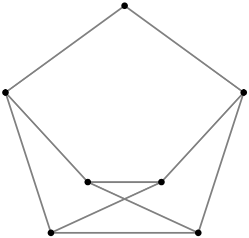
A generalisation of the Erdős–Nešetřil conjecture*

Ross J. Kang

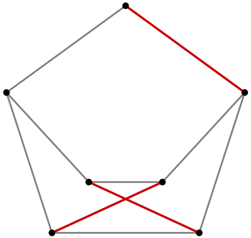
Radboud University Nijmegen

Leiden 12/2016

* This research was made possible thanks to a Van Gogh grant.

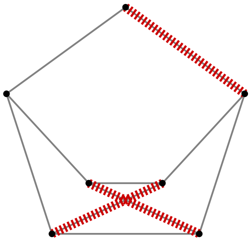


Edge-colouring



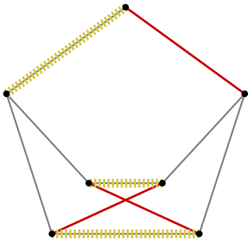
matching : set of non-interfering edges

Edge-colouring

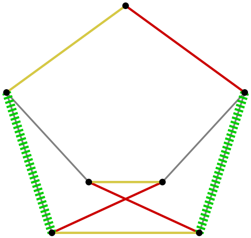


matching : set of non-interfering edges

Edge-colouring

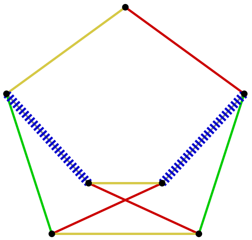


matching : set of non-interfering edges



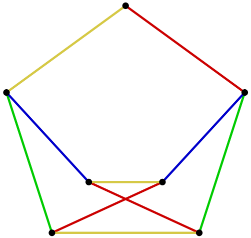
matching : set of non-interfering edges

Edge-colouring



matching : set of non-interfering edges

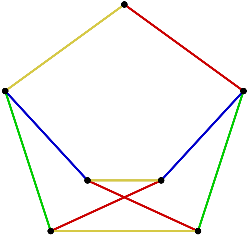
Edge-colouring



matching : set of non-interfering edges

edge-colouring : edge partition into matchings

Edge-colouring



4 colours

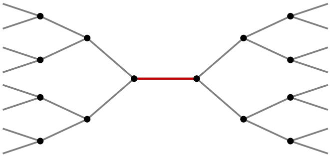
matching : set of non-interfering edges

edge-colouring : edge partition into matchings

chromatic index χ' : least number of parts needed

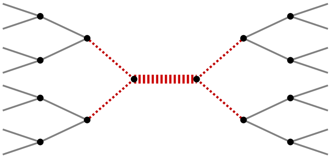
Strong edge-colouring

Strengthen notion of matching: transmission gives stronger interference



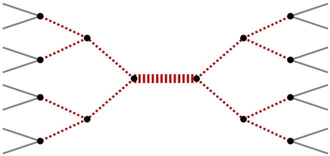
Strong edge-colouring

Strengthen notion of matching: transmission gives stronger interference

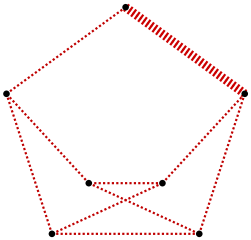


Strong edge-colouring

Strengthen notion of matching: transmission gives stronger interference

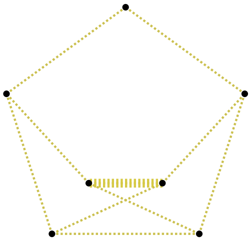


Strong edge-colouring



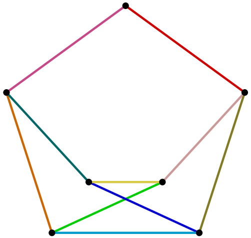
strong matching : set of edges that *induce* a matching

Strong edge-colouring



strong matching : set of edges that *induce* a matching

Strong edge-colouring



10 colours

strong matching : set of edges that *induce* a matching

strong edge-colouring : edge partition into strong matchings

strong chromatic index χ'_s : least number of parts needed

A basic question

Let $\Delta(G)$ denote the maximum degree in a graph G.

