Introduction to Quantum Field Theory

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0.1 Preface

Quantum field theory is our description of the basic forces between elementary particles. There is a close connection in its methods with condensed matter physics, classical and quantal. In a compromise with the requirement of conciseness the following approach has emerged, assuming knowledge of classical electrodynamics, special relativity, quantum mechanics and some group theory:

- The electromagnetic field is quantized canonically in the Coulomb gauge and its quanta are interpreted in terms of particles, the photons. Then the amplitude that the vacuum persists under influence of an external source (vacuum amplitude) is calculated and reexpressed in a general covariant gauge. This introduces functional techniques, propagators and the quantum version of the action functional, generally known as the effective action.

- Having seen that a quantized field gives a description of particles, the real scalar field is introduced as the simplest example. The complex scalar field is coupled to the electromagnetic field using the principle of gauge invariance and the system is canonically quantized, without going into details. Instead, the φ^4 theory is used for showing that operator field equations imply equations for the vacuum amplitude and Dyson-Schwinger equations for the effective action. Feynman diagrams provide a natural representation of various mathematical expressions. The iterative solution of the Dyson-Schwinger equations generates the loop expansion in powers of $\hbar$. We concentrate on the semiclassical approximation (no loop diagrams), in which the effective action has the form of a classical action.

- Using external sources for emission and absorption of particles, scattering amplitudes are derived in terms of correlation functions (connected Green functions). The resulting expressions also apply to bound states and are on the same footing as the LSZ (Lehmann-Symanzik-Zimmermann) formulas. Applications in scalar electrodynamics illustrate how it works.

- For the description of spin 1/2 particles spinor fields are introduced. We start here from the particles and derive the action and field equations from the vacuum amplitude. It is shown how Lorentz invariance and locality lead to Fermi-Dirac statistics, the Dirac equation and anticommuting variables. The presentation is initially in terms of hermitian spinor fields (Majorana fields). The subsequent introduction of complex fields (Dirac fields) and the coupling to the electromagnetic field follows closely the steps taken earlier for the scalar field.

- For the derivation of Feynman rules the stage has been set already by the example of scalar electrodynamics, and the presentation concentrates on putting minus signs in the appropriate places.

- The path integral is a spinoff giving a representation of the solution of the Dyson-Schwinger equations as a functional Fourier transform. This does not do justice to the path integral as an independent fundamental formulation of quantum theory, but it is quick.

A space favoured metric is used, $g_{11} = g_{22} = g_{33} = -g_{00} = 1$, with correspondig Dirac matrices. This may be compared with the convention used by the influential books of Bjorken and Drell: $g_{\mu\nu} = -(g_{\mu\nu})_{\mathrm{BD}}$, $i\gamma^\mu = (\gamma^\mu)_{\mathrm{BD}}$. The charge of the electron is $-e$, $e = |e|$.

The following books on quantum field theory are refered to in the text by name of authors:

J.D. Bjorken and S.D. Drell,
I: Relativistic Quantum Mechanics, McGraw-Hill (1964);
II: Relativistic Quantum Fields, McGraw-Hill (1965).

C. Itzykson and J.-B. Zuber, Quantum Field Theory, McGraw-Hill (1980).

B. de Wit and J. Smith, Field Theory in Particle Physics I,
North-Holland (1986).

L. Ryder, Quantum Field Theory, Cambridge University Press 1985.

L.S. Brown, Quantum Field Theory, Cambridge University Press 1992.

Furthermore mentioned are

S. Weinberg, The quantum theory of massless particles,
in Lectures on Particles and Field Theory, Pentice-Hall 1965
(Brandeis Summer Institute in Theoretical Physics).

A. Pais, Inward Bound, Of Matter and Forces in the Physical World,
Prentice-Hall 1965.

The following references are included for completeness, they are not recommended for study at an introductory level.

B.S. DeWitt, Dynamical Theory of Groups and Fields
in Relativity Groups and Topology, Les Houches 1963, and separate book by Gordon and Breach 1964.

