Frugal, acyclic and star colourings of graphs

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Abstract

Given a graph $G = (V, E)$, a vertex colouring of V is t-*frugal* if no colour appears more than t times in any neighbourhood and is *acyclic* if each of the bipartite graphs consisting of the edges between any two colour classes is acyclic. For graphs of bounded maximum degree, Hind, Molloy and Reed [14] studied proper t-frugal colourings and Yuster [22] studied acyclic proper 2-frugal colourings. In this paper, we expand and generalise this study.

Keywords: graph colouring, frugal colouring, acyclic colouring, star colouring, improper colouring, graphs of bounded maximum degree

1. Introduction

In this paper, a *(vertex) colouring* of a graph $G = (V, E)$ is any map $f : V \to \mathbb{N}$. The *colour classes* of a colouring f are the preimages $f^{-1}(i) := \{v \in V : f(v) = i\}$. Recall that a colouring of a graph is *proper* if adjacent vertices receive distinct colours. In this article, we will consider colourings that are not necessarily proper, but satisfy another condition.

For $t \geq 1$, a colouring of G is t-*frugal* if no colour appears more than t times in any neighbourhood. The notion of frugal colouring was introduced in 1997 by Hind, Molloy and Reed [14]. They investigated proper t-frugal colourings as a way to improve bounds related to the Total Colouring Conjecture (cf. [15]). Note that a proper 1-frugal colouring of G is equivalent to a proper colouring of the *square* G^2 of G, i.e. the graph formed from G by adding the edges between any two vertices at distance two from each other. A brief account of optimal proper 1-frugal colourings for bounded degree graphs is given in [5]. We also remark that 1-frugal colourings were studied by Hahn *et al.* [13] (and they refer to such colourings as *injective*); in particular, they characterised the extremal examples for 1-frugal colouring of bounded degree graphs. In Section 3, we consider the asymptotic difference between optimal t-frugal and proper t-frugal colourings for various choices of t.

In Section 4, we make a similar comparison while imposing an additional condition that is well-studied in the graph colouring literature. A colouring of V is *acyclic* if each of the bipartite graphs consisting of the edges between any two colour classes is acyclic. In other words, a colouring of G is acyclic if G contains no *alternating cycle* (that is, an even cycle that alternates between two distinct colours). See, e.g. [12, 3, 8, 4], for more on acyclic proper colouring. We also consider an even stronger condition. A *star colouring* of G is a colouring such that no path of length three (i.e. with four vertices) is alternating; in other words, each bipartite subgraph consisting of the edges between two colour classes is a disjoint union of stars. Clearly, every star colouring is acyclic. With respect to graphs of bounded maximum degree, the study of acyclic proper colourings was instigated by Erdős (cf. [3]) and settled asymptotically by Alon, McDiarmid and Reed [4]; proper star colourings were studied by Fertin, Raspaud and Reed [11]. Extending the work of Alon *et al.*, Yuster [22] investigated acyclic proper 2-frugal colourings (but called them linear colourings, since

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the edges between any pair of colour classes induce a disjoint union of paths). In Section 4, we expand this study to different values of t and colourings that are not necessarily proper.

Before we describe our results, let us outline the notation used throughout the paper, as well as some straightforward observations. As usual, the *chromatic number* $\chi(G)$ denotes the least number of colours needed in a proper colouring. The *acyclic chromatic number* $\chi_a(G)$ denotes the least number of colours needed in an acyclic proper colouring, and the *star chromatic number* $\chi_s(G)$ of a graph G is the least number of colours needed in a proper star colouring. For $t \geq 1$, the superscript t will denote that we require colourings to be t-frugal. That is, $\chi^t(G)$ will denote the least number of colours used in a proper, t-frugal colouring. Similarly, $\chi_a^t(d)$ denotes the least number of colours in a proper, acyclic, t-frugal colouring and $\chi_s^t(d)$ denotes the least number of colours in a proper, star, t-frugal colouring.

Often we will drop the constraint that the colourings be proper. In this case we will use the notation φ instead of χ. The *t-frugal chromatic number* $\varphi^t(G)$ denotes the least number of colours needed in a t-frugal colouring that is not necessarily proper. We define the *acyclic t-frugal chromatic number* $\varphi_a^t(G)$ and the *t-frugal star chromatic number* $\varphi_s^t(G)$ analogously.

We will be interested in studying these parameters for graphs of bounded degree. To this end, let $\chi(d)$ denote the maximum possible value of $\chi(G)$ over all graphs G whose maximum degree $\Delta(G)$ is d. We analogously define $\chi_a(d)$, $\varphi^t(d)$, $\chi^t(d)$, $\varphi_a^t(d)$, $\chi_a^t(d)$, $\varphi_s^t(d)$ and $\chi_s^t(d)$.

We frequently use the monotonicity of these parameters with respect to d: $\chi(d-1) \leq \chi(d)$, $\chi_a(d-1) \leq \chi_a(d)$, $\varphi^t(d-1) \leq \varphi^t(d)$, and so on. The parameters are also monotone with respect to t: $\varphi^{t+1}(G) \leq \varphi^t(G)$, $\chi^{t+1}(G) \leq \chi^t(G)$, and so on. The proofs for the next proposition are left to the reader.