What is the worst value among those G with $\Delta(G) = d$?

A basic question

Let $\Delta(G)$ denote the maximum degree in a graph G.

What is the worst value among those G with $\Delta(G) = d$?

I.e.

What is $\chi'(d) := \sup\{\chi'(G) \mid \Delta(G) = d\}$*?*

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

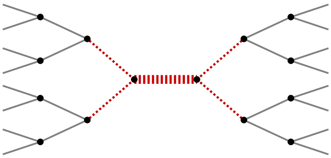
Chromatic index

What is $\chi'(d) := \sup\{\chi'(G) \mid \Delta(G) = d\}$*?*

Chromatic index

What is $\chi'(d) := \sup\{\chi'(G) \mid \Delta(G) = d\}$*?*

Trivial: $\chi'(d) \le 2(d-1)+1 = 2d-1$. Greedy.

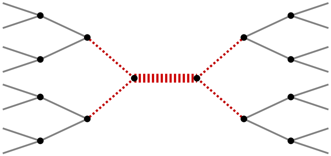


Easy: $\chi'(d) \ge d$. All edges around a vertex must get different colours.

Chromatic index

What is $\chi'(d) := \sup\{\chi'(G) \mid \Delta(G) = d\}$*?*

Trivial: $\chi'(d) \leq 2(d-1)+1 = 2d-1$. Greedy.



Easy: $\chi'(d) \geq d$. All edges around a vertex must get different colours.

Classic: $\chi'(d) \leq d+1$. Recolouring argument by Gupta and by Vizing (1960s).

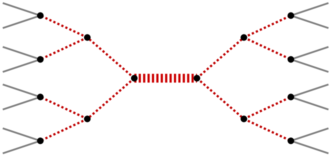
Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

Trivial: $\chi'_s(d) \leq 2d(d-1)+1 = 2d^2 - 2d + 1$. Greedy.



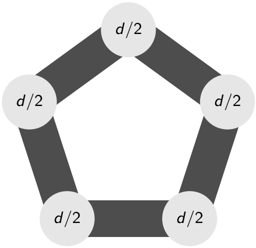
Lower bound?

Better upper bound?

Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

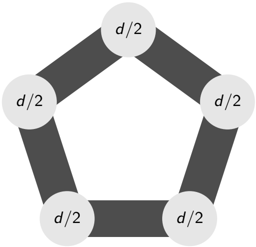
Example: $\chi'_s(d) \geq 5d^2/4$, d even.



Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

Example: $\chi'_s(d) \geq 5d^2/4$, d even.



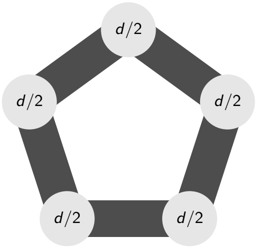
Conjecture (Erdős & Nešetřil 1980s)

$\chi'_s(d) \leq 5d^2/4.$

Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

Example: $\chi'_s(d) \geq 5d^2/4$, d even.



Conjecture (Erdős & Nešetřil 1980s)

$\chi'_s(d) \leq 5d^2/4$. *(Or even just* $\chi'_s(d) \leq (2-\varepsilon)d^2$ *for some absolute* $\varepsilon > 0$*?)*

Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

This remains wide open, except

Theorem (Molloy & Reed 1997)

$\chi'_s(d) \leq (2 - \varepsilon)d^2$ *for some absolute* $\varepsilon > 0$*.* ($\varepsilon \ll 0.002$.)

Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

This remains wide open, except

Theorem (Molloy & Reed 1997)

$\chi'_s(d) \leq (2-\varepsilon)d^2$ *for some absolute* $\varepsilon > 0$*.* ($\varepsilon \ll 0.002$.)

Theorem (Andersen 1992, Horák, He & Trotter 1993)

$\chi'_s(3) = 10$.

Confirms first non-trivial case. Running example certifies sharpness.

