

ACYCLIC T -IMPROPER COLOURINGS OF GRAPHS WITH BOUNDED MAXIMUM DEGREE

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Abstract. For graphs of bounded maximum degree, we consider acyclic t -improper colourings, that is, colourings in which each bipartite subgraph consisting of the edges between two colour classes is acyclic and each colour class induces a graph with maximum degree at most t .

In the first part, we show that all subcubic graphs are acyclically 1-improperly 3-choosable, thus extending a result of Boiron, Sopena and Vignal (1997, DMTCS 49, 1–10). In the second part, we consider the supremum, over all graphs of maximum degree at most d , of the acyclic t -improper chromatic number and provide t -improper analogues of results by Alon, McDiarmid and Reed (1991, RSA 2(3), 277–288) and Fertin, Raspaud and Reed (2004, JGT 47(3), 163–182).

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1. Introduction. Recall that a colouring is *acyclic* if each bipartite subgraph consisting of the edges between two colour classes is acyclic. In other words, a colouring is acyclic if it contains no *alternating cycle* (that is, a cycle which alternates between two distinct colours). The starting point of our study is the work of Boiron, Sopena and Vignal [3, 4]. These authors considered acyclic colourings whose colour classes satisfy certain hereditary properties (their main motivation being the connection of such colourings with oriented colourings). The property we are predominantly concerned with here is the following: given an integer $t \geq 0$, a set of vertices is *t -dependent* if it induces a subgraph of maximum degree at most t . A colouring is *t -improper* if its colour classes are t -dependent. Given a graph G , we let $\chi_a^t(G)$ denote the acyclic t -improper chromatic number of G , that is, the least number of colours in an acyclic colouring of the vertices of G such that each colour class is t -dependent. Note $\chi_a^0(G)$ is just the usual acyclic chromatic number $\chi_a(G)$, since a 0-dependent set is an independent set. Let $\chi_a^t(d)$ denote the maximum possible value of $\chi_a^t(G)$ over all graphs G with maximum degree d . We observe that $\chi_a(G) = \chi_a^0(G) \geq \chi_a^1(G) \geq \dots \geq \chi_a^{\Delta(G)}(G) = 1$ and also that $\chi_a(d) = \chi_a^0(d) \geq \chi_a^1(d) \geq \dots \geq \chi_a^d(d) = 1$. We will investigate the behaviour of $\chi_a^t(d)$ viewed as a function of t .

In [3], Boiron *et al.* considered the problem of acyclically improperly colouring subcubic graphs, i.e. graphs of maximum degree at most three. In particular, they proved that $\chi_a^1(3) = 3$ and conjectured that $\chi_a^2(3) = 2$. The list colouring variant of this problem is our first main topic. Given a graph G , a *list assignment* L is a

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mapping that assigns to every vertex v a list $L(v)$ of colours. We say G is L -colourable if there exists a proper colouring such that each vertex v is assigned a colour chosen from $L(v)$. Given an integer $\ell > 0$, G is ℓ -choosable if, for any list assignment L such that $|L(v)| \geq \ell$ for each vertex v , the graph is L -colourable. The *choice number* $\text{ch}(G)$ of G is the smallest ℓ such that G is ℓ -choosable. As well as being interesting in its own right, studying the choice number is rewarding because (a) for all graphs G , $\text{ch}(G) \geq \chi(G)$, so upper bounds on the choice number yield upper bounds on the chromatic number, and (b) inductive proofs often succeed in bounding $\text{ch}(G)$ where similar approaches to bounding $\chi(G)$ directly fail. The choice number has been well-studied since it was introduced by Erdős, Rubin and Taylor [6]. In parallel with the definition of $\chi_a^t(G)$ and $\chi_a^t(d)$, we define $\text{ch}_a^t(G)$ and $\text{ch}_a^t(d)$ in the natural way. Note that, for any graph G and any integers t, d with $0 \leq t < d$, $\text{ch}_a^t(G) \geq \text{ch}_a^{t+1}(G)$, $\text{ch}_a^t(d) \geq \text{ch}_a^{t+1}(d)$, $\text{ch}_a^t(G) \geq \chi_a^t(G)$ and $\text{ch}_a^t(d) \geq \chi_a^t(d)$ for any $t \geq 0$.

Consider the complete bipartite graph $G = K_{d,d}$. Suppose each vertex from one stable set is assigned the list $\{1, 2, \dots, d-1\}$ and each vertex from the other stable set is assigned the list $\{d, d+1, \dots, 2d-2\}$. Then, in any colouring of G using colours from these lists, some colour must be repeated in each stable set, so there is an alternating cycle. It follows that $\text{ch}_a^d(d) \geq \text{ch}_a^d(K_{d,d}) \geq d$ and, in particular, that $\text{ch}_a^1(3) \geq \text{ch}_a^2(3) \geq \text{ch}_a^3(3) \geq 3$, so no list colouring analogue of the above-mentioned conjecture of Boiron *et al.* that $\chi_a^2(3) = 2$ can hold. On the other hand, as our first main result, we prove the following strengthening of their upper bound:

THEOREM 1. *Every subcubic graph is acyclically 1-improperly 3-choosable.*

In other words, $\text{ch}_a^1(3) \leq 3$. We note that Montassier, Ochem and Raspaud [12] showed $\text{ch}_a(3) = 4$ and $\text{ch}_a(4) = 5$.

Our second main topic is to look at bounds on $\chi_a^t(d)$ for large values of d and t . For some intuition, note that $\chi_a^t(d) \leq \chi_a(d) \leq d^2 + 1$: we can greedily colour each vertex v by picking a colour different from those already assigned at distance two from v , thus guaranteeing that no alternating cycles shall arise. In 1976 (cf. [1]), Erdős conjectured that it would be possible to do asymptotically better, that $\chi_a(d) = o(d^2)$. Fifteen years later, Alon, McDiarmid and Reed [2] proved this conjecture by showing that $\chi_a(d) \leq cd^{4/3}$, for some fixed constant $c \leq 50$, using the Lovász Local Lemma; hence, $\chi_a^t(d) = O(d^{4/3})$. They also showed that $\chi_a(d) = \Omega(d^{4/3}/(\ln d)^{1/3})$ using probabilistic methods.