Proposition 1.1. *For any graph G and any $t \geq 1$, the following hold:*
- (i) $\chi^1(G) = \chi_a^1(G) = \chi_s^1(G) = \chi(G^2)$;
- (ii) $\varphi^t(G) \leq \chi^t(G)$, $\varphi_a^t(G) \leq \chi_a^t(G)$, $\varphi_s^t(G) \leq \chi_s^t(G)$;
- (iii) $\varphi^t(G) \leq \varphi_a^t(G) \leq \varphi_s^t(G)$, $\chi^t(G) \leq \chi_a^t(G) \leq \chi_s^t(G)$;
- (iv) $\varphi^t(G) \geq \Delta(G)/t$.

We are now prepared to highlight some of our findings in fuller detail. In Section 3, we study the asymptotic behaviour of $\varphi^t(d)$ (for t-frugal colourings) and compare it with that of $\chi^t(d)$ (for proper t-frugal colourings). We find that these quantities are roughly of the same order when t grows reasonably slowly as a function of d, but that they differ substantially for more quickly growing choices of t.

Theorem 1.2. *Let $t = t(d) \geq 1$ be any sequence of positive integers. As $d \to \infty$, the following hold for $\varphi^t(d)$ and $\chi^t(d)$:*
- (i) *if $t = o(\ln d/ \ln \ln d)$, then both $\varphi^t(d)$ and $\chi^t(d)$ are $\Theta(d^{1+1/t}/t)$;*
- (ii) *if $t = \omega(\ln d)$, then $\varphi^t(d) = (1 + o(1))d/t$ while $\chi^t(d) = d + 1$ for d sufficiently large.*

Part (i) follows from Corollary 3.4 and Theorem 3.2. Part (ii) follows from Theorem 3.7 and Theorem 3.1. We determine that, asymptotically, $\varphi^t(d)$ and $\chi^t(d)$ diverge when $t = \Theta(\ln d/ \ln \ln d)$, but our characterisation of $\varphi^t(d)$ is not tight in the range between $t = \Theta(\ln d/ \ln \ln d)$ and $t = \Theta(\ln d)$ (cf. Theorem 3.6).

In Section 4, we consider $\chi_a^t(d)$ (for acyclic proper t-frugal colourings) and $\chi_s^t(d)$ (for proper t-frugal star colourings), to flesh out the next theorem.

Theorem 1.3. *As $d \to \infty$, the following hold for $\chi_a^t(d)$ and $\chi_s^t(d)$:*
- (i) *$\chi_a^1(d)$ and $\chi_s^1(d)$ are both $\Theta(d^2)$;*
- (ii) *$\chi_a^2(d)$ and $\chi_s^2(d)$ are both $\Theta(d^{3/2})$;*
- (iii) *for any $3 \leq t \leq d$, we have $\chi_a^t(d) = \tilde{O}(d^{4/3})$, whereas $\chi_s^t(d) = \tilde{O}(d^{3/2})$.*

This theorem follows from Proposition 1.1(i), Corollary 3.4, Theorem 4.4 and Theorem 4.5.

We also in Section 4 consider $\varphi_a^t(d)$ (for acyclic t-frugal colourings) and $\varphi_s^t(d)$ (for t-frugal star colourings), and our result, a corollary of Theorem 4.8 below, extends previous work on acyclic t-improper colouring (a notion that we define later).

Theorem 1.4. *$\varphi_s^t(d) = O(d \ln d + (d - t)d)$.*

For a compact overview of this work, the reader may consult Tables 1, 3 and 4. Some open problems are mentioned at the end of the paper.

2. Probabilistic and asymptotic preliminaries

We make use of two standard versions of the Lovász Local Lemma [9].

Symmetric Lovász Local Lemma ([9], cf. [19], page 40)**.** *Let $\mathcal{E}$ be a set of (typically bad) events such that for each $A \in \mathcal{E}$*

(i) $\Pr(A) \le p < 1$, *and*
(ii) *A is mutually independent of a set of all but at most δ of the other events.*

If $ep(\delta+1) < 1$, then with positive probability none of the events in $\mathcal{E}$ occur.

General Lovász Local Lemma ([9], cf. [19], page 222)**.** *Let $\mathcal{E}$ be a set $\{A_1, \ldots, A_n\}$ of (typically bad) events such that for each A_i there exists a set $\mathcal{D}_i \subseteq \mathcal{E}$ such that A_i is mutually independent of all A_j not in $\mathcal{D}_i$. If there are real weights $0 \le x_i < 1$ such that for all i*

$$\Pr(A_i) \le x_i \prod_{A_j \in \mathcal{D}_i} (1 - x_j),$$

then with positive probability none of the events in $\mathcal{E}$ occur.

We will also use the following bound on the upper tail of the binomial distribution $\mathrm{BIN}(n,p)$.

A Chernoff Bound (cf. Equation (2.5) of [17])**.** *If $t \ge 0$, then*

$$\Pr(\mathrm{BIN}(n,p) \ge np + t) \le \exp\left(-\frac{t^2}{2(np + t/3)}\right).$$

3. Frugal colourings

Notice that $\chi^t(K_{d+1}) = d+1$, implying $\chi^t(d) \ge d+1$. As a tool to improve bounds for total colouring (cf. [15]), Hind *et al.* [14], showed for sufficiently large d that $\chi^{(\ln d)^5}(d) = d+1$. Recently, this was improved as follows.