Strong chromatic index

What is $\chi'_s(d) := \sup\{\chi'_s(G) \mid \Delta(G) = d\}$*?*

This remains wide open, except

Theorem (Molloy & Reed 1997)

$\chi'_s(d) \le (2-\varepsilon)d^2$ *for some absolute* $\varepsilon > 0$. ($\varepsilon \ll 0.002$.)

Lemma (sparse neighbourhoods colouring)

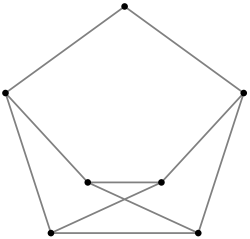
If every neighbourhood is sparse enough and degree bound is large enough, then vertices can be coloured with < 1 *factor lower than the trivial number.*

Lemma (square of line graph neighbourhood sparsity)

The auxiliary graph implicit in strong edge-colouring of bounded degree graph has sparse enough neighbourhoods.

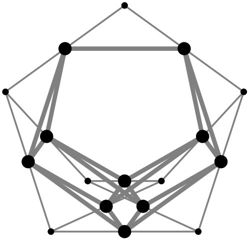
Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



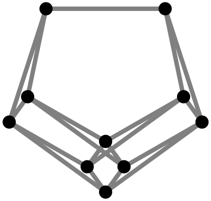
Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



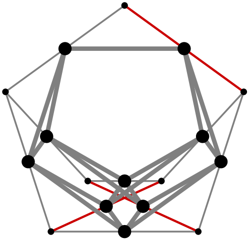
Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



Line graphs

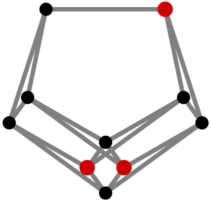
The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



matching in G

Line graphs

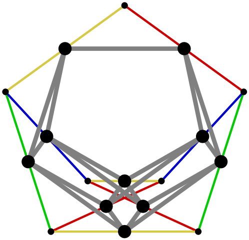
The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



matching in G $\equiv$ stable set in $L(G)$

Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.

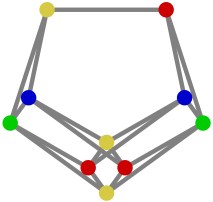


matching in $G \equiv$ stable set in $L(G)$

edge-colouring in G

Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.

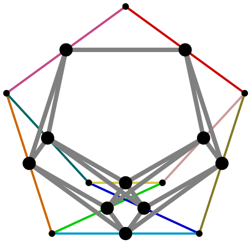


matching in $G \equiv$ stable set in $L(G)$

edge-colouring in $G \equiv$ vertex-colouring in $L(G)$

Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



matching in $G \equiv$ stable set in $L(G)$

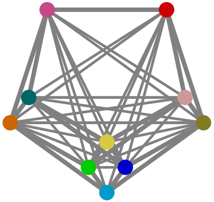
edge-colouring in $G \equiv$ vertex-colouring in $L(G)$

strong matching in G

strong edge-colouring in G

Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



matching in $G \equiv$ stable set in $L(G)$

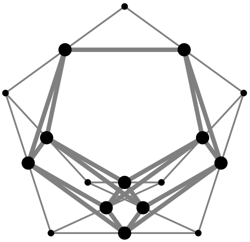
edge-colouring in $G \equiv$ vertex-colouring in $L(G)$

strong matching in $G \equiv$ stable set in $(L(G))^2$

strong edge-colouring in $G \equiv$ vertex-colouring in $(L(G))^2$

Line graphs

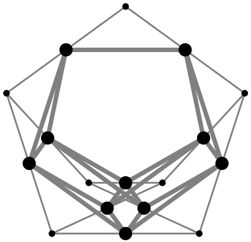
The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



maximum degree Δ of G

Line graphs

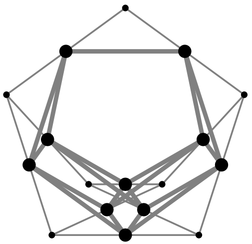
The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



maximum degree Δ of $G \equiv$ clique number ω of $L(G)$

Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.

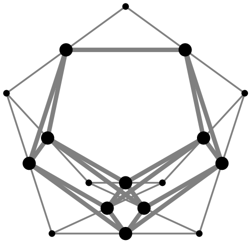


maximum degree Δ of $G \equiv$ clique number ω of $L(G)$

Gupta–Vizing: $\chi'(G) = \chi(L(G)) \in \{\omega(L(G)), \omega(L(G)) + 1\}$.