We begin by considering lower bounds on $\chi_a^t(d)$. Observe that, for any graph G , $\chi_a^t(G) \geq \frac{\chi_a(G)}{ct^{4/3}}$: given an acyclic t -improper colouring, we can acyclically colour each colour class with at most $ct^{4/3}$ new colours to obtain an acyclic colouring of the entire graph. Hence, $\chi_a^t(d) = \Omega((d/t)^{4/3}/(\ln d)^{1/3})$. Our second main result is to show that this basic lower bound on $\chi_a^t(d)$ can be much improved upon asymptotically, as long as $d - t \geq 10\sqrt{d \ln d}$. More fully,

THEOREM 2. *If $t \leq d - 10\sqrt{d \ln d}$, then $\chi_a^t(d) = \Omega((d-t)^{4/3}/(\ln d)^{1/3})$.*

In particular, if $t = (1 - \varepsilon)d$ for any fixed constant ε , $0 < \varepsilon \leq 1$, then we obtain the same asymptotic lower bound as Alon *et al.* Comparing this lower bound with the upper bound $\chi_a^t(d) = O(d^{4/3})$, we see the surprising fact that even allowing $t = \Omega(d)$ does not greatly reduce the number of colours needed for improper acyclic colourings of graphs with large maximum degree.

Lastly, we consider bounds on $\chi_a^t(d)$ when $d - t = O(d^{1/2})$. At some point, $\chi_a^t(d)$ must drop significantly as t increases, because $\chi_a^d(d) = 1$. Although we are unable to pin down the behaviour of $\chi_a^t(d)$ viewed as a function of t , we can improve upon the upper bound of Alon *et al.* when t is very close to d . More precisely, our third main

result is the following:

THEOREM 3. $\chi_a^t(d) = O(d \ln d + (d-t)d)$.

As for lower bounds in the regime $d-t = O(d^{1/2})$, first note that Boiron *et al.* showed $\chi_a^{d-2}(d) \geq 3$; we can straightforwardly generalise this result by showing that $\chi_a^t(d) \geq d-t+1$. This is done as follows: if K_{d+1} is the complete graph on $d+1$ vertices, then $\chi_a^t(K_{d+1}) \geq d-t+1$, since, in any acyclic t -improper colouring of K_{d+1} , at most one colour class has more than one vertex and no colour class has more than $t+1$ vertices. We can, however, improve upon this further and, in the final section, we exhibit a set of examples showing the following lower bound.

THEOREM 4. $\chi_a^{d-1}(d) = \Omega(d^{2/3})$.

We would like to reduce the gaps between the lower and upper bounds on $\chi_a^t(d)$. For $t = d-1$, the problem is particularly tantalising, and, in this case, the lower bound of Theorem 4 and the upper bound of Theorem 3 differ by a factor of $d^{1/3} \ln d$. For this choice of t , the problem also includes the conjecture of Boiron *et al.* that every subcubic graph is acyclically 2-improperly 2-colourable.

In the rest of the paper, we use the following notation. The degree of a given vertex v is denoted by $d(v)$. A k -vertex (resp. a $\leq k$ -vertex) is a vertex of degree k (resp. degree at most k). We denote by $N(v)$ the set of the neighbours of v . A k -cycle (resp. a $\geq k$ -cycle) is a cycle containing k vertices (resp. at least k vertices). For a graph G and a vertex $v \in V(G)$, we denote by $G \setminus \{v\}$ the graph obtained from G by removing v and its incident edges; for an edge uv of $E(G)$, $G \setminus \{uv\}$ denotes the graph obtained from G by removing the edge uv . These notions are extended to sets of vertices and edges in an obvious way. Let G be a graph and f be a colouring of G . For a given vertex v of G , we denote by $\text{im}_f(v)$, or simply $\text{im}(v)$ when the colouring is clear from the context, the number of neighbours of v having the same colour as v and call this quantity the *impropriety* of the vertex v . In all the figures depicting configurations, we use the following drawing convention: a vertex whose neighbours are totally specified is white, whereas a vertex whose neighbours are partially specified is black. For notation not defined here, we refer the reader to [13].

2. Upper bound for $\text{ch}_a^1(3)$. In this section, we prove Theorem 1, i.e. we show that $\text{ch}_a^1(3) \leq 3$. Our approach is to consider a minimal counterexample H to the theorem, i.e. H has maximum degree three and list assignment L such that $|L(v)| \geq 3$ for any $v \in V(H)$, H is not acyclically 1-improperly L -colourable and, subject to these conditions, H is minimal with respect to $|V(H)|$. We first show in Subsection 2.1 that H is a 2-connected cubic graph. Then, in Subsection 2.2, we provide an inductive approach to give an acyclic 1-improper L -colouring of H . This contradiction gives us the theorem. We remark that this proof technique was also used in [8].

2.1. H is a 2-connected cubic graph. The aim here is Lemma 6 below, by using the following lemma.

LEMMA 5. *Let G be a connected subcubic graph with list assignment L such that each list size is at least three.*

(a) *Suppose that $v \in V(G)$ is a ≤ 2 -vertex and that $G \setminus \{v\}$ has an acyclic 1-improper L -colouring f . Then there is an acyclic 1-improper L -colouring f' of G such that every vertex v' at distance at least three from v satisfies $f'(v') = f(v')$.*

(b) *Suppose that $uv \in E(G)$ is a cut-edge and that $G \setminus \{uv\}$ has an acyclic 1-improper L -colouring f . Then there is an acyclic 1-improper L -colouring f' of G .*

Before proving Lemma 5, let us use it to prove the following.

LEMMA 6. *Suppose H , together with list assignment L , is a minimal counterexample to Theorem 1. Then H is a 2-connected cubic graph.*

Proof. H is clearly connected; suppose H has a cut-vertex u . Since H is subcubic, there is a cut-edge uv . Let H_1 and H_2 be the connected components of $H \setminus \{uv\}$. By minimality of H , there is an acyclic 1-improper L -colouring f_1 (resp. f_2) of H_1 (resp. H_2). Now, $f = f_1 \cup f_2$ is an acyclic 1-improper L -colouring of $H \setminus \{uv\}$ and thus, by Lemma 5(b), there is an acyclic 1-improper L -colouring of H , contradicting that H is a counterexample.

The fact that H is cubic follows from Lemma 5(a). \square

Proof of Lemma 5(a). If $d(v) = 1$, then f can be extended to G by selecting any colour $f'(v) \in L(v)$ for v that is different from its neighbour's. So we may suppose that v has exactly two neighbours u and w . If $f(u) \neq f(w)$, then we can just choose a colour $f'(v)$ from $L(v) \setminus \{f(u), f(w)\}$; thus, from now on we assume $f(u) = f(w) = a$.

Let S be the set of colours appearing on vertices of $\{u\} \cup N(u) \setminus \{v\}$. If $|S| < 2$, then we may choose $f'(v)$ from $L(v) \setminus S$; this prevents a new alternating cycle and does not increase the impropriety of u or w . We may thus assume that $|S| = 3$, so u has two neighbours u_1, u_2 aside from v , and $u_1 \neq w \neq u_2$. Symmetrically, we may assume that w has neighbours w_1 and w_2 aside from v , and $w_1 \neq u \neq w_2$. This is depicted in Figure 2.1.