J. Schwinger,
I: Particles, Sources and Fields I, Addison-Wesley 1970,
II: Particles, Sources and Fields II, Addison-Wesley 1973,
III: Quantum Kinematics and Dynamics, Benjamin 1970.

Chapter 1

Quantized electromagnetic field

In this chapter we will quantize the electromagnetic field by canonical methods and derive the interpretation of the quanta as particles, the photons. The classical field is recovered as an expectation value of the quantum field in suitable states in Hilbert space. Subsequently we study the amplitude for the vacuum to persist under influence of an external source, as well as amplitudes for emission and absorption of photons by the source. These amplitudes will be basic tools in our presentation. An interesting application is the radiation of an indefinite number of photons by an external source. We end with a discussion of the princple of locality in quantum field theory.

1.1 Canonical quantization

Suppose we have a system described by coordinates $q_k(t)$ and a Lagrange function $L(q(t), \dot{q}(t))$, which may also depend explicitly on time ($\dot{q}_k \equiv dq_k/dt$). A simple example is a particle at position $\mathbf{q} = (q_1, q_1, q_3)$ in a potential $V(q)$,

$$L = \frac{1}{2} m \dot{q}_k \dot{q}_k - V(q), \tag{1.1}$$

where m is the mass of the particle. We use the convention in which a summation is implied over two repeated indices (unless otherwise idicated). The action functional of the system is given by

$$S(q) = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t)). \tag{1.2}$$

Requiring the action to be stationary under variations $\delta q(t)$ leads to the equations of motion. Keeping only terms linear in δq we have

$$\delta S \equiv S(q + \delta q) - S(q) = \int_{t_1}^{t_2} dt\, \delta L(q, dq/dt)$$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} dt\,(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial(dq_k/dt)}\frac{d}{dt}\delta q_k) \\
&= \left[\frac{\partial L}{\partial(dq_k/dt)}\delta q_k\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\,(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial(dq_k/dt)})\delta q_k
\end{aligned} \tag{1.3}
$$

Requiring the $\delta S = 0$ for arbitrary δq which vanish at the boundaries, $\delta q(t_{1,2}) = 0$, thus gives the equations of motion in Lagrange form

$$
\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0. \tag{1.4}
$$

For our example these look like

$$
m\ddot{q}_k + \frac{\partial V(q)}{\partial q_k} = 0. \tag{1.5}
$$

Let us introduce at this point the notion of a fuctional derivative $\delta S/\delta q_k$. The action is a functional of $q_k(t)$, i.e. it gives a number to any point in a space of functions $q_k(t)$. The functional derivative is easiest to understand as a generalization of the partial derivative, viewing t as a continuous index. Making a variation $\delta q_k(t)$ it is defined by writing δS in the form

$$
\delta S = \int dt\,\frac{\delta S}{\delta q_k}\,\delta q_k. \tag{1.6}
$$

Hence, for the specific form (1.2) of the action,

$$
\frac{\delta S}{\delta q_k(t)} = \frac{\partial L}{\partial q_k(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k(t)}. \tag{1.7}
$$

The canonical momenta p_k are defined as

$$
p_k = \frac{\partial L}{\partial \dot{q}_k}, \tag{1.8}
$$

and the Hamilton function $H(p,q)$ is defined by a Legendre transformation from $\dot{q}_k$ to p_k,

$$
H(p,q) = p_k\dot{q}_k - L(q,\dot{q}). \tag{1.9}
$$

To be able to express the hamiltonian H in terms of the canonical coordinates and momenta we have to solve for $\dot{q}_k$, $\dot{q}_k = \dot{q}_k(q,p)$. The equations of motion can now be expressed in Hamilton form,

$$
\begin{aligned}
\frac{\partial H}{\partial q_k} &= -\frac{\partial L}{\partial q_k} + p_l\frac{\partial \dot{q}_l}{\partial q_k} - \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial q_k} \\
&= -\frac{\partial L}{\partial q_k} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = -\dot{p}_k, \qquad (1.10) \\
\frac{\partial H}{\partial p_k} &= \dot{q}_k + p_l\frac{\partial \dot{q}_l}{\partial p_k} - \frac{\partial L}{\partial \dot{q}_l}\frac{\partial \dot{q}_l}{\partial p_k} = \dot{q}_k. \qquad (1.11)
\end{aligned}
$$