Theorem 3.1 (Molloy and Reed [20, 21])**.** *For sufficiently large d,*

$$\chi^{50 \ln d/\ln\ln d}(d) = d+1.$$

Since $\chi^t(K_{d+1}) \ge d+1$, it follows that $\chi^t(d) = d+1$ for $t = t(d) \ge 50 \ln d/\ln\ln d$. For smaller frugalities, Hind *et al.* [14] also showed the following.

Theorem 3.2 (Hind *et al.* [14])**.** *For any $t \ge 1$ and sufficiently large d,*

$$\chi^t(d) \le \max\left\{(t+1)d, \left\lceil e^3 \frac{d^{1+1/t}}{t} \right\rceil\right\}.$$

Note that $\chi^1(d) \sim d^2$ by Proposition 1.1(i). By Proposition 1.1(iii), the following example due to Alon (cf. [14]) shows that Theorem 3.2 is asymptotically correct up to a constant multiple when $t = o(\ln d/\ln\ln d)$.

Proposition 3.3. *For any $t \ge 1$ and any prime power n,*

$$\varphi^t(n^t + \cdots + 1) \ge \frac{n^{t+1} + \cdots + 1}{t}.$$

Proof. Set $d = n^t + \cdots + 1$ and $m = n^{t+1} + \cdots + 1$. Let P be a $(t+2)$-dimensional projective geometry with m points (see for instance [18]). We form a bipartite graph G with parts A and B, where A is the set of points in P and B is the set of $(t+1)$-flats (hyperplanes), and an edge between two vertices $a \in A, b \in B$ if the point a lies in the hyperplane b.

Every hyperplane contains exactly d points in P and every point is in exactly d hyperplanes. So G has maximum degree d. Since every set of $t+1$ points lies in a $(t+1)$-flat, no colour can appear more than t times on A in any t-frugal colouring of G (*whether proper or not*); thus, at least m/t colours are required. $\square$

Corollary 3.4. *Suppose that $t = t(d) \geq 1$ and $t = o(\ln d/\ln\ln d)$. Then*

$$\varphi^t(d) \geq (1+o(1))\frac{d^{1+1/t}}{t}$$

for sufficiently large d.

Proof. Let x solve $d = x^{t(d)} + \cdots + 1$ where d is chosen large enough to satisfy certain inequalities specified below. Set $m = x^{t(d)+1} + \cdots + 1$. Note that $x \to \infty$ as $d \to \infty$, since $t = o(\ln d)$. It follows that $d = (1+o(1))x^{t(d)}$ and $x = (1+o(1))d^{1/t(d)}$.

Due to a classical result of Ingham [16] on the gaps between primes, there is a prime n between $x - Cx^{5/8}$ and x, for some absolute constant C. Let $d' = n^{t(d)} + \cdots + 1$ and $m' = n^{t(d)+1} + \cdots + 1$. We have, using Proposition 3.3,

$$\varphi^{t(d)}(d) \geq \varphi^{t(d)}(d') \geq \frac{m'}{t(d)} \geq \left(1 - \frac{C}{x^{3/8}}\right)^{t(d)+1} \frac{m}{t(d)}. \tag{1}$$

Since $x = (1+o(1))d^{1/t(d)}$ and $t(d) = o(\ln d/\ln\ln d)$, we have

$$\left(1 - \frac{C}{x^{3/8}}\right)^{t(d)+1} \geq 1 - \frac{C(t(d)+1)}{x^{3/8}} \geq 1 - \frac{2C(t(d)+1)}{d^{0.375/t(d)}} = 1 + o(1).$$

Also, using $d = (1+o(1))x^{t(d)}$ and $x = (1+o(1))d^{1/t(d)}$, we have that

$$m = (x^{t(d)+2} - 1)/(x-1) = (1+o(1))x^{t(d)+1} = (1+o(1))d^{1+1/t(d)}.$$

Combining these last two inequalities with Inequality (1), we obtain that

$$\varphi^{t(d)}(d) \geq (1+o(1))\frac{d^{1+1/t(d)}}{t(d)},$$

as claimed. $\square$

Theorems 3.1 and 3.2 and Corollary 3.4 determine the behaviour of $\chi^t(d)$ up to a constant multiple for all t except for some range of $t = \Theta(\ln d/\ln\ln d)$.

Next, we provide some upper bounds on $\varphi^t(d)$ which show that this parameter is asymptotically lower than $\chi^t(d)$, if t grows quickly enough as a function of d.

Theorem 3.5. *For positive integers t and d with $t \leq d$, let γ be such that*

$$\gamma > \left(\frac{e(t+1)}{\sqrt{2\pi t}e^{1/(12t+1)}}\right)^{1/t} e.$$

Then

$$\varphi^t(d) \leq \left\lceil \gamma \frac{d^{1+1/t}}{t} \right\rceil.$$

Proof. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \lceil \gamma d^{1+1/t}/t \rceil$. Let $f : V \to \{1, \dots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \dots, x\}$.

For vertices $v_1, \dots, v_{t+1}$ with $\{v_1, \dots, v_{t+1}\} \subseteq N(v)$ for some v, let $A_{\{v_1,\dots,v_{t+1}\}}$ be the event that $f(v_1) = \dots = f(v_{t+1})$. If none of these events hold, then f is t-frugal.