Note that possibly $u_1 \in \{w_1, w_2\}$ or $u_2 \in \{w_1, w_2\}$, but this does not affect our arguments. If there is no colour from $L(v)$ that would extend f to G , then w.l.o.g. the following holds: $L(v) = \{a, b, c\}$ and assigning b (resp. c) to v would create an alternating cycle through $u_1 u v w w_1$ (resp. $u_2 u v w w_2$); in particular, $f(u_1) = f(w_1) = b$, $f(u_2) = f(w_2) = c$, and the colour a appears at least twice in $N(u_1)$, $N(u_2)$, $N(w_1)$, and $N(w_2)$.

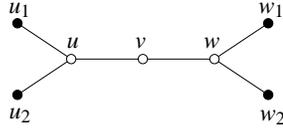


FIG. 2.1. A worst configuration for Lemma 5(a)

Assume u cannot be recoloured to obtain an acyclic 1-improper L -colouring f' of $G \setminus \{v\}$ with $f'(u) \neq f'(w)$; otherwise, this could be easily extended to a valid colouring of G . Then we have that $L(u) = \{a, b, c\}$ and $\text{im}(u_1) = \text{im}(u_2) = 1$. By recolouring u_1 with a colour chosen from $L(u_1) \setminus \{a, b\}$ we obtain a colouring with $\text{im}(u_1) = 0$. Since we possibly created an alternating cycle containing u , we also recolour u with the colour b . This new partial colouring f' of $G \setminus \{v\}$ is acyclic and 1-improper with $f'(u) \neq f'(w)$ and so is easily extended to G . Since we recoloured only vertices at distance at most two from v , we are done. \square

Proof of Lemma 5(b). Let G_1 and G_2 be the two connected components of $G \setminus \{uv\}$ with $u \in G_1$ and $v \in G_2$. If we have $f(u) \neq f(v)$, or if we have $f(u) = f(v)$ with $\text{im}(u) = 0$ in G_1 and $\text{im}(v) = 0$ in G_2 , then f is also an acyclic 1-improper L -colouring of G . So suppose $f(u) = f(v) = a$ with $\text{im}(u) = 1$ in G_1 . Let u_1, u_2 be the two neighbours of u in G_1 and assume $f(u_1) = f(u) = a$. Clearly, $f(u_2) \neq a$ and we can recolour u with a colour from $L(u) \setminus \{a, f(u_2)\}$ without creating any alternating cycle in G_1 . The resulting acyclic 1-improper L -colouring f' of $G \setminus \{uv\}$ is a valid L -colouring of G , since $f'(u) \neq f'(v)$. \square

2.2. Inductive colouring of H . Here, we will complete the proof of the theorem by using the following lemma.

LEMMA 7. Let G be a 2-connected cubic graph with list assignment L such that each list size is at least three. Suppose G contains two adjacent vertices u^* and v^* such that $L(u^*) \neq L(v^*)$.

(a) There exists a vertex order x_1, x_2, \dots, x_n for G such that $x_1 = u^*$, $x_n = v^*$ and for every $1 \leq i < n$, the vertex x_i is adjacent to some vertex x_j with $j > i$.

Let $G_i = G \setminus \{x_{i+1}, \dots, x_n\}$ for $1 \leq i \leq n$.

(b) If $2 \leq i \leq n-1$ and f_{i-1} is an acyclic 1-improper L -colouring of G_{i-1} , then there is an acyclic 1-improper L -colouring f_i of G_i such that $f_i(x_1) = f_{i-1}(x_1)$.

(c) If f_{n-1} is an acyclic 1-improper L -colouring of G_{n-1} such that $f_{n-1}(x_1) \notin L(x_n)$, then there is an acyclic 1-improper L -colouring f of G .

Before we prove the lemma, we first show how it is used to prove the theorem.

Proof of Theorem 1. Let H , together with list assignment L , be a minimal counterexample to the theorem. By Lemma 6, H is 2-connected and cubic. Recall that every subcubic graph is acyclically 1-improperly 3-colourable [3]. We can therefore assume that H contains two adjacent vertices u^* and v^* such that $L(u^*) \neq L(v^*)$.

Therefore, Lemma 7 applies and we inductively colour H as follows. At Step 1, set $f_1(x_1) = c$ for some $c \in L(x_1) \setminus L(x_n)$. Then, at each Step i , $2 \leq i \leq n-1$, extend the colouring without changing the colour of x_1 , by Lemma 7(b). At Step n , use Lemma 7(c). The resulting L -colouring f of H is acyclic and 1-improper, a contradiction. \square

Proof of Lemma 7(a). Such an ordering is used in a standard proof of Brooks' Theorem; the proof can be found in a standard reference, e.g. [13]. \square

Proof of Lemma 7(b). All vertices of G_i except x_i are coloured by f_{i-1} . Notice that x_i is a ≤ 2 -vertex in G_i by definition of the order x_1, \dots, x_n . The vertex x_1 is also a ≤ 2 -vertex in G_i since it is adjacent to x_n in G . If x_1 and x_i are distance at least three apart, then Lemma 5(a) applies with $v = x_i$ and we are done. Furthermore, by closely examining the arguments in Lemma 5(a), we see that the only case in which we might recolour any vertex (as opposed to an extension of f_{i-1} to $v = x_i$) is when the neighbours of v are as in Figure 2.1, i.e. they both have degree 3 and are not adjacent. Since x_i has degree 2, we may thus assume that x_1 and x_i are not adjacent. By symmetry, we need only consider the case that $x_1 = u_1$ (see Figure 2.2(a)). As before, we can assume the following: $L(v) = \{a, b, c\}$, $f_{i-1}(u) = f_{i-1}(w) = a$, $f_{i-1}(u_1) = f_{i-1}(w_1) = b$, $f_{i-1}(u_2) = f_{i-1}(w_2) = c$, and the colour a appears at least twice in $N(u_1)$, $N(u_2)$, $N(w_1)$, and $N(w_2)$. Since x_1 is a 2-vertex, it follows that $\text{im}(x_1) = 0$, and we can recolour u with a colour from $L(u) \setminus \{a, c\}$. We then obtain an acyclic 1-improper colouring f_i which can be easily extended to G since $f_i(u) \neq f_i(w)$. \square

Proof of Lemma 7(c). Let y and z be the neighbours of x_n distinct from x_1 and let y_1 and y_2 (resp. z_1 and z_2) be the neighbours of y (resp. z) distinct from x_n (see Figure 2.2(b)). Let $a = f_{n-1}(x_1)$. We consider some cases depending on the colours appearing in the neighbourhood of x_n .