Line graphs

The *line graph* $L(G)$ of a graph G has vertices corresponding to G-edges and edges if the two corresponding G-edges have a common G-vertex.



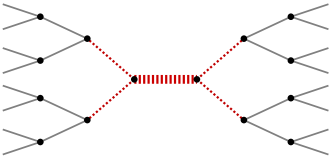
maximum degree Δ of $G \equiv$ clique number ω of $L(G)$

Gupta–Vizing: $\chi'(G) = \chi(L(G)) \in \{\omega(L(G)), \omega(L(G)) + 1\}$.

Molloy–Reed: $\chi'_s(G) = \chi((L(G))^2) \leq (2 - \varepsilon)\omega(L(G))^2$.

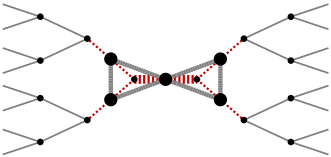
Structure of line graphs

In a line graph, every (vertex) neighbourhood partitions into two cliques.



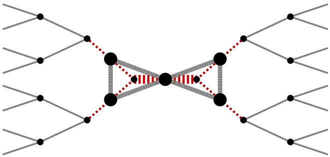
Structure of line graphs

In a line graph, every (vertex) neighbourhood partitions into two cliques.



Structure of line graphs

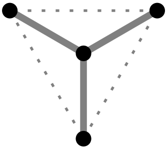
In a line graph, every (vertex) neighbourhood partitions into two cliques.



Call any G with this property *quasiline*.

Structure of line graphs

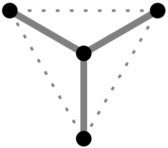
Any quasiline graph, and thus any line graph, contains no claw.



Call any G with this property *claw-free*.

Structure of line graphs

Any quasiline graph, and thus any line graph, contains no claw.

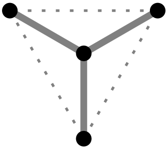


Call any G with this property *claw-free*.

line graphs $\subseteq$ quasiline graphs $\subseteq$ claw-free graphs

Structure of line graphs

Any quasiline graph, and thus any line graph, contains no claw.

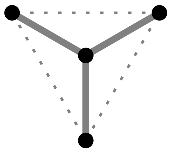


Call any G with this property *claw-free*.

$$\text{line graphs} \subsetneq \text{quasiline graphs} \subsetneq \text{claw-free graphs}$$

Structure of line graphs

Any quasiline graph, and thus any line graph, contains no claw.



Call any G with this property *claw-free*.

line graphs $\subsetneq$ quasiline graphs $\subsetneq$ claw-free graphs

How do line graph results extend to claw-free graphs?

Colouring of claw-free graphs

edge-colouring of graph $\rightsquigarrow$ colouring of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G)$ among those claw-free G with $\omega(G) = \omega$?

Colouring of claw-free graphs

edge-colouring of graph $\rightsquigarrow$ colouring of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G)$ among those claw-free G with $\omega(G) = \omega$?

Gupta, Vizing (1960s): $\chi(G) \in \{\omega, \omega + 1\}$ if G line graph.

Colouring of claw-free graphs

edge-colouring of graph $\rightsquigarrow$ colouring of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G)$ among those claw-free G with $\omega(G) = \omega$?

Gupta, Vizing (1960s): $\chi(G) \in \{\omega, \omega + 1\}$ if G line graph.

Chudnovsky & Ovetsky (2007): $\chi(G) \leq 3\omega/2$ if G quasiline. Sharp.

Colouring of claw-free graphs

edge-colouring of graph $\rightsquigarrow$ colouring of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G)$ among those claw-free G with $\omega(G) = \omega$?

Gupta, Vizing (1960s): $\chi(G) \in \{\omega, \omega + 1\}$ if G line graph.

Chudnovsky & Ovetsky (2007): $\chi(G) \leq 3\omega/2$ if G quasiline. Sharp.