Clearly, $\Pr\left(A_{\{v_1,\dots,v_{t+1}\}}\right) = 1/x^t$. Furthermore, each vertex participates in at most $d\binom{d-1}{t}$ of these events; thus, each event is independent of all but at most $(t+1)d\binom{d-1}{t}$ other events. We have that

$$\begin{aligned} &e \Pr\left(A_{\{v_1,\dots,v_{t+1}\}}\right)\left((t+1)d\binom{d-1}{t} + 1\right) \\ &\qquad < e\frac{t^t}{\gamma^t d^{t+1}}(t+1)\frac{d^{t+1}}{t!} = \frac{e(t+1)t^t}{\gamma^t t!}. \end{aligned}$$

By a precise form of Stirling's formula (cf. (1.4) of [7]),

$$t! \geq (t/e)^t \sqrt{2\pi t} e^{1/(12t+1)};$$

therefore,

$$\frac{e(t+1)t^t}{\gamma^t t!} \leq \frac{e(t+1)}{\sqrt{2\pi t} e^{1/(12t+1)}} \left(\frac{e}{\gamma}\right)^t.$$

It follows that $e \Pr\left(A_{\{v,v_1,\dots,v_{t+1}\}}\right)\left((t+1)d\binom{d-1}{t} + 1\right) < 1$; thus, by the Symmetric Lovász Local Lemma, f is a t-frugal colouring with positive probability. $\square$

To get a more concrete feeling of this bound, note that, for example,

- when $t = 2$, we need $\gamma > \left(\frac{3}{\sqrt{4\pi}} e^{24/25}\right)^{1/2} e \approx 1.487e$,
- when $t = 3$, we need $\gamma > \left(\frac{4}{\sqrt{6\pi}} e^{36/37}\right)^{1/3} e \approx 1.346e$,
- when $t = 4$, we need $\gamma > \left(\frac{5}{\sqrt{8\pi}} e^{48/49}\right)^{1/4} e \approx 1.277e$, and
- when $t = 1000$, we need $\gamma > \left(\frac{1001}{\sqrt{2000\pi}} e^{12000/12001}\right)^{1/1000} e \approx 1.004e$.

If $t \to \infty$ as $d \to \infty$, then we can choose $\gamma = e + \epsilon$ for any fixed ϵ. By Corollary 3.4, this result is asymptotically correct up to a constant multiple when $t = o(\ln d / \ln \ln d)$. When $t \to \infty$ as $d \to \infty$ and $t = o(\ln d / \ln \ln d)$, this result is asymptotically correct up to a multiplicative factor of e.

We now provide some better bounds on $\varphi^t(d)$ when t grows faster than $\ln d / \ln \ln d$. Recall from the Proposition 1.1(iv) that $\varphi^t(d) \geq d/t$. The following result follows from a routine modification of the proof of Theorem 3.5 and so we omit the proof.

Theorem 3.6. *For any fixed $\epsilon > 0$, if $\epsilon t > 2 \ln d / \ln \ln d$, then*

$$\varphi^t(d) \leq \left\lceil \frac{d}{t^{1-\epsilon}} \right\rceil.$$

From this result we can conclude that asymptotically $\varphi^t(d)$ becomes smaller than $\chi^t(d)$ at around $t = \Theta(\ln d / \ln \ln d)$, since necessarily $\chi^t(d) = \Omega(d)$. Now, for the case $t = \omega(\ln d)$, we give an essentially optimal upper bound for $\varphi^t(d)$.

Theorem 3.7. *Suppose $t = \omega(\ln d)$. Then $\varphi^t(d) = (1 + o(1))\frac{d}{t}$.*

Proof. Let $\epsilon > 0$ be arbitrary (but fixed). Let G be any graph with maximum degree d and let $x = \lceil (1+\epsilon)d/t \rceil$. Let $f : V \to \{1, \dots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \dots, x\}$.

For a vertex v and a colour $i \in \{1, \dots, x\}$, let $A_{v,i}$ be the event that v has more than t neighbours with colour i. If none of these events hold, then f is t-frugal. Each event is independent of all but at most $d^2 x \ll d^3$ other events.

By a Chernoff bound, we have that

$$\begin{aligned}\Pr(A_{v,i}) &= \Pr(\mathrm{BIN}(d, 1/x) > t) \le \Pr(\mathrm{BIN}(d, 1/x) > d/x + ct) \\ &\le \exp\left(-c^2 t^2 / (2d/x + 2ct/3)\right)\end{aligned}$$

where $c = \epsilon/(1+\epsilon)$. Thus, $e \Pr(A_{v,i})(d^3+1) = \exp(-\Omega(t))d^3 < 1$ for large enough d, and by the Symmetric Lovász Local Lemma, f is t-frugal with positive probability for large enough d. $\square$

Table 1 gives a rough overview of the behaviour we have outlined in this section.

Table 1: Bounds for $\varphi^t(d)$ and $\chi^t(d)$.