1. If y, z, x_1 have pairwise distinct colours, then f_{n-1} easily extends to G since $a \notin L(x_n)$.

2. Now suppose that exactly one colour distinct from a , say b , appears in the neighbourhood of x_n . We consider two cases depending on the colours of y and z .

(a) Suppose that $f_{n-1}(y) = f_{n-1}(z) = b$. If we can choose a colour in $L(x_n)$ for x_n without creating an alternating cycle or having $\text{im}(v) > 1$ for some v , then we are done. Otherwise, we may assume w.l.o.g. that $L(x_n) = \{b, c, d\}$ with $f_{n-1}(y_1) = f_{n-1}(z_1) = c$, $f_{n-1}(y_2) = f_{n-1}(z_2) = d$, and the colour b appears at least twice in

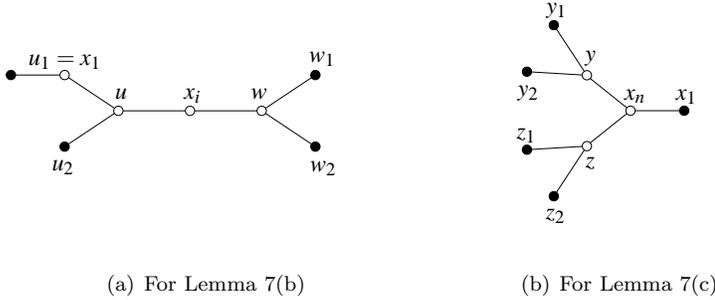


FIG. 2.2. Configurations for Lemma 7

$N(y_1)$, $N(y_2)$, $N(z_1)$, and $N(z_2)$. We first try to recolour y with a suitable colour distinct from b . (This would not create an alternating cycle since $f_{n-1}(y_1) \neq f_{n-1}(y_2)$.) If this colouring succeeds, then we assign x_n the colour b . Otherwise, $L(y) = \{b, c, d\}$ and $\text{im}(y_1) = \text{im}(y_2) = 1$. In this case, we recolour y_1 with a colour from $L(y_1) \setminus \{b, c\}$ and recolour y with the colour c . As the neighbours of y_2 all now have distinct colours, there is no alternating cycle through y_2 nor through y_1 , and so we have not created an alternating cycle. Furthermore, y, z, x_1 have three distinct colours and we are back in Case 1.

(b) Now suppose that $f_{n-1}(y) = a$ and $f_{n-1}(z) = b$. If we can choose a colour for x_n without creating an alternating cycle or having $\text{im}(v) > 1$ for some v , then we are done. Otherwise, we may assume w.l.o.g. that $L(x_n) = \{b, c, d\}$ with $\text{im}(z) = 1$, $f_{n-1}(y_1) = c$, $f_{n-1}(y_2) = d$, and the colour a appears at least twice in $N(y_1)$ and $N(y_2)$. We first try to recolour y with a colour distinct from a . If this is possible, we are in the situation of Case 1 or of Case 2(a). Otherwise, $L(y) = \{a, c, d\}$ and $\text{im}(y_1) = \text{im}(y_2) = 1$, so we recolour y_1 with a colour from $L(y_1) \setminus \{a, c\}$ and recolour y with the colour c ; we are back in Case 1.

3. Finally, suppose that $f_{n-1}(y) = f_{n-1}(z) = f_{n-1}(x_1) = a$. Let $L(x_n) = \{b, c, d\}$ (and recall that $a \notin L(x_n)$). If no colour from $L(x_n)$ can be used to colour x_n , then each colour could create an alternating cycle containing x_n . We may assume w.l.o.g. that $f_{n-1}(y_1) = b$, $f_{n-1}(y_2) = c$ and the colour a appears at least twice in $N(y_1)$ and $N(y_2)$. As usual, we first try to recolour y with a colour distinct from a . If successful, then we return to one of the previous cases. Otherwise, $L(y) = \{a, b, c\}$ and $\text{im}(y_1) = \text{im}(y_2) = 1$, so we recolour y_1 with a colour in $L(y_1) \setminus \{a, b\}$ and recolour y with the colour b ; we then return to Case 2.

□

The analysis above can be easily adapted to give an algorithm to acyclically 1-improperly list colour any subcubic graph from lists of size at least three in time $O(|V(G)|^2)$. Indeed, our arguments, those of Boiron *et al.*, and a well-known linear algorithm for finding cut-vertices may be straightforwardly combined to yield a linear-time algorithm, but we decline to give the details here.

3. A probabilistic lower bound for $\chi_a^t(d)$. In this section, we prove Proposition 10 below, a more explicit version of Theorem 2. Our argument mirrors that of Alon *et al.* but uses upper bounds on the t -dependence number α^t , the size of a largest t -dependent set, in the random graph $G_{n,p}$. For more precise upper bounds on $\alpha^t(G_{n,p})$, consult [10].

LEMMA 8. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n/n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n/p^4 \rfloor$. Then asymptotically almost surely and uniformly over p in the above range, any colouring of $G_{n,p}$ with $k \leq (n-m)/4$ colours and in which each colour class contains at most m vertices contains an alternating 4-cycle.*

Proof. As there are at most $k^n \leq n^n$ possible k -colourings of $G_{n,p}$, to prove the lemma it suffices to show that for any fixed k -colouring of the vertices of $G_{n,p}$ (which we denote $\{v_1, \dots, v_n\}$) with colour classes C_1, \dots, C_k in which $|C_i| \leq m$ for all $1 \leq i \leq k$, the probability that $G_{n,p}$ does not contain an alternating 4-cycle is $o(n^{-n})$.

Fix a colouring as above, and let q be minimal such that $|C_1 \cup \dots \cup C_q| \geq (n-m)/2$. Let $A = C_1 \cup \dots \cup C_q$ and let $B = C_{q+1} \cup \dots \cup C_k$. As no colour class has size greater than m , $|A| \leq (n+m)/2$ and so $|B| \geq (n-m)/2$. By symmetry, we may also assume that $|A| \geq n/2$.

Next, let $P = \{\{x_1, x'_1\}, \dots, \{x_r, x'_r\}\}$ be a maximal collection of pairs of elements of A such that for $1 \leq i \leq r$, x_i and x'_i have the same colour, and for $1 \leq i < j \leq r$, $\{x_i, x'_i\}$ and $\{x_j, x'_j\}$ are disjoint. As we may place all but perhaps one vertex from each colour class C_i in some such pair (with one vertex left over precisely if $|C_i|$ is odd), it follows that

$$r \geq \frac{1}{2} (|A| - q) \geq \frac{1}{2} \left(\frac{n}{2} - k \right) \geq \frac{n}{8}.$$

Similarly, let $Q = \{\{y_1, y'_1\}, \dots, \{y_s, y'_s\}\}$ be a maximal collection of pairs of elements of B satisfying identical conditions; by an identical argument to that above, it follows that $s \geq (n-m)/8$.