Chudnovsky & Seymour VI (2010):
$\chi(G) \leq 2\omega$ if G connected with stable set of size 3. Sharp.

Without stable set condition, $\chi(G) \leq \omega^2$ but
$\chi(G)$ can be $\Omega(\omega^2/\log\omega)$ as $\omega \to \infty$ in suitable Ramsey graphs.

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Erdős & Nešetřil (1980s): $\chi(G^2) \leq 5\omega^2/4$ if G line graph?

Molloy & Reed (1997): $\chi(G^2) \leq (2 - \varepsilon)\omega^2$ if G line graph.

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Erdős & Nešetřil (1980s): $\chi(G^2) \le 5\omega^2/4$ if G line graph?

Molloy & Reed (1997): $\chi(G^2) \le (2-\varepsilon)\omega^2$ if G line graph.

Does $\chi(G^2)$ get worse approaching claw-free from line (like for $\chi(G)$)?

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq (2 - \varepsilon)\omega^2$.

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \le (2 - \varepsilon)\omega^2$. *Same small unspecified* $\varepsilon > 0$.

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq (2 - \varepsilon)\omega^2$. *Same small unspecified $\varepsilon > 0$.*

Conjecture (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq 5\omega^2/4$.

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \le (2 - \varepsilon)\omega^2$. *Same small unspecified* $\varepsilon > 0$.

Conjecture (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \le 5\omega^2/4$.

Theorem (Cames van Batenburg & K 2016+)

$\chi(G^2) \le 10$ *if* $\omega = 3$.

Colouring squares of claw-free graphs

strong edge-colouring of graph $\rightsquigarrow$ colouring of square of claw-free graph

maximum degree of graph $\rightsquigarrow$ clique number of claw-free graph

What is the worst $\chi(G^2)$ among those claw-free G with $\omega(G) = \omega$?

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq (2-\varepsilon)\omega^2$. *Same small unspecified $\varepsilon > 0$.*

Conjecture (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq 5\omega^2/4$. $\rightarrow$ **Suffices to prove this for G line graph of multigraph.**

Theorem (Cames van Batenburg & K 2016+)

$\chi(G^2) \leq 10$ *if* $\omega = 3$.

Greedy colouring

Recall “trivial” bound max degree$+1$, colouring greedily one by one.

What if we colour the smallest degree element last?

Greedy colouring in squares

Recall "trivial" bound max degree+1, colouring greedily one by one.

What if we colour the smallest degree element last?

Lemma (double greedy)

Fix $K \geq 0$ and $\mathcal{C}_1$, $\mathcal{C}_2$ graph classes. Assume every $G \in \mathcal{C}_2$ has $\chi(G^2) \leq K+1$. Assume $\mathcal{C}_1$ contains singleton, closed under vertex-deletion and for any $G \in \mathcal{C}_1$

- *G belongs to $\mathcal{C}_2$, or*
- *there is vertex $v \in G$ with square degree $\deg_{G^2}(v) \leq K$ such that those G-neighbours x with $\deg_{G^2}(x) > K+2$ induce a clique in $(G \setminus v)^2$.*

Then every $G \in \mathcal{C}_1$ has $\chi(G^2) \leq K+1$.

Colouring squares of claw-free graphs

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq (2-\varepsilon)\omega^2$ *for claw-free* G *with* $\omega(G) = \omega$.

Colouring squares of claw-free graphs

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq (2-\varepsilon)\omega^2$ *for claw-free* G *with* $\omega(G) = \omega$.

Lemma (claw-free $\rightarrow$ quasiline)

For claw-free G *with* $\omega(G) = \omega$*, either* G *is quasiline or there is* $v \in G$ *with* $\deg_{G^2}(v) \leq \omega^2 + (\omega+1)/2$ *s.t. neighbours induce clique in* $(G \setminus v)^2$.

Colouring squares of claw-free graphs

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \le (2-\varepsilon)\omega^2$ *for claw-free* G *with* $\omega(G) = \omega$.

Lemma (claw-free $\to$ quasiline)

For claw-free G *with* $\omega(G) = \omega$*, either* G *is quasiline or there is* $v \in G$ *with* $\deg_{G^2}(v) \le \omega^2 + (\omega+1)/2$ *s.t. neighbours induce clique in* $(G \setminus v)^2$.