<table>
<tr><th rowspan="2" colspan="2">t</th><th colspan="2">$\varphi^t(d)$</th><th colspan="2">$\chi^t(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td colspan="2">$O\left(\frac{\ln d}{\ln\ln d}\right)$</td><td>$\Omega\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$O\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$\Omega\left(\frac{d^{1+1/t}}{t}\right)$</td><td>$O\left(\frac{d^{1+1/t}}{t}\right)$</td></tr>
<tr><td rowspan="2">$\omega\left(\frac{\ln d}{\ln\ln d}\right)$</td><td>$O(\ln d)$</td><td rowspan="2">$\left\lceil \frac{d}{t} \right\rceil$</td><td>$\left\lceil \frac{d}{t^{1-\epsilon}} \right\rceil$</td><td rowspan="2">$d+1$</td><td rowspan="2">$d+1$</td></tr>
<tr><td></td><td>$\left\lceil (1+\epsilon)\frac{d}{t} \right\rceil$</td></tr>
</table>

4. Acyclic frugal colourings

Alon *et al.* [4] tackled the question of the asymptotic behaviour of $\chi_a(d)$. They found a nearly optimal upper bound for the acyclic chromatic number of graphs of maximum degree d, answering a long-standing question of Erdős (cf. [3]). Using the Lovász Local Lemma, they showed the following.

Theorem 4.1 (Alon *et al.* [4])**.** $\chi_a(d) \le \lceil 50 d^{4/3} \rceil$.

Using a probabilistic construction, they showed this upper bound to be asymptotically correct up to a logarithmic multiple.

Theorem 4.2 (Alon *et al.* [4])**.** $\chi_a(d) = \Omega\left(d^{4/3}/(\ln d)^{1/3}\right)$.

Yuster [22] considered acyclic proper 2-frugal colourings of graphs and showed $\chi_a^2(d) = \Theta(d^{3/2})$. In particular, by an adaptation of Theorem 4.1, he showed the following.

Theorem 4.3 (Yuster [22])**.** $\chi_a^2(d) \le \lceil \max\{50 d^{4/3}, 10 d^{3/2}\} \rceil$.

For acyclic frugal colourings, we start by considering the smallest cases $t = 1, 2, 3$ and establish upper bounds for acyclic proper frugal colourings. Later in the section, we consider larger values of t and focus on acyclic frugal colourings that are not necessarily proper.

For $t = 1, 2, 3$, notice that Corollary 3.4 implies the bounds $\varphi_a^1(d) \geq (1-\epsilon)d^2$, $\varphi_a^2(d) \geq (1/2-\epsilon)d^{3/2}$ and $\varphi_a^3(d) \geq (1/3-\epsilon)d^{4/3}$, for fixed $\epsilon > 0$ and large enough d. By Proposition 1.1(i), it follows that $\varphi_a^1(d) \sim \chi_a^1(d) \sim d^2$.

Theorem 4.3 implies that $\varphi_a^2(d) = \Theta(d^{3/2})$ and $\chi_a^2(d) = \Theta(d^{3/2})$. We give two extensions to Theorem 4.3, one for the case $t \geq 2$ with a slightly stronger notion of acyclic colouring, and the other for the case $t \geq 3$. For the first extension, the slightly stronger notion we use is that of star colouring. It was shown by Fertin *et al.* [11] that the star chromatic number satisfies $\mathrm{im}_s(d) = O(d^{3/2})$ and $\mathrm{im}_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$.

Theorem 4.4. $\chi_s^2(d) \leq \lceil 12d^{3/2} \rceil$.

In showing this, we provide a slightly simpler proof of the fact that $\chi_a^2(d) = O(d^{3/2})$. Our proof is an extension of the proof of Theorem 8.1 in Fertin *et al.* [11]. For $t = 2$, this bound is correct up to a constant multiple since $\chi_s^2(d) \geq \varphi^2(d) \geq (1/2-\epsilon)d^{3/2}$. Furthermore, for $t \geq 3$, this bound is correct up to a logarithmic multiple since $\chi_s^t(d) \geq \mathrm{im}_s(d) = \Omega\left(d^{3/2}/(\ln d)^{1/2}\right)$. Note that $\chi_s^1(G) = \chi(G^2) \leq (\Delta(G))^2 + 1$ for any graph G, so that $\chi_s^1(d) \sim d^2$ using Corolllary 3.4.

Theorem 4.5. $\chi_a^3(d) \leq \lceil 50d^{4/3} \rceil$.

Our proof is an extension of the proof of Theorem 1.1 in Alon *et al.* [4]. For $t = 3$, this bound is correct up to a constant multiple since $\chi_a^t(d) \geq (1/3-\epsilon)d^{4/3}$. For $t \geq 4$, this bound is correct up to a logarithmic multiple since $\chi_a^t(d) \geq \chi_a(d) = \Omega\left(d^{4/3}/(\ln d)^{1/3}\right)$. This partially answers a question by Esperet, Montassier and Raspaud [10].

In our proofs of Theorems 4.4 and 4.5, we employ the General Lovász Local Lemma. Since they are fairly similar, we just provide the full details for one of them.

Proof of Theorem 4.4. Let $G = (V, E)$ be any graph with maximum degree d and let $x = \lceil 12d^{3/2} \rceil$. Let $f : V \to \{1, \ldots, x\}$ be a random colouring of the vertices of G where for each $v \in V$, $f(v)$ is chosen uniformly and independently at random from the set $\{1, \ldots, x\}$. We define three types of events, the first two of which are from Fertin *et al.* [11].

I For adjacent vertices u, v, let $A_{\{u,v\}}$ be the event that $f(u) = f(v)$.

II For a path of length three $v_1v_2v_3v_4$, let $B_{\{v_1,\ldots,v_4\}}$ be the event that $f(v_1) = f(v_3)$ and $f(v_2) = f(v_4)$.