Let E be the event that for all $1 \leq i \leq r$, $1 \leq j \leq s$, $\{x_i, y_j, x'_i, y'_j\}$ is not an alternating 4-cycle, and let E' be the event that $G_{n,p}$ contains no alternating 4-cycle; clearly $E' \subseteq E$. For fixed $1 \leq i \leq r$ and $1 \leq j \leq s$, the probability that $\{x_i, y_j, x'_i, y'_j\}$ is not an alternating 4-cycle is $(1-p^4)$ and this event is independent from all other such events. As $(n-m) \geq 128 \ln n/p^4$ it follows that

$$\begin{aligned} \Pr(E') &\leq \Pr(E) \leq (1-p^4)^{rs} \leq e^{-p^4 rs} \\ &\leq \exp \left\{ -\frac{p^4 n(n-m)}{64} \right\} \leq e^{-2n \ln n} = o(n^{-n}), \end{aligned}$$

as required. \square

Using this lemma, we next bound the acyclic t -improper chromatic number of $G_{n,p}$ for p in the range allowed in Lemma 8.

LEMMA 9. *Fix an integer $n \geq 1$ and $p \in \mathbb{R}$ with $4(\ln n/n)^{1/4} \leq p \leq 1$. Let $m = \lfloor n - 128 \ln n/p^4 \rfloor$ and let $t(n, p) = p(m-1) - 2\sqrt{np}$. Then asymptotically almost surely, for all integers $t \leq t(n, p)$, $\chi_a^t(G_{n,p}) \geq 32 \ln n/p^4$, uniformly over p and t in the above ranges.*

Proof. Fix n and p as above, and choose $t \leq t(n, p)$. We will show that asymptotically almost surely $G_{n,p}$ contains no t -dependent set of size greater than m , from which the claim follows immediately by applying Lemma 8 as $(n-m)/4 \geq 32 \ln n/p^4$. Let $G[m]$ represent the subgraph of $G_{n,p}$ induced by $\{v_1, \dots, v_m\}$. By a union bound and symmetry, we have

$$\Pr(\alpha^t(G_{n,p}) \geq m) \leq \binom{n}{m} \Pr(\Delta(G[m]) \leq t) \leq 2^m \Pr(\Delta(G[m]) \leq t).$$

Since, if $\Delta(G[m]) \leq t$ then $G[m]$ has at most $tm/2$ edges, it follows that

$$\begin{aligned} \Pr(\alpha^t(G_{n,p}) \geq m) &\leq 2^n \Pr\left(E(G[m]) \leq \frac{tm}{2}\right) \\ &\leq 2^n \Pr\left(E(G[m]) - p\binom{m}{2} \leq \frac{tm}{2} - p\binom{m}{2}\right) \end{aligned}$$

Finally, by a Chernoff bound and by the definition of $t(n, p)$, we conclude that

$$\begin{aligned} \Pr(\alpha^t(G_{n,p}) \geq m) &\leq 2^n \exp\left\{-\left(\frac{tm}{2} - p\binom{m}{2}\right)^2 \cdot \left(2p\binom{m}{2}\right)^{-1}\right\} \\ &\leq 2^n \exp\left\{-\frac{(t - p(m-1))^2}{4p}\right\} \leq (2/e)^n = o(1), \end{aligned}$$

as claimed. \square

Using Lemma 9, it is a straightforward calculation to bound $\chi_a^t(d)$ for d sufficiently large and t sufficiently far from d .

PROPOSITION 10. *For all sufficiently large integers d and all non-negative integers $t \leq d - 10\sqrt{d \ln d}$,*

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$

Proof. Choose n so that

$$2^{13}n^3 \ln n \leq d^3(d-t) \leq 2^{14}n^3 \ln n; \quad (3.1)$$

such a choice of n clearly exists as long as d is large enough. Let $p = (d - 4\sqrt{d \ln d})/n$; we first check that p and t satisfy the requirements of Lemma 9. Presuming d is large enough that $np \geq d/2$, by the lower bound in (3.1) and the fact that $d(d-t) \leq d^2$ we have

$$p \geq \frac{d}{2n} \geq \frac{(d^3(d-t))^{1/4}}{2n} \geq \frac{8n^{3/4}(\ln n)^{1/4}}{2n} = 4\left(\frac{\ln n}{n}\right)^{1/4}. \quad (3.2)$$

Furthermore, letting $m = \lfloor n - 128 \ln n/p^4 \rfloor$, we have

$$\begin{aligned} p(m-1) - 2\sqrt{np} &\geq np - \frac{128 \ln n}{p^3} - 2\sqrt{np} - 2 = d - 4\sqrt{d \ln d} - 2\sqrt{np} - 2 - \frac{128 \ln n}{p^3} \\ &\geq d - 8\sqrt{d \ln d} - \frac{128 \ln n}{p^3}. \end{aligned} \quad (3.3)$$

Since $p > d/2n$ and by the lower bound in (3.1),

$$\frac{128 \ln n}{p^3} < \frac{2^{10}n^3 \ln n}{d^3} \leq \frac{d-t}{8},$$

which combined with (3.3) yields

$$\begin{aligned} p(m-1) - 2\sqrt{np} &> d - 8\sqrt{d \ln d} - \frac{(d-t)}{8} \\ &= t + \frac{7(d-t)}{8} - 8\sqrt{d \ln d} > t, \end{aligned} \quad (3.4)$$

the last inequality holding since $t \leq d - 10\sqrt{d \ln d}$. As (3.2) and (3.4) hold we may apply Lemma 9 to bound $\chi_a^t(G_{n,p})$ with this choice of t and p ; as $n > d$, it follows that as long as d is sufficiently large,

$$\Pr \left(\chi_a^t(G_{n,p}) \geq \frac{32 \ln n}{p^4} \right) \geq \frac{3}{4}, \quad (3.5)$$

say. Furthermore, by a union bound and a Chernoff bound,

$$\begin{aligned} \Pr(\Delta(G_{n,p}) > d) &\leq n \Pr \left(\text{BIN} \left(n, \frac{d - 4\sqrt{d \ln d}}{n} \right) > d \right) \\ &\leq n e^{-16 \ln d / 3} \leq \frac{1}{n}, \end{aligned} \quad (3.6)$$

the last inequality holding as $\ln d \geq \ln n / 2$ (which is an easy consequence of (3.1)). Combining (3.5) and (3.6), we obtain that

$$\Pr \left(\chi_a^t(G_{n,p}) \geq \frac{32 \ln n}{p^4}, \Delta(G_{n,p}) \leq d \right) \geq \frac{3}{4} - \frac{1}{n} \geq \frac{1}{2}$$

as long as $n \geq 4$, so there is some graph G with maximum degree at most d and with $\chi_a^t(G) \geq 32 \ln n / p^4$. Since χ_a^t is monotonically increasing in d , it follows that

$$\chi_a^t(d) \geq \frac{32 \ln n}{p^4} > \frac{32 n^4 \ln n}{d^4}. \quad (3.7)$$