In our example

$$\begin{aligned} p_k &= m\dot{q}_k, &(1.12)\\ H(p,q) &= \frac{p_k p_k}{m} - L(q,\frac{p}{m}) = \frac{p_k p_k}{2m} + V(q), &(1.13)\\ \dot{p}_k &= -\frac{\partial V}{\partial q_k}, \quad \dot{q}_k = \frac{p_k}{m}. &(1.14)\end{aligned}$$

Hamilton's equations can be rewritten in terms of Poisson brackets, defined for general $A = A(q,p)$ and $B = B(q,p)$ by

$$(A,B) = \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p_k}. \tag{1.15}$$

The canonical Poisson brackets are

$$(q_k, p_l) = \delta_{kl}, \quad (q_k, q_l) = (p_k, p_l) = 0, \tag{1.16}$$

and in bracket form the Hamilton equations read

$$\dot{p}_k = (p_k, H), \quad \dot{q}_k = (q_k, H). \tag{1.17}$$

In the canonical quantization method the quantum mechanical description of the system is based on the correspondence: commutator $[A,B] \leftrightarrow$ Poisson bracket (A,B) , such that in the formal classical limit $\hbar \to 0$:

$$[A,B]/i\hbar \to (A,B). \tag{1.18}$$

In practise the recipe for quantization amounts to assuming p_k and q_k to be operators in Hilbert space with the canonical commutation relations

$$[q_k, p_l] = i\hbar\delta_{kl}, \quad [q_k, q_l] = [p_k, p_l] = 0. \tag{1.19}$$

A familiar representation is $p_k \to -i\hbar\partial/\partial q_k$, $q_k \to q_k$, acting on wave functions $\psi(q,t)$, the coordinate representation. In the Schrödinger picture the time dependence is carried by the wave function and the canonical operators do not depend on time. In the Heisenberg picture the time dependence is carried by the operators and the wave function is time indpendent. Then the p's and q's depend on time and the canonical commutators are supposed to hold only at equal times,

$$[q_k(t), p_l(t')] = i\hbar\delta_{kl}, \quad \text{etc. at } t = t'. \tag{1.20}$$

For $t \neq t'$ the commutators may be different and follow from the Heisenberg equations of motion

$$\frac{d}{dt}p_k = [p_k, H]/i\hbar, \quad \frac{d}{dt}q_k = [q_k, H]/i\hbar. \tag{1.21}$$

Let us recall finally the special case of the harmonic oscillator, e.g.

$$V(q) = \frac{1}{2} m\omega^2 q_k q_k, \tag{1.22}$$

as such systems for which the hamiltonian is quadratic in the canonical variables play an important role in the following. The hamiltonian is diagonalized by the introduction of creation and annihilation operators, $a_k^\dagger$ and a_k,

$$a_k = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega q_k + ip_k), \quad a_k^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega q_k - ip_k), \tag{1.23}$$

with

$$[a_k, a_l^\dagger] = \delta_{kl}, \quad [a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0, \tag{1.24}$$

$$H = \hbar\omega \sum_k (a_k^\dagger a_k + \frac{1}{2}). \tag{1.25}$$

The eigenstates of H may be labeled by occupation numbers n_k $(= 0, 1, 2, \ldots)$,

$$|n_1 n_2 n_3\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \frac{(a_3^\dagger)^{n_3}}{\sqrt{n_3!}} |0\rangle, \tag{1.26}$$

$$H|n_1 n_2 n_3\rangle = \hbar\omega(n_1 + n_2 + n_3 + \frac{3}{2})|n_1 n_2 n_3\rangle, \tag{1.27}$$

where $|0\rangle = |000\rangle$ is the ground state (lowest energy state) which satisfies $a_k|0\rangle = 0$.

