A precise threshold for quasi-Ramsey numbers

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α, ω

A clique has all possible edges and a stable set has none.

The *clique number* ω is the number of vertices in a largest clique.

The *stability number* α is the number of vertices in a largest stable set.

$G_{n,p}$

The *binomial random graph* $G_{n,p}$, championed by Erdős and Rényi 1959/1960:

$V(G_{n,p})$: $[n] = \{1, \ldots, n\}$

$E(G_{n,p})$: each of $\binom{n}{2}$ possible edges included independently with probability $p = p(n)$

Due to its elegance and interesting properties, $G_{n,p}$ has been widely studied.

We want properties of $G_{n,p}$ to hold *asymptotically almost surely* (a.a.s.),
i.e. with probability $\to 1$ as $n \to \infty$.

$\alpha(G_{n,1/2})$, $\omega(G_{n,1/2})$

$\alpha(G_{n,1/2}) \sim 2\log_2 n$ and $\omega(G_{n,1/2}) \sim 2\log_2 n$ a.a.s.

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Here are some classic applications.

- Erdős 1947 (also Spencer 1977): The best asymptotic lower bound on diagonal Ramsey numbers to date, $R(k,k) \geq \Omega(k2^{k/2})$ as $k \to \infty$. (Conlon 2009: $R(k,k) \leq 2^{2k-\Omega(\log^2 k/\log\log k)}$ as $k \to \infty$.)

  $R(k,k)$ is the least n such that $\forall G, |V(G)| = n$: $\alpha(G) \geq k$ or $\omega(G) \geq k$.

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- Erdős 1959: Construction of graphs of high girth and chromatic number.
- Erdős and Fajtlowicz 1981: Short disproof of Hajós's conjecture.

A sharp two-point formula is known: for every $\varepsilon > 0$,

$$\alpha(G_{n,1/2}) = \left\lfloor 2\log_2 n - 2\log_2\left(\frac{2\log_2 n}{e}\right) + 1 \pm \varepsilon \right\rfloor \text{ a.a.s.}$$

Matula 1972 (cf. Bollobás and Erdős 1976).

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Let $t \geq 0$ parameterise how close a set must be to perfectly dense or sparse, in terms of minimum or maximum degree.

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Let $t \geq 0$ parameterise how close a set must be to perfectly dense or sparse, in terms of minimum or maximum degree.

t-clique has min degree $\geq k - 1 - t$; t-stable set has max degree $\leq t$.

ω^t is the number of vertices in a largest t-clique.

α^t is the number of vertices in a largest t-stable set.

(Note that $t = 0$ is clique or stable set, while $t = k - 1$ is anything.)

α^t

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For all G (not random),

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$$\chi^t(G) \le \frac{\Delta(G)+1}{t+1} \qquad (\text{due to Lovász 1966})$$

$$\frac{|V(G)|}{\alpha^t(G)} \le \chi^t(G)$$

$$\frac{(t+1)|V(G)|}{\Delta+1} \le \alpha^t(G) \qquad (**)$$

$\alpha^t(G_{n,1/2})$

In particular, $(*)$ and $(**)$ imply a.a.s.

$$\alpha^t(G_{n,1/2}) \geq \alpha(G_{n,1/2}) \sim 2\log_2 n,$$
$$\alpha^t(G_{n,1/2}) \geq \frac{(t+1)n}{\Delta(G_{n,1/2})+1} \sim 2t,$$

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but could $\alpha^t(G_{n,1/2})$ be much bigger?

Proposition (Kang and McDiarmid 2007/2010)

- *If $t = o(\log n)$, then $\alpha^t(G_{n,1/2}) \sim 2\log_2 n$ a.a.s.*
- *If $t = \omega(\log n)$ and $t = o(n)$, then $\alpha^t(G_{n,1/2}) \sim 2t$ a.a.s.*

(And this extends with $2\log_b(np)$ and t/p nearly down to $p = \Theta(1/n)$.)