An easy calculation using the upper bound in (3.1) and the fact that $\ln n < 2 \ln d$ gives the bound

$$d^4 < \frac{2^{19} n^4 (\ln d)^{4/3}}{(d-t)^{4/3}},$$

so $32 n^4 \ln n / d^4 > (d-t)^{4/3} / 2^{14} (\ln d)^{1/3}$. By (3.7), it follows that

$$\chi_a^t(d) \geq \frac{(d-t)^{4/3}}{2^{14} (\ln d)^{1/3}},$$

as claimed. \square

4. A probabilistic upper bound for $\chi_a^t(d)$. In this section, we study the situation when t is even closer to d , when $d-t = o(d^{1/2})$ in particular. Theorem 3 is a corollary of our main result here.

We analyse a different parameter from, but one that is closely related to, the acyclic t -improper chromatic number. A *star colouring* of G is a colouring such that no path of length three (i.e. with four vertices) is alternating; in other words, each bipartite subgraph consisting of the edges between two colour classes is a disjoint union of stars. The *star chromatic number* $\chi_s(G)$ is the least number of colours needed in a proper star colouring of G . We analogously define the parameters $\chi_s^t(G)$ and $\chi_s^t(d)$ in the natural way. The star chromatic number was one of the main motivations for the original study of acyclic colouring [9]. Clearly, any star colouring is acyclic; thus, $\chi_a^t(d) \leq \chi_s^t(d)$. Fertin, Raspaud and Reed [7] showed that $\chi_s(d) = O(d^{3/2})$ and that $\chi_s(d) = \Omega(d^{3/2} / (\ln d)^{1/2})$. We note that a natural adaptation to star colouring of the argument given in the last section gives the following:

THEOREM 11. *There exists a fixed constant $C > 0$ such that, if $t \leq d - C\sqrt{d \ln d}$, then $\chi_s^t(d) = \Omega((d-t)^{3/2}/(\ln d)^{1/2})$.*

Given a graph G of maximum degree d , the idea behind our method for improved upper bounds is to find a dominating set \mathcal{D} and a function $g = g(d) = o(d^{3/2})$ such that $|(N(v) \cup N^2(v)) \cap \mathcal{D}| \leq g$ for all $v \in V(G)$. Given such a set \mathcal{D} in G , we assign colours to the vertices in \mathcal{D} by greedily colouring \mathcal{D} in the square of G (i.e. vertices in \mathcal{D} at distance at most two in G receive different colours) with at most $g + 1$ colours; then we give the vertices of $G \setminus \mathcal{D}$ the colour $g + 2$. It can be verified that this colouring prevents any alternating paths of length three (and so prevents alternating cycles) and ensures that every vertex has at least one neighbour of a different colour. Furthermore, we can generalise this idea by prescribing that our set \mathcal{D} is k -dominating — each vertex outside of \mathcal{D} has at least k neighbours in \mathcal{D} — to give a bound on $\chi_s^{d-k}(d)$.

THEOREM 12. $\chi_s^t(d) = O(d \ln d + (d-t)d)$.

This result provides an asymptotically better upper bound than $\chi_s^t(d) = O(d^{3/2})$ when $d-t = o(d^{1/2})$. It also provides a better bound than $\chi_a^t(d) = O(d^{4/3})$ when $d-t = o(d^{1/3})$. Theorem 12 is an easy consequence of the following lemma:

LEMMA 13. *Given a d -regular graph G and an integer $k \geq 1$, let $\psi(G, k)$ be the least integer $k' \geq k$ such that there exists a k -dominating set \mathcal{D} for which, for all $v \in V(G)$, $|N(v) \cap \mathcal{D}| \leq k'$. Let $\psi(d, k)$ be the maximum over all d -regular graphs G of $\psi(G, k)$. Then, for all d sufficiently large, $\psi(d, k) \leq \max\{3k, 31 \ln d\}$.*

We postpone the proof of this lemma, first using it to prove Theorem 12:

Proof of Theorem 12. We first remark that the function χ_s^t is monotonic with respect to graph inclusion in the following sense: if G and G' are graphs with $V(G) = V(G')$, $\Delta(G) = \Delta(G')$ and $E(G) \subset E(G')$, then $\chi_s^t(G) \leq \chi_s^t(G')$. As any graph G of maximum degree d is contained in a d -regular graph, to prove that $\chi_s^t(d) = O(d \ln d + (d-t)d)$ it therefore suffices to show that $\chi_s^t(G) = O(d \ln d + (d-t)d)$ for d -regular graphs G . We hereafter assume G is d -regular and d is large enough to apply Lemma 13. Let $k = d - t$. We will show that $\chi_s^t(G) \leq d\psi(d, k) + 2$, which proves the theorem.

By Lemma 13, there is a k -dominating set \mathcal{D} such that $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for all $v \in V(G)$. Fix such a dominating set \mathcal{D} and form the auxiliary graph H as follows: let H have vertex set \mathcal{D} and let uv be an edge of H precisely if u and v have graph distance at most two in G . As $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for all $v \in V(G)$, H has maximum degree at most $d\psi(d, k)$.

To colour G , we first greedily colour H using at most $d\psi(d, k) + 1$ colours, and assign each vertex v of \mathcal{D} the colour it received in H . We next choose a new colour not used on the vertices of \mathcal{D} , and assign this colour to all vertices of $V(G) \setminus \mathcal{D}$. We remind the reader that $\text{im}(v)$ denotes the number of neighbours of v of the same colour as v . If $v \in \mathcal{D}$ then $\text{im}(v) = 0$, and if $v \in V \setminus \mathcal{D}$ then $\text{im}(v) \leq d - |N(v) \cap \mathcal{D}| \leq d - k = t$, so the resulting colouring is t -improper.