III For vertices v_1, v_2, v_3 with $\{v_1, v_2, v_3\} \subseteq N(v)$ for some v, let $A_{\{v_1,v_2,v_3\}}$ be the event that $f(v_1) = f(v_2) = f(v_3)$.

It is clear that if none of these events occur, then f is a proper 2-frugal star colouring. Furthermore, $\Pr(A) = 1/x$ and $\Pr(B) = \Pr(C) = 1/x^2$, where A, B, C are events of Types I, II, III, respectively. Also, since G has maximum degree d, each vertex participates in at most $d \cdot \binom{d-1}{2} < d^3/2$ events of Type III. It is routine to check that in Table 2, the (i, j) entry is an upper bound on the number of nodes corresponding to events of Type j which are adjacent in the dependency graph to a node corresponding to an event of Type i.

Table 2: Upper bounds in the dependency graph for Theorem 4.4.

	I	II	III
I	$2d$	$4d^3$	d^3
II	$4d$	$8d^3$	$2d^3$
III	$3d$	$6d^3$	$3d^3/2$

We define the weight x_i of each event i to be twice its probability and we want to show that each of the following inequalities hold:

$$\frac{1}{x} \le \frac{2}{x}\left(1-\frac{2}{x}\right)^{2d}\left(1-\frac{2}{x^2}\right)^{5d^3}, \tag{2}$$

$$\frac{1}{x^2} \le \frac{2}{x^2}\left(1-\frac{2}{x}\right)^{4d}\left(1-\frac{2}{x^2}\right)^{10d^3}, \text{ and} \tag{3}$$

$$\frac{1}{x^2} \le \frac{2}{x^2}\left(1-\frac{2}{x}\right)^{3d}\left(1-\frac{2}{x^2}\right)^{15d^3/2}. \tag{4}$$

Inequalities (2), (3) and (4) correspond to events of Type I, II and III, respectively. Inequality (3) implies the other two and it is valid for sufficiently large d since

$$\left(1-\frac{2}{x}\right)^{4d}\left(1-\frac{2}{x^2}\right)^{10d^3} \ge \left(1-\frac{8}{12\sqrt{d}}\right)\left(1-\frac{20}{12^2}\right) > \frac{1}{2}.$$

Therefore, by the General Lovász Local Lemma, f is an acyclic proper 3-frugal colouring with positive probability. □

Proof outline for Theorem 4.5. Our proof is an extension of the proof of Theorem 4.1 in which we add a fifth event to ensure that the random colouring f is 3-frugal:

V For vertices v, v_1, v_2, v_3, v_4 with $\{v_1, v_2, v_3, v_4\} \subseteq N(v)$, let $E_{\{v_1,\ldots,v_4\}}$ be the event that $f(v_1) = f(v_2) = f(v_3) = f(v_4)$.

□

For acyclic proper t-frugal colourings and proper t-frugal star colourings, Theorems 4.5 and 4.4, respectively, basically give the correct behaviour for graphs of (large) bounded maximum degree, so now we would like to consider what happens when we no longer prescribe that the colourings be proper.

For lower bounds, we defer to lower bounds on a weaker parameter. For $t \ge 0$, a colouring of G is *t-improper* if every vertex v has at most t neighbours with the same colour as v. Clearly, every t-frugal colouring is t-improper. The *acyclic t-improper chromatic number* $\mathrm{im}_a^t(G)$ (respectively, the *t-improper star chromatic number* $\mathrm{im}_s^t(G)$) of a graph G is the least number of colours needed in an acyclic t-improper colouring (respectively, a t-improper star colouring). We define $\mathrm{im}_a^t(d)$ and $\mathrm{im}_s^t(d)$ similarly as before.

We now mention some lower bounds on $\mathrm{im}_a^t(d)$ and $\mathrm{im}_s^t(d)$ which are clearly also lower bounds on $\varphi_a^t(d)$ and $\varphi_s^t(d)$, respectively. First, by a probabilistic construction applying bounds on the t-dependence number of random graphs, Addario-Berry *et al.* [1] showed the following non-trivial extensions of the lower bounds for $\chi_a(d)$ and $\mathrm{im}_s(d)$ of Alon *et al.* [4] and Fertin *et al.* [11], respectively.

Theorem 4.6 (Addario-Berry *et al.* [1]). *For any $t = t(d) \le d - 10\sqrt{d \ln d}$ and sufficiently large d,*

$$\mathrm{im}_a^t(d) \ge \frac{(d-t)^{4/3}}{2^{14}(\ln d)^{1/3}}.$$

For any $t = t(d) \le d - 16\sqrt{d \ln d}$ and sufficiently large d,

$$\mathrm{im}_s^t(d) \ge \frac{(d-t)^{3/2}}{2^{12}(\ln d)^{1/2}}.$$

An important implication of this is that, even if $t = (1-\epsilon)d$ for some fixed $\epsilon > 0$, the bounds of Theorems 4.5 and 4.4 correctly describe the behaviours of $\varphi_a^t(d)$ and $\varphi_s^t(d)$, respectively, up to a logarithmic multiple.

A deterministic construction by Addario-Berry, Kang and Müller [2] based on a "doubled" grid shows that $\mathrm{im}_a^t(d)$ (and hence both $\varphi_a^t(d)$ and $\varphi_s^t(d)$) grow significantly even if t is as close to d as possible (without being equal to d, in which case all of the parameters $\mathrm{im}_a^t(d)$, $\mathrm{im}_s^t(d)$, $\varphi_a^t(d)$ and $\varphi_s^t(d)$ are trivially 1).

