On distance edge-colourings and matchings

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Problem definition

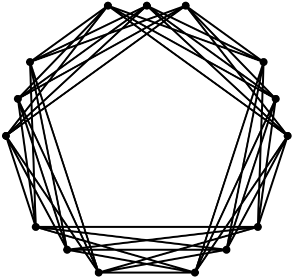
Let $G = (V, E)$ be a (simple) graph.

The distance between two vertices in G is the number of edges in a shortest path in G between them.

The distance between two edges in G is the number of *vertices* in a shortest path between them. Incident edges have distance 1.

Problem definition

A *distance-t matching* of G is a set of edges no two of which are within distance t in G.



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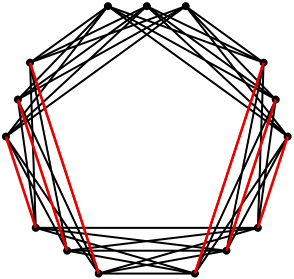


A *distance-t edge-colouring* is an edge-colouring of G such that each colour class induces a distance-t matching.

Problem definition

The *distance-t chromatic index* $\chi'_t(G)$ of G is the least integer k such that there exists a distance-t edge-colouring of G using k colours.

The *distance-t matching number* $\mu_t(G)$ of G is the largest integer k such that there exists a distance-t matching in G with k edges.

A crucial link: $\chi'_t(G) \geq |E|/\mu_t(G)$.

Problem definition

Remarks:

- For $t = 1$, $\chi'_t(G)$ is the chromatic index $\chi'(G)$ of G and $\mu_t(G)$ is the size of a largest matching.
- For $t = 2$, $\chi'_t(G)$ is the strong chromatic index of G and $\mu_t(G)$ is the size of a largest induced matching.
- There is a close connection to $(L(G))^t$, the t^{th} power of the line graph of G — a distance-t matching in G is equivalent to an independent set in $(L(G))^t$ — and hence to work on powers of graphs, e.g. of Alon and Mohar (2002) and Atkinson and Frieze (2004).

Scope of current work

Two main settings

1. We consider $\chi'_t(G)$ for graphs G of maximum degree Δ, in particular, the asymptotics, as $\Delta \to \infty$, of $\chi'_t(\Delta) := \inf\{\chi'_t(G) : \Delta(G) \le \Delta\}$.
2. We also consider $\chi'_t(G(n,p))$ and $\mu_t(G(n,p))$ where $G(n,p)$ denotes the Erdős-Rényi random graphs — a graph formed on $\{1,\dots,n\}$ by including each of the possible $\binom{n}{2}$ edges independently at random with probability p.

(Recall that a property holds *asymptotically almost surely* (a.a.s.) if it holds with probability tending to 1 as $n \to \infty$.)

Background and motivation

$t = 1.$

Vizing's Theorem: $\chi'_1(G) \in \{\Delta, \Delta + 1\}$.

$\implies \chi'_1(G(n, p)) = (1 + o(1))np$ a.a.s. (when p large enough)

The maximum matching number may be computed by, for example, Edmond's algorithm, in polynomial time.

Background and motivation

$t = 2$.

Erdős-Nešetřil proposed the problem of determining $\chi'_2(\Delta)$ in 1985. They presented the example of a multiplied 5-cycle, which provides a lower bound of $\chi'_2(\Delta) \geq 5\Delta^2/4$ for arbitrarily large Δ.

Molloy and Reed (1997) showed $\chi'_2(\Delta) \leq 1.998\Delta^2$ for large enough Δ.

A series of papers — from El Maftouhi and Gordones (1994) to Frieze, Krivelevich, Sudakov (2005) — have considered $\chi'_2(G(n,p))$, showing for example for p fixed that a.a.s.

$$\frac{nd}{2\log_b n} \leq \chi'_2(G(n,p)) \leq \frac{3nd}{4\log_b n}$$

where $d = np$ and $b = 1/(1-p)$.

Background and motivation

$t = 2$ continued.

The induced matching number has been studied intensively by Cameron and others since the 1980's.

$t > 2$.

The distance-t chromatic index was studied by Ito, Kato, Zhou, Nishizeki (2007) and the distance-t matching number was studied by Stockmeyer and Vazirani (1978).