Furthermore, given any path $P = v_1v_2v_3v_4$ of length three in G , either two consecutive vertices v_i, v_{i+1} of P are not in \mathcal{D} (in which case $c(v_i) = c(v_{i+1})$ and P is not alternating), or two vertices v_i, v_{i+2} are in \mathcal{D} (in which case $c(v_i) \neq c(v_{i+2})$ and P is not alternating). Thus, the above colouring is a star colouring G of impropriety at most t and using at most $d(3k + 31 \ln d) + 2$ colours; as G was an arbitrary d -regular graph, it follows that $\chi_s^t(d) \leq d\psi(d, k) + 2$, as claimed. \square

We next prove Lemma 13 with the aid of the following symmetric version of the Lovász Local Lemma:

LEMMA 14 ([5], cf. [11], page 40). *Let \mathcal{A} be a set of bad events such that for each $A \in \mathcal{A}$*

1. $\Pr(A) \leq p < 1$, and
2. *A is mutually independent of a set of all but at most δ of the other events.*

If $4p\delta \leq 1$, then with positive probability, none of the events in \mathcal{A} occur.

Proof of Lemma 13. We may clearly assume that k is at least $(31/3) \ln d$, since, if the claim of the lemma holds for such k , then it also holds for smaller k . Let $p = 2k/d$ and let \mathcal{D} be a random set obtained by independently choosing each vertex v with probability p . We claim that, with positive probability, \mathcal{D} is a k -dominating set such that $|N(v) \cap \mathcal{D}| \leq 3k$ for all $v \in V(G)$; we will prove our claim using the local lemma.

For $v \in V(G)$, let A_v be the event that either $|N(v) \cap \mathcal{D}| < k$ or $|N(v) \cap \mathcal{D}| > 3k$. By the mutual independence principle, cf. [11], page 41, A_v is mutually independent of all but at most d^2 events A_w (with $w \neq v$). Furthermore, since $|N(v) \cap \mathcal{D}|$ has a binomial distribution with parameters d and p , we have by a Chernoff bound that

$$\Pr(A_v) = \Pr(|N(v) \cap \mathcal{D}| - \mathbf{E}(|N(v) \cap \mathcal{D}|) > k) \leq 2e^{-k/5} = o(d^{-2})$$

so $4\Pr(A_v)d^2 < 1$ for d large enough. By applying Lemma 14 with $\mathcal{A} = \{A_v \mid v \in V\}$, it follows that with positive probability none of the events A_v occur, i.e. \mathcal{D} has the desired properties. \square

5. A deterministic lower bound for $\chi_a^{d-1}(d)$. In this section, we concentrate on the case $t = d - 1$ and exhibit an example G_n which gives the asymptotic lower bound of Theorem 4. Given a positive integer n , we construct the graph G_n as follows: G_n has vertex set $\{v_{ij} : i, j \in \{1, \dots, n\}\} \cup \{w_{ij} : i, j \in \{1, \dots, n\}\}$. For $i, j \in \{1, \dots, n\}$ we let $\mathcal{V}_{ij} = \{v_{ij}, w_{ij}\}$. We can think of the set of vertices as an n -by- n matrix, each entry of which has been “doubled”. Within each column $\mathcal{C}_i = \bigcup_{j=1}^n \mathcal{V}_{ij}$ and within each row $\mathcal{R}_j = \bigcup_{i=1}^n \mathcal{V}_{ij}$ we add all possible edges. The graph G_n has $2n^2$ vertices and is regular with degree $d = 4n - 3$. We will prove the following proposition, which directly implies Theorem 4:

PROPOSITION 15. $\chi_a^{d-1}(G_n) \geq \frac{n}{n^{1/3}+1} + 1$.

Proof. Let $f : G_n \rightarrow \{1, \dots, k\}$ be an acyclic $(d - 1)$ -improper colouring of G_n ; we will show that necessarily $k \geq \frac{n}{n^{1/3}+1}$. Since $n \geq 1$ it follows that $n/2 \geq \frac{n}{n^{1/3}+1}$ and thus we may assume that $k < n/2$. Clearly, some colour – say a_1 – appears on two vertices $x, x' \in \mathcal{C}_1$. We call the colour a_1 “black” and refer to vertices receiving colour a_1 as black vertices. If $y, y' \in \mathcal{C}_1$ both receive colour $i \neq a_1$, then $xyx'y'$ forms an alternating cycle, so a_1 is the only colour appearing twice in \mathcal{C}_1 . It follows that at most $k - 1$ vertices in \mathcal{C}_1 are not black.

Applying the same logic to any column \mathcal{C}_i , we see that all but $k - 1$ vertices in \mathcal{C}_i must receive the same colour, say a_i . Since $k < n/2$, it is easily seen, then, that there must be a row \mathcal{R}_k such that v_{k1} and w_{k1} are both black, and v_{ki} and w_{ki} are both coloured a_i . This implies that $a_i = a_1$, since otherwise $v_{k1}v_{ki}w_{k1}w_{kj}$ would be an alternating cycle. It follows that in all columns, at most $k - 1$ vertices receive a colour other than a_1 . Symmetrically, there is a colour b such that in all rows, at most $k - 1$ vertices receive a colour other than b ; clearly, it must be the case that $b = a_1$.

If there are $i, j \in \{1, \dots, n\}$ such that both \mathcal{R}_i and \mathcal{C}_j are entirely coloured black, then all the neighbours of v_{ij}, w_{ij} are coloured with a_1 and the colouring is not $(d - 1)$ -improper; therefore, it must be the case that either all rows, or all columns, contain a non-black vertex. Without loss of generality, we may assume that all rows contain a non-black vertex.

Let x_1, \dots, x_r be non-black vertices receiving the same colour, say a , and let $x_i \in \mathcal{V}_{\ell_i, m_i}$, for $1 \leq i \leq r$. As previously noted, no two of x_1, \dots, x_r may lie in the same row or column; i.e., for $i \neq j$, $\ell_i \neq \ell_j$ and $m_i \neq m_j$.