Theorem 4.7 (Addario-Berry, Kang and Müller [2]). $\mathrm{im}_a^{d-1}(d) = \Omega(d)$.

Observe that the lower bound for $\varphi_a^t(d)$ (respectively, $\varphi_s^t(d)$) of Theorem 4.6 is stronger than that of Theorem 4.7 when $d - t = \omega(d^{3/4}(\ln d)^{1/4})$ (respectively, $d - t = \omega(d^{2/3}(\ln d)^{1/3})$).

We can obtain an asymptotic improvement compared to Theorems 4.4 and 4.5 only when $t = t(d)$ is *very* close to d. We adapt an argument of Addario-Berry *et al.* [1]. The following theorem, for example, implies that $\varphi_a^t(d)$ is asymptotically smaller than $\chi_a^t(d)$ when $d - t(d) = o\left(d^{1/3}/(\ln d)^{1/3}\right)$.

Theorem 4.8. *For any $t = t(d) \geq 1$ and sufficiently large d,*

$$\varphi_s^t(d) \leq d \cdot \max\{3(d-t), 31 \ln d\} + 2.$$

The proof of this result relies on the following lemma which is essentially from Addario-Berry *et al.* [1]. A *total k-dominating set* $\mathcal{D}$ in a graph is a set of vertices such that each vertex has at least k neighbours in $\mathcal{D}$. Given a d-regular graph $G = (V, E)$ and $1 \leq k \leq d$, let $\psi(G, k)$ be the least integer $k \leq k' \leq d$ such that there exists a total k-dominating set $\mathcal{D}$ for which $|N(v) \cap \mathcal{D}| \leq k'$ for all $v \in V$. The quantity $\psi(G, k)$ is well-defined due to the fact that V is a total k-dominating set in G for all $k \leq d$. Let $\psi(d, k)$ be the maximum over all d-regular graphs G of $\psi(G, k)$.

Lemma 4.9. *For any $1 \leq k \leq d$ and sufficiently large d,*

$$\psi(d, k) \leq \max\{3k, 31 \ln d\}.$$

Because this is only a slight modification to an analogous lemma in Addario-Berry *et al.* [1], we omit its proof here and mention that it is an application of the Symmetric Lovász Local Lemma.

Proof of Theorem 4.8. We first remark that if G is a subgraph of G', then $\varphi_s^t(G) \leq \varphi_s^t(G')$. As any graph of maximum degree d is contained in a d-regular graph, it therefore suffices to show the theorem holds for d-regular graphs. We hereafter assume $G = (V, E)$ is d-regular and d is large enough to apply Lemma 4.9. Let $k = d - t$. We will show that $\varphi_s^t(G) \leq d\psi(d, k) + 2$, which proves the theorem.

By the definition of $\psi(d, k)$, if d is sufficiently large, there is a set $\mathcal{D}$ such that $k \leq |N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for any $v \in V$. Fix such a set $\mathcal{D}$ and form the auxiliary graph H as follows: let H have vertex set $\mathcal{D}$ and let uv be an edge of H precisely if u and v have graph distance at most two in G. As $|N(v) \cap \mathcal{D}| \leq \psi(d, k)$ for any $v \in V$, H has maximum degree at most $d\psi(d, k)$.

To colour G, we first properly colour H by using the greedy algorithm to choose colours from the set $\{1, \ldots, d\psi(d, k) + 1\}$ and then assign each vertex v of $\mathcal{D}$ the colour it received in H. We next assign colour $d\psi(d, k) + 2$ to all vertices of $V \setminus \mathcal{D}$. Since $k \leq |N(v) \cap \mathcal{D}|$ for any $v \in V$, colour $d\psi(d, k) + 2$ appears at most $d - k = t$ times in any neighbourhood. Since the vertices of H at distance two have distinct colours, each colour other than $d\psi(d, k) + 2$ appears at most once in any neighbourhood. So the resulting colouring is t-frugal.

Furthermore, given any path $P = v_1 v_2 v_3 v_4$ of length three in G, either two consecutive vertices v_i, v_{i+1} of P are not in $\mathcal{D}$ (in which case v_i and v_{i+1} have the same colour and P is not alternating), or two vertices v_i, v_{i+2} are in $\mathcal{D}$ (in which case v_i and v_{i+2} have different colours and P is not alternating). Thus, the above colouring is a t-frugal star colouring of G using at most $d\psi(d, k) + 2$ colours. $\square$

What we have demonstrated in this section is, first, that the asymptotic behaviour of the acyclic proper t-frugal chromatic number and the proper t-frugal star chromatic number can be determined up to at most a logarithmic multiple. Second, we showed that the asymptotic behaviour of the acyclic t-frugal chromatic number (respectively, t-frugal star chromatic number) of graphs of bounded maximum degree seems closely tied to that of their acyclic t-improper chromatic number (respectively, t-improper star chromatic number) as long as $t \geq 3$ (respectively, $t \geq 2$). Tables 3 and 4 give a summary of the bounds we have obtained.

Table 3: Asymptotic bounds for $\chi_a^t(d)$ and $\chi_s^t(d)$.