1.2 Action for the electromagnetic field

The action for the electromagnetic field $A_\mu(x)$ coupled to an external current $J_\mu(x)$ is given by

$$S = \int d^4x \, [-\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + J^\mu(x) A_\mu(x)], \tag{1.28}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{1.29}$$

is the electromagnetic field strength tensor and the integration is over all of space-time. An external current is a current which is not a dynamical variable, it influences the electromagnetic field but does not suffer a back reaction from the field. It is an idealization of a real current produced by particle motion. By prescribing the current as we choose we can probe the field and study some elementary dynamics without. We use Lorents-Heaviside electromagnetic units (rationalized

Gauss units), which is customary in relativistic quantum field theory, and the conventions

$$\begin{aligned}
x &= (x^1, x^2, x^3, x^0) = (\mathbf{x}, x^0), \quad x^0 = ct, && (1.30)\\
x_\mu &= g_{\mu\nu} x^\nu, \quad F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}, && (1.31)\\
g_{11} &= g_{22} = g_{33} = -g_{00} = +1, && (1.32)\\
F_{mn} &= \epsilon_{mnk} B_k, \quad F_{m0} = F^{0m} = E_m, && (1.33)\\
\partial_\mu &= \frac{\partial}{\partial x^\mu}, \quad \Box = \partial_\mu \partial^\mu, \quad \Delta = \partial_m \partial_m, && (1.34)
\end{aligned}$$

where Greek indices run from 0 to 3 and Latin indices from 1 to 3. Notice that $x^0 = -x_0$ and $x^m = x_m$, $m = 1, 2, 3$. We shall furthermore use units in which the velocity of light $c = 1$.

The equations of motion (Maxwell's equations) follow from the principle of stationary action. Under a variation δA_μ of A_μ we have

$$\begin{aligned}
\delta S &\equiv S(A + \delta A) - S(A) \\
&= \int d^4x\, \delta(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu), && (1.35)\\
\delta F_{\mu\nu} &= \partial_\mu (A_\nu + \delta A_\nu) - \partial_\nu (A_\mu + \delta A_\mu) - F_{\mu\nu} \\
&= \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu, && (1.36)\\
\delta(F_{\mu\nu} F^{\mu\nu}) &= 2F^{\mu\nu} \delta F_{\mu\nu} = 4F^{\mu\nu} \partial_\mu \delta A_\nu, && (1.37)\\
\delta S &= \int d^4x\, (-F^{\mu\nu} \partial_\mu \delta A_\nu + J^\mu \delta A_\mu) \\
&= \int d^4x\, (\partial_\mu F^{\mu\nu} + J^\nu) \delta A_\nu. && (1.38)
\end{aligned}$$

We made a partial integration in the last step and assumed that the surface term is zero, which is correct if we impose that $\delta A_\mu(x)$ vanishes outside some large but finite domain in spacetime. Requiring $\delta S = 0$ for arbitrary variations in this domain gives Maxwell's equations

$$0 = \frac{\delta S}{\delta A_\nu} = \partial_\mu F^{\mu\nu} + J^\nu = (\partial^2 g_\mu^\nu - \partial_\mu \partial^\nu) A^\mu + J^\nu. \tag{1.39}$$