CLAIM 1. *At least $3\binom{r}{2}$ vertices of $\bigcup_{1 \leq i < j \leq r} \mathcal{V}_{\ell_i, m_j}$ receive a non-black colour other than a .*

Proof. No vertices in $\bigcup_{1 \leq i < j \leq r} \mathcal{V}_{\ell_i, m_j}$ receive colour a as each such vertex is in the same row as one of x_1, \dots, x_r . On the other hand, for each pair i, j with $1 \leq i < j \leq r$, at least three of the vertices in $\mathcal{V}_{\ell_i, m_j} \cup \mathcal{V}_{\ell_j, m_i}$ must receive a colour other than a_1 . For if $y, y' \in \mathcal{V}_{\ell_i, m_j} \cup \mathcal{V}_{\ell_j, m_i}$ both receive colour a_1 , then $x_i y x_j y'$ forms an alternating cycle. The result follows as there are $\binom{r}{2}$ pairs i, j with $1 \leq i < j \leq r$. \square

CLAIM 2. *At least r distinct non-black colours appear on $\bigcup_{1 \leq i < j \leq r} \mathcal{V}_{\ell_i, m_j}$.*

Proof. By an argument just as above, each of $\mathcal{V}_{\ell_1, m_2}, \dots, \mathcal{V}_{\ell_1, m_r}$ must contain a vertex receiving a colour other than a_1 or a . These colours must all be distinct as $\mathcal{V}_{\ell_1, m_2}, \dots, \mathcal{V}_{\ell_1, m_r}$ are all contained within \mathcal{R}_{ℓ_1} . \square

Let $\{a_2, a_3, \dots, a_k\}$ be the set of non-black colours. Let $x_1^2, \dots, x_{r_2}^2$ be the vertices receiving colour a_2 , and for $i = 3, \dots, k$ let $x_1^i, \dots, x_{r_i}^i$ be the vertices receiving colour a_i which are in a different row from all vertices in $\bigcup_{j < i} \bigcup_{s \leq r_j} x_s^j$. As every row contains a non-black vertex, $\sum_{i=2}^k r_i = n$; it is possible that $r_i = 0$ for certain i , if there is a vertex coloured with one of a_2, \dots, a_i in every row.

For $i \in \{2, \dots, k\}$ and $s \in \{1, \dots, r_i\}$, say vertex $x_s^i \in \mathcal{V}_{\ell_s^i, m_s^i}$, and let

$$A_i = \bigcup_{1 \leq s < t \leq r_i} \mathcal{V}_{\ell_s^i, m_t^i} \cup \mathcal{V}_{\ell_t^i, m_s^i}.$$

By Claim 1, at least $3\binom{r_i}{2}$ vertices of A_i are non-black. Furthermore, if $i \neq i'$ then for any $s \in \{1, \dots, r_i\}$, $s' \in \{1, \dots, r_{i'}\}$, x_s^i and $x_{s'}^{i'}$ are in different rows – so A_i and $A_{i'}$ are disjoint. It follows that in $\bigcup_{i=2}^k A_i \cup \{x_1^i, \dots, x_{r_i}^i\}$, at least

$$\sum_{i=2}^k \left(3\binom{r_i}{2} + r_i \right) \geq \sum_{i=2}^k r_i^2 \quad (5.1)$$

vertices are non-black. As $\sum_{i=2}^k r_i = n$, it is easily seen that

$$\sum_{i=2}^k r_i^2 \geq (k-1) \left(\left\lfloor \frac{n}{k-1} \right\rfloor \right)^2.$$

As there are only $k-1$ non-black colours, it follows that some non-black colour – say a_2 – appears at least $(\lfloor n/(k-1) \rfloor)^2$ times. If $(\lfloor n/(k-1) \rfloor)^2 \geq n^{2/3}$, then by Claim 2, at least $n^{2/3} + 1 > \frac{n}{n^{1/3}+1} + 1$ colours appear on G_n , so we may assume that $n^{2/3} > (\lfloor n/(k-1) \rfloor)^2 \geq (n/(k-1) - 1)^2$. But then $k > \frac{n}{n^{1/3}+1} + 1$, as claimed. \square

Since $d = 4n - 3$, the above proposition yields $\chi_a^{d-1}(G_n) \geq (1 + o(1))2^{-4/3}d^{2/3}$. It is worth noting that the correct asymptotic order of $\chi_a^{d-1}(G_n)$ is unknown; it is even conceivable that $\chi_a^{d-1}(G_n) = \Theta(d)$. For improper star colouring, a construction and accompanying argument that are similar to the above gives $\chi_s^{d-1}(d) \geq (1 + o(1))2^{-1/6}d^{2/3}$.

6. Conclusion. In this paper, we studied the problem of acyclically t -improperly colouring graphs with maximum degree at most d . We first considered the list colouring variant of the problem for $d = 3$ and showed that every subcubic graph is acyclically 1-improperly 3-choosable. This strengthens a result of Boiron *et al.* A natural

question is to consider the case of $d = 4$; however, it seems unlikely that the method used in this paper easily extends. In light of [12], it might be fruitful to study the relationship between acyclic t -improper colourings and maximum average degree.

We next considered the behaviour of the acyclic t -improper chromatic number in the case of large d . We showed that the same asymptotic lower bound for ordinary acyclic chromatic number by Alon *et al.* could also be established for the acyclic t -improper chromatic number for any $t = t(d)$ satisfying $d - t = \Theta(d)$. We remark that, in this case, the upper bound $\chi_a(d) \leq cd^{4/3}$ can be easily adapted to list colouring, i.e. $\text{ch}_a(d) \leq c'd^{4/3}$ for some absolute constant c' . This means that, for $d - t = \Theta(d)$, Theorem 2 is asymptotically tight up to a factor of $(\ln d)^{1/3}$, even for list colouring.

Lastly, we studied the case of large d and t very close to d . For this case, we showed Theorem 12 using the Lovász Local Lemma. This theorem improves upon upper bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$ implied by the results of Alon *et al.* and Fertin *et al.*, respectively, giving for instance that $\chi_s^t(d) = O(d \ln d)$ for $d - t = O(\ln d)$. On the other hand, we showed that $\chi_a^{d-1}(d) = \Omega(d^{2/3})$ by a deterministic construction.

$d - t$	$\chi_a^t(d)$		$\chi_s^t(d)$	
	lower	upper	lower	upper
$\Theta(d)$	$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$	$O(d^{4/3})$	$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$	$O(d^{3/2})$
$\omega(\sqrt{d \ln d})$	$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$		$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$	
$O(d^{1/2})$	$\Omega(d^{2/3})$	$O((d-t)d)$	$\Omega(d^{2/3})$	$O((d-t)d)$
$O(d^{1/3})$		$O(d \ln d)$		$O(d \ln d)$
$O(\ln d)$				
0	1	1	1	1

TABLE 1
Asymptotic bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$.

There is much remaining work in the case $d - t = o(d)$. Table 1 is a rough summary of the current bounds on $\chi_a^t(d)$ and $\chi_s^t(d)$ when d is large. Of particular interest, it is unknown if $\chi_a^{d-1}(d)$ is $\Theta(d^{2/3})$, $\Theta(d \ln d)$ or somewhere strictly between these extremes.

CONJECTURE 16. $\chi_a^{d-1}(d) = \Theta(d)$ and $\chi_s^{d-1}(d) = \Theta(d)$.

Another line of enquiry would be to consider the list colouring analogue of this problem. For instance, the first question one might consider is whether $\text{ch}_a^d(d)$ is closer to $\Theta(d)$ or $\Theta(d^{4/3})$. To our knowledge, there has been no progress on this question to date.

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