A distance-t version of the Erdős-Nešetřil problem

The following is a trivial upper bound:

$$\chi'_t(\Delta) \le \Delta((L(G))^t) \le 2\sum_{j=1}^{t}(\Delta-1)^j + 1.$$

Note that $2\sum_{j=1}^{t}(\Delta-1)^j \sim 2\Delta^t$ as $\Delta \to \infty$.

Problem

Determine the value of $\lim_{\Delta\to\infty} \chi'_t(\Delta)/\Delta^t$, *if it exists.*

Two constructive lower bounds

Proposition

Fix an integer $t \geq 2$. For arbitrarily many Δ, there exists a regular graph with degree Δ such that $\chi'_t(G) > \Delta^t / \left(2(t-1)^{t-1}\right)$.

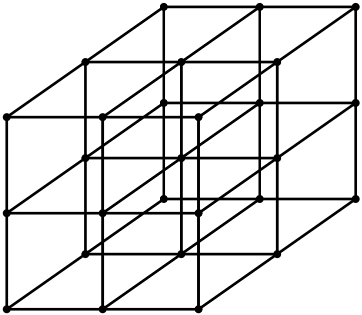
Proof.

V: $(x_1, \ldots, x_{t-1}) \in \{1, \ldots, x\}^{t-1}$ for some integer $x > 1$.
E: $(x_1, \ldots, x_{t-1}) \sim (x'_1, \ldots, x'_{t-1})$ iff they differ in one coordinate.
Any two edges are at distance at most t. □

For $t = 2$, this example is simply a clique and the resultant bound is weaker than the multiplied 5-cycle. We cannot hope for this example to be tight in general; however, it verifies for $t > 2$ that $\chi'_t(\Delta) = \Theta(\Delta^t)$.

Two constructive lower bounds

$t = 4$, $x = 3$:



Two constructive lower bounds

Proposition

For arbitrarily many Δ, there exists a bipartite, regular graph with degree Δ such that $\chi'_3(G) = \Delta^3 - \Delta^2 + 2\Delta$.

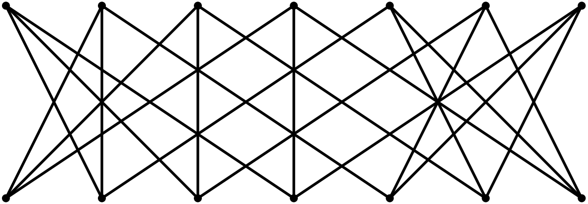
Proof.

Let P be the projective plane with $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Let G be the point-line incidence graph for P. Then any two edges are at distance at most 3 and $|E| = (q+1)(q^2 + q + 1)$. □

This beats the general example as well.

Two constructive lower bounds

$q = 2$:



An upper bound on $\mu_t(G(n,p))$

Theorem

Let $\varepsilon > 0$ *and* $p = d/n$ *with* $d \geq d_0$ *for some large fixed* d_0*. Then a.a.s.*

$$\mu_t\left(G(n,p)\right) \leq \frac{n}{2d^{t-1}}\left(t \log d - \log\log d - \log et + \varepsilon\right).$$

The proof is inspired by work of Atkinson and Frieze and uses Janson's Inequality. The following lower bound for $t > 2$ follows:

$$\chi'_t(G(n,p)) \geq (1+o(1)) \min\left\{\frac{nd}{2}, \frac{d^t}{t\log d}\right\}.$$

An upper bound on $\chi'_t(G(n,p))$

Define

$$\Delta_t(G) = \begin{cases} \max\left\{\deg_{t/2}(e) : e \in E\right\} & \text{if } t \text{ is even} \\ \max\left\{\deg_{(t+1)/2}(v) : v \in V\right\} & \text{if } t \text{ is odd} \end{cases}.$$

Note that $\Delta_t(G)$ is the size of a largest clique in $(L(G))^t$.

Theorem

If $100np \le \left(\frac{\log n}{\log\log n}\right)^{1/t}$, *then a.a.s.* $\chi'_t(G(n,p)) = \Delta_t(G(n,p))$.

Conclusion

Forthcoming work on graphs with maximum degree at most Δ and girth at least g, continuing the work of Mahdian (2000), Alon and Mohar (2002).

Speculation

For $t > 2$, a.a.s.

$$\chi'_t(G(n,p)) = (1 + o(1)) \min \left\{ \frac{nd}{2}, \frac{d^t}{t \log d} \right\}.$$