Lemma (quasiline $\to$ line graph of multigraph)

For quasiline G *with* $\omega(G) = \omega$*, either* G *is line graph of multigraph or there is* $v \in G$ *with* $\deg_{G^2}(v) \le \omega^2 + \omega$ *s.t. neighbours* x *with* $\deg_{G^2}(x) > \omega^2 + \omega$ *induce clique in* $(G \setminus v)^2$.

Colouring squares of claw-free graphs

Theorem (de Joannis de Verclos, K & Pastor 2016+)

$\chi(G^2) \leq (2-\varepsilon)\omega^2$ *for claw-free* G *with* $\omega(G) = \omega$.

Lemma (claw-free $\rightarrow$ quasiline)

For claw-free G *with* $\omega(G) = \omega$, *either* G *is quasiline or there is* $v \in G$ *with* $\deg_{G^2}(v) \leq \omega^2 + (\omega+1)/2$ *s.t. neighbours induce clique in* $(G \setminus v)^2$.

Lemma (quasiline $\rightarrow$ line graph of multigraph)

For quasiline G *with* $\omega(G) = \omega$, *either* G *is line graph of multigraph or there is* $v \in G$ *with* $\deg_{G^2}(v) \leq \omega^2 + \omega$ *s.t. neighbours* x *with* $\deg_{G^2}(x) > \omega^2 + \omega$ *induce clique in* $(G \setminus v)^2$.

Lemma (line graph of multigraph)

$\chi(G^2) \leq (2-\varepsilon)\omega^2$ *if* G *line graph of multigraph with* $\omega(G) = \omega$.

Colouring squares of claw-free graphs

Theorem (Cames van Batenburg & K 2016+)

$\chi(G^2) \leq 10$ *if* G *claw-free with* $\omega(G) = 3$.

Colouring squares of claw-free graphs

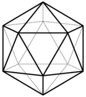
Theorem (Cames van Batenburg & K 2016+)

$\chi(G^2) \leq 10$ if G claw-free with $\omega(G) = 3$.

Lemma

If G connected claw-free with $\omega(G) = 3$, then

- *G is icosahedron;*
- *G is line graph of a 3-regular graph; or*
- *there is $v \in G$ with $\deg_{G^2}(v) \leq 9$ s.t. $\deg_{G^2}(x) \leq 11$ for all neighbours x.*



Colouring squares of claw-free graphs

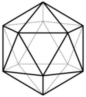
Theorem (Cames van Batenburg & K 2016+)

$\chi(G^2) \leq 10$ *if* G *claw-free with* $\omega(G) = 3$.

Lemma

If G *connected claw-free with* $\omega(G) = 3$*, then*

- G *is icosahedron;*
- G *is line graph of a 3-regular graph; or*
- *there is* $v \in G$ *with* $\deg_{G^2}(v) \leq 9$ *s.t.* $\deg_{G^2}(x) \leq 11$ *for all neighbours* x.



Similar techniques to achieve optimal reduction for $\omega(G) = 4$.

For further consideration

The Erdős–Nešetřil conjecture itself!

For further consideration

The Erdős–Nešetřil conjecture itself!

Other optimisation/extremal problems where claw-free reduces to (multi)line?

For further consideration

The Erdős–Nešetřil conjecture itself!

Other optimisation/extremal problems where claw-free reduces to (multi)line?

Superclasses of claw-free graphs?

For further consideration

The Erdős–Nešetřil conjecture itself!

Other optimisation/extremal problems where claw-free reduces to (multi)line?

Superclasses of claw-free graphs?

For $t \geq 3$, how does $\chi(G^t)$ behave in terms of $\omega(G)$ for claw-free G?
(For line graph G and large fixed t this is already a difficult problem.)

Announcement

26 and 27 January 2017

STAR Workshop on Random Graphs.

in Utrecht

Speakers include:

Mihyun Kang (Graz),
Marián Boguña (Barcelona),
Nick Wormald (Melbourne),
Vincent Tassion (Zürich).

Thank you!