$t = \Theta(\log n)$

What happens at the transition $t = \Theta(\log n)$?

Theorem (Kang and McDiarmid 2010)

There is a function $\kappa = \kappa(\tau)$, continuous and strictly increasing for $\tau \in [0, \infty)$, with $\kappa(0) = 2/\log 2$ and $\kappa(\tau) \sim 2\tau$ as $\tau \to \infty$, such that, if $t \sim \tau \log n$, then

$$\alpha^t(G_{n,1/2}) \sim \kappa(\tau) \log n \text{ a.a.s.}$$

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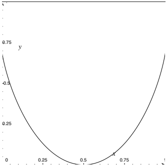
κ is defined using the Fenchel-Legendre transform of logarithmic moment generating function of Bernoulli(1/2),

$$\Lambda^*(x) = \begin{cases} x \log(2x) + (1-x)\log(2(1-x)) & x \in [0,1] \\ \infty & \text{otherwise} \end{cases},$$



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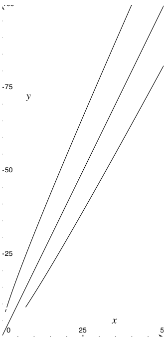
κ is based on the following being around 1:

$$\binom{n}{k} \exp\left(-\binom{k}{2}\Lambda^*\left(\frac{t}{k-1}\right)\right),$$

where $k = \kappa \log n$ and $t = \tau \log n$.



Back to Ramsey numbers†

Define the *homogeneous number* $h := \max\{\alpha, \omega\}$.

The bounds on $R(k, k)$ due to Erdős and Szekeres 1935, Erdős 1947 show

- $h(G) \geq \frac{1}{2} \log_2 |V(G)|$ for all G and
- $h(G) \leq 2 \log_2 |V(G)|$ for some G with $|V(G)|$ large enough.

† Picture borrowed from the cover of Soifer 2009.

Quasi-Ramsey problem

Define the *t-homogeneous number* $h^t := \max\{\alpha^t, \omega^t\}$.

Observe that

- $h^t(G) \geq \frac{1}{2}\log_2 |V(G)|$ for all G for all $t \geq 0$ and
- $h^0(G) \leq 2\log_2 |V(G)|$ for some G with $|V(G)|$ large enough,
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As we increase t, when could we expect a linear lower bound on h^t? Moreover, when could we expect a polynomial lower bound on h^t?

A rough threshold

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About halfway!

Proposition (Erdős and Pach 1983)

Let $t = t(k) = c(k-1)$ for some fixed $0 \leq c \leq 1$.

- $h^t(G) = O(\log_2 |V(G)|)$ *for some G with $|V(G)|$ large enough if $c < 1/2$.*
- $h^t(G) = \Omega(|V(G)|)$ *for all G if $c > 1/2$.*

Erdős and Pach also obtained a polynomial lower bound at *precisely* $c = 1/2$.

A rough threshold

What is an intuition for the threshold being around halfway?

One can try to extend the Erdős 1947 probabilistic construction, using the sharp estimates on $\alpha^t(G_{n,1/2}) \sim \omega^t(G_{n,1/2})$, hence on $h^t(G_{n,1/2})$.

$\kappa(\tau)$ from earlier is always greater than 2τ and $\kappa(\tau) \to 2\tau$ as $\tau \to \infty$.

So $h^t(G_{n,1/2}) < \kappa_n \log n$ a.a.s. if $\kappa_n \geq (2+\varepsilon)\tau_n$ and τ_n is large enough wrt ε, however this is *not* true for $\kappa_n \leq 2\tau_n$.

We suspect that any improvement of this bound in this regime would yield a corresponding improvement for the $t = 0$ case!

A precise threshold

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Let us first see where $h^t(G_{n,1/2})$ (and large deviations) leads us.

Proposition (Kang, Pach, Patel and Regts 2014+)

Let $t = t(k) = \frac{1}{2}(k-1) - \nu\sqrt{(k-1)\log k}$ for some fixed $\nu \geq 0$.

- $h^t(G) = O\left(|V(G)|^{\frac{1}{\nu^2+1}}\right)$ *for some G with $|V(G)|$ large enough.*

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Notes:

- This bound is useless when $\nu = 0$.
- For any $\nu = \nu(k) \to \infty$ as $k \to \infty$, there are graphs with sub-polynomial t-homogeneous numbers.

A precise threshold

Theorem (Kang, Pach, Patel and Regts 2014+)

Let $t = t(k) = \frac{1}{2}(k-1) - \nu\sqrt{(k-1)\log k}$ *for some fixed* $\nu \geq 0$*.*

- $h^t(G) = O\left(|V(G)|^{\frac{1}{\nu^2+1}}\right)$ *for some* G *with* $|V(G)|$ *large enough.*
- $h^t(G) = \Omega\left(\frac{|V(G)|^{\frac{1}{C\nu^2+1}}}{\log|V(G)|}\right)$.

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Notes:

- The bound is $\Omega\left(\frac{|V(G)|}{\log|V(G)|}\right)$ when $\nu = 0$: the logarithmic term is needed.
- If $\nu = o(1)$ as $k \to \infty$, then G has nearly linear t-homogeneous sets.

Graph discrepancy

Proof relies on an extremal result for edge count in a set of bounded order.

Recall that, given G, the *discrepancy* of a set $X \subseteq V(G)$ is

$$D(X) = |E(G[X])| - \frac{1}{2}\binom{|X|}{2}.$$

Lemma (Erdős and Spencer 1974, monograph)

For n large enough, if $\ell \in \{1, \ldots, n\}$, then any graph G, $|V(G)| = n$, has

$$\max_{S \subseteq V(G), |S| \leq \ell} |D(S)| \geq \frac{\ell^{3/2}}{1000}\sqrt{\log \frac{5n}{\ell}}.$$

Sketch proof

Theorem (Kang, Pach, Patel, Regts 2014+)

Fix $\nu \geq 0$, $c > 4/3$. For large enough j and any G with $|V(G)| \geq j^{c10^6\nu^2+4/3}$, we have $h^t(G) \geq j$ for $t(k) = \frac{1}{2}(k-1) - \nu\sqrt{(k-1)\log k}$.

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Sketch proof.

Define a skew form of discrepancy. For any $X \subseteq V(G)$,

$$D_\nu(X) = |D(X)| - \nu\sqrt{|X|^3 \log |X|}.$$

Taking X with $D_\nu(X)$ maximum, assuming wlog $D(X) > 0$, we can easily derive

$$\deg(x) \geq \tfrac{1}{2}(|X|-1) + \nu\sqrt{|X| \ln |X|} \text{ for any } x \in X.$$

Applying discrepancy lemma with $\ell = j^{4/3}$, we get a set Y with $D(Y) \geq \nu j^2\sqrt{c \log j}$. Consider the skew term of $D_\nu(Y)$: it is $-\nu j^2\sqrt{4/3 \log j}$ and so $\ll D(Y)$ as $j \to \infty$.

Thus $D_\nu(X) \geq j^2$, from which we conclude $|X| \geq j$. □

An open problem

Problem (Erdős and Pach 1983)

Determine $R^*_{1/2}(k,k)$, defined as

$$\min\left\{n \,:\, |V(G)| = n \implies G \text{ has } \left(\tfrac{1}{2}(k-1)\right)\text{-homogenous } k\text{-set}\right\}.$$

They showed

$$R^*_{1/2}(k,k) = \Omega\left(\frac{k\log k}{\log\log k}\right) \text{ and } R^*_{1/2}(k,k) = O(k^2).$$

Thank you!

And to Tobias:

祝你生日快乐