We recall at this point the gauge invariance of the theory. Under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x) \tag{1.40}$$

the field strength $F_{\mu\nu}(x)$ is invariant. The term involving the external current is also invariant,

$$\begin{aligned}
\int d^4x\, J^\mu A_\mu &\to \int d^4x\, J^\mu (A_\mu + \partial_\mu \omega) = \int d^4x\, (J^\mu A_\mu - \omega \partial_\mu J^\mu) \\
&= \int d^4x\, J^\mu A_\mu, && (1.41)
\end{aligned}$$

provided the current is conserved[1]

$$\partial_\mu J^\mu = 0. \tag{1.42}$$

Note that in making the partial integration above we also assumed that ω vanishes outside a finite spacetime domain. The gauge invariance implies that the solution of the field equations is not unique. A unique solution for A_μ is obtained by imposing a gauge condition, such as the Lorentz condition

$$\partial_\mu A^\mu = 0. \tag{1.43}$$

Another frequently used condition is the radiation or Coulomb gauge, in which

$$\partial_m A^m = \nabla \cdot \mathbf{A} = 0. \tag{1.44}$$

We recall here also the energy-momentum tensor of the electromagnetic field,

$$T^{\mu\nu} = F^\mu{}_\alpha F^{\nu\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \tag{1.45}$$

which describes the energy density

$$T^{00} = F^0{}_a F^{0a} + \frac{1}{2} F_{a0} F^{a0} + \frac{1}{4} F_{ab} F^{ab} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{1.46}$$

and the momentum density

$$T^{0n} = F^0{}_a F^{na} = E_a \epsilon_{nab} B_b = (\mathbf{E} \times \mathbf{B})_n, \tag{1.47}$$

also known as the Poynting vector. The local balance equation

$$\partial_\mu T^{\mu\nu} = -F^\nu{}_\alpha J^\alpha \tag{1.48}$$

expresses the conservation of the total energy-momentum in the field

$$P^\nu = \int d^3x \, T^{0\nu}. \tag{1.49}$$

If the external current vanishes, P^ν is time independent,

$$\partial_0 P^\nu = \int d^3x \, \partial_0 T^{0\nu} = -\int d^3x \, \partial_m T^{m\nu} = 0. \tag{1.50}$$

See the text books for the derivation of the energy-momentum tensor.

The action S can be written in the form

$$S = \int d^4x \, \mathcal{L}, \tag{1.51}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu, \tag{1.52}$$

[1] The terminology: a current $j^\mu(x)$ is 'conserved', simply means: $\partial_\mu j^\mu(x) = 0$. It is of course the total charge $Q = \int d^3x \, j^0(x)$ which is conserved.

in which the Lagrange density $\mathcal{L}$ is a scalar under Lorentz transformations, provided that A_μ is a Lorentz vector (or a vector modulo a gauge transformation). This nice manifest Lorentz invariance is broken in the canonical formalism which treats the time and the time derivatives in a special way. A manifestly covariant description is possible with functional techniques and the path integral formalism which we shall introduce later. At this stage however the canonical formalism is instructive for a first exploration of the quantum properties of the electromagnetic field.

1.3 Quantization in the Coulomb gauge

We write the action in the form

$$\begin{aligned} S &= \int dt\, L, && (1.53)\\ L &= \int d^3x\,(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu) && (1.54)\\ &= \int d^3x\,(\frac{1}{2}\dot{A}_m\dot{A}_m - \dot{A}_m\partial_m A_0 \\ &\quad + \frac{1}{2}\partial_n A_m\partial_m A_n - \frac{1}{2}\partial_m A_n\partial_m A_n + \frac{1}{2}\partial_m A_0\partial_m A_0 \\ &\quad + J^0 A_0 + J^m A_m) && (1.55) \end{aligned}$$

We note the analogy with the quantum mechanics of a particle with coordinates $q_k(t)$, $k = 1, 2, 3$: the label k is analogous to (μ, x_1, x_2, x_3) in $A_\mu(x_1, x_2, x_3, t)$:

$$A_\mu(\mathbf{x}, t) \leftrightarrow q_k(t), \quad (\mu, \mathbf{x}) \leftrightarrow k. \tag{1.56}$$

Since $\mathbf{x}$ can take an infinite number of different values, the field corresponds to an infinite number of degrees of freedom. There are now several complications:

- the index $\mathbf{x}$ is continuous;
- