers

$$R^*_c(k) = \min\{n : |V(G)| = n \Rightarrow \min\{\alpha_c(G), \omega_c(G)\} = k\} \text{ and}$$
$$R_c(k) = \min\{n : |V(G)| = n \Rightarrow \min\{\alpha_c(G), \omega_c(G)\} \geq k\},$$

- identified a sharp transition for $R_c(k)$ at $c = 1/2 + \Theta(\sqrt{\log \ell/\ell})$, and
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- Hypergraphs? (See next page.)

Thank you!

[KPR16+] Ross J. Kang, Viresh Patel, and Guus Regts.
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Heterogeneously weighted random graph

Theorem (Erdős and Pach 1983, cf. Kang, Pach, Patel and Regts 2015)
$R_{1/2}(k) = \Omega(k \log k / \log\log k)$.

> *i.e. there is some graph on* $Ck \log k / \log\log k$ *vertices such that any set of* $\ell \geq k$ *vertices is excessive in neither the graph nor complement.*

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Let $z = \dfrac{\zeta \log k}{\log\log k}$ for some suitably chosen fixed $\zeta > 0$.

Let $V = V_1 \cup \cdots \cup V_z$ where $|V_1| = \cdots = |V_z| = \left(1 - \dfrac{1}{2z}\right) k$.

Generate E randomly for any $v_i \in V_i$ and $v_j \in V_j$ by

$$\mathbb{P}(v_i v_j \in E) = \begin{cases} \frac{1}{2} - (2z)^{-4(i+j)-1} & \text{if } i \neq j; \\ \frac{1}{2} + (2z)^{-8i} & \text{if } i = j. \end{cases}$$

There is a chance the graph $G = (V, E)$ has the desired properties.

Proof ingredient 2: set system discrepancy

Theorem (Spencer 1985, Lovász, Spencer and Vesztergombi 1986)
For $A_1, \dots, A_n \subseteq [n]$ and $p \in [0,1]$, there exists $Y \subseteq [n]$ such that for all i $||A_i \cap Y| - p|A_i|| \le 6\sqrt{n}$.

For $\mathcal{H} = \{A_1, \dots, A_n\} \subseteq 2^{[n]}$, define the *discrepancy* of $\mathcal{H}$ as

$$\text{disc}(\mathcal{H}) := \min_{\chi \in \{-1,1\}^V} \max_{S \in \mathcal{H}} \textstyle\sum_{i \in S} \chi(i).$$

Spencer showed that $\text{disc}(\mathcal{H}) \le 6\sqrt{n}$ for any such $\mathcal{H}$.

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If A is the incidence matrix of $\mathcal{H}$, i.e. A is the $n \times n$ matrix given by

$$A_{ij} = \begin{cases} 1 & \text{if } j \in A_i, \\ 0 & \text{otherwise.} \end{cases},$$

then the *linear discrepancy* is

$$\mathrm{lindisc}(\mathcal{H}) := \max_{c \in [0,1]^V} \min_{x \in \{0,1\}^V} \|A(x-c)\|_\infty.$$

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Apply this result with c the all p vector.

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Pigeonhole guarantees some $|X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k$ if $C \geq \frac{25}{\log(6/5)}$.

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Claim 1 *For any $j \in [m]$, H_{i_j} has minimum degree $\geq \frac{1}{2}(|X_{i_j}| - 1) + \nu\sqrt{|X_{i_j}| - 1}$.*

Claim 2 *For any $\ell \in [m-3]$, $D(X_{i_{\ell+3}}) \leq \frac{5}{6} D(X_{i_\ell})$.*

Then $(\frac{5}{6})^{(m-1)/3} D(X_{i_1}) \geq D(X_{i_m}) \geq 1 \Rightarrow m - 1 \leq 3 \log_{6/5} D(X_{i_1}) \leq 6 \log_{6/5} k$.
Pigeonhole guarantees some $|X_{i_j}| \geq \frac{|V| \log(6/5)}{25 \log k} = \frac{C \log(6/5)}{25} k \geq k$ if $C \geq \frac{25}{\log(6/5)}$.
Claim 1 implies that X_{i_j} is the desired subset. □

Claim 1 *For any* $j \in [m]$, H_{i_j} *has minimum degree* $\geq \frac{1}{2}(|X_{i_j}| - 1) + \nu\sqrt{|X_{i_j}| - 1}$.

Proof of Claim 1.
If not there exists $x \in X_{i_j}$ with $\deg_{H_{i_j}}(x) < \frac{1}{2}(|X_{i_j}| - 1) + \nu\sqrt{|X_{i_j}| - 1}$.
Let $X'_{i_j} = X_{i_j} \setminus \{x\}$. Since $D(X_{i_j}) > 0$,

$$\begin{aligned} D_\nu(X'_{i_j}) &= e(X'_{i_j}) - \tfrac{1}{2}\tbinom{|X_{i_j}|-1}{2} - \nu(|X_{i_j}| - 1)^{3/2} \\ &> e(X_{i_j}) - \tfrac{1}{2}\tbinom{|X_{i_j}|}{2} - \nu\sqrt{|X_{i_j}| - 1} - \nu(|X_{i_j}| - 1)^{3/2} \\ &> e(X_{i_j}) - \tfrac{1}{2}\tbinom{|X_{i_j}|}{2} - \nu|X_{i_j}|^{3/2} = D_\nu(X_{i_j}) \end{aligned}$$

(since $n^{3/2} > \sqrt{n-1} + (n-1)^{3/2}$), contradicting maximality of $D_\nu(X_{i_j})$. $\square$