<table>
<tr><th rowspan="2">t</th><th colspan="2">$\chi_a^t(d)$</th><th colspan="2">$\chi_s^t(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td>1</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td></tr>
<tr><td>2</td><td>$\Omega\left(d^{3/2}\right)$</td><td>$O\left(d^{3/2}\right)$</td><td>$\Omega\left(d^{3/2}\right)$</td><td rowspan="3">$O(d^{3/2})$</td></tr>
<tr><td>3</td><td>$\Omega\left(d^{4/3}\right)$</td><td rowspan="2">$O(d^{4/3})$</td><td rowspan="2">$\Omega\left(\frac{d^{3/2}}{(\ln d)^{1/2}}\right)$</td></tr>
<tr><td>≥ 4</td><td>$\Omega\left(\frac{d^{4/3}}{(\ln d)^{1/3}}\right)$</td></tr>
</table>

Table 4: Asymptotic bounds for $\varphi_a^t(d)$ and $\varphi_s^t(d)$.

<table>
<tr><th rowspan="2">$d-t$</th><th colspan="2">$\varphi_a^t(d)$</th><th colspan="2">$\varphi_s^t(d)$</th></tr>
<tr><th>lower</th><th>upper</th><th>lower</th><th>upper</th></tr>
<tr><td>$d-1$</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td><td>$\Omega\left(d^2\right)$</td><td>$O\left(d^2\right)$</td></tr>
<tr><td>$d-2$</td><td>$\Omega\left(d^{3/2}\right)$</td><td>$O\left(d^{3/2}\right)$</td><td>$\Omega\left(d^{3/2}\right)$</td><td rowspan="4">$O(d^{3/2})$</td></tr>
<tr><td>$d-3$</td><td>$\Omega\left(d^{4/3}\right)$</td><td rowspan="4">$O(d^{4/3})$</td><td rowspan="3">$\Omega\left(\frac{(d-t)^{3/2}}{(\ln d)^{1/2}}\right)$</td></tr>
<tr><td>$\omega(d^{3/4}(\ln d)^{1/4})$</td><td>$\Omega\left(\frac{(d-t)^{4/3}}{(\ln d)^{1/3}}\right)$</td></tr>
<tr><td>$\omega(d^{2/3}(\ln d)^{1/3})$</td><td rowspan="4">$\Omega\left(d\right)$</td></tr>
<tr><td>$O(d^{1/2})$</td><td rowspan="3">$\Omega\left(d\right)$</td><td rowspan="2">$O((d-t)d)$</td></tr>
<tr><td>$O(d^{1/3})$</td><td>$O((d-t)d)$</td></tr>
<tr><td>$O(\ln d)$</td><td>$O(d \ln d)$</td><td>$O(d \ln d)$</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
</table>

5. Concluding remarks and open problems

We believe the following conjecture to be natural in light of the results we obtained in Section 3.

Conjecture 5.1. $\varphi^t(d) \sim \lceil d^{1+1/t}/t \rceil$ *for any* $t = t(d) \geq 1$.

This conjecture essentially holds for $t = \omega(\ln d)$, but, when $t = O(\ln d)$, the upper and lower bounds that we outlined are separated by at least a constant multiple.

In Section 4, by dropping the condition that the colourings be proper, we demonstrated what seems to be a close qualitative link between acyclic t-frugal and acyclic t-improper colourings for t large enough. We believe that there is a threshold for t above which the acyclic t-frugal chromatic number is asymptotically equal to the acyclic t-improper chromatic number. Indeed, we conjecture the following.

Conjecture 5.2. $\varphi_a^t(d) = \Theta(\mathrm{im}_a^t(d))$ *for any* $t = t(d) \geq 1$ *unless* $t \in \{1, 2\}$*. Analogously,* $\varphi_s^t(d) = \Theta(\mathrm{im}_s^t(d))$ *for any* $t = t(d) \geq 1$ *unless* $t = 1$.

We point out here that, in the setting of planar graphs, such an asymptotic "convergence" does not occur. The acyclic t-improper chromatic number of planar graphs is bounded by a constant (namely, 5); whereas, the t-frugal chromatic number of a planar graph G is at least $\Delta(G)/t$ and thus can be arbitrarily large. It remains interesting to determine, for fixed t,

$$\varphi^t_{\mathrm{planar}}(d) := \max\{\varphi^t(G) \mid G \text{ is planar and } \Delta(G) \leq d\}$$

and, in particular, what is the smallest constant $K \geq 1$ such that $\varphi^t_{\mathrm{planar}}(d) \leq Kd/t + o(d)$. That such a constant K exists is implied by work of Amini, Esperet and van den Heuvel [6].

Another interesting line of inquiry is to determine the threshold for t above which $\varphi^t(d)$ is asymptotically smaller than $\chi^t(d)$, or $\varphi^t_a(d)$ (respectively, $\varphi^t_s(d)$) is asymptotically smaller than $\chi^t_a(d)$ (respectively, $\chi^t_s(d)$). Theorem 3.6 implies that the answer in the former case is in the range $t = \Theta(\ln d/\ln\ln d)$. Theorems 4.6 and 4.8 suggest that t, for the latter question, is in the range such that $d - t = \Omega(d^{1/3}/(\ln d)^{1/3})$ and $d - t = o(d)$ (respectively, $d - t = \Omega(d^{1/2}/(\ln d)^{1/2})$ and $d - t = o(d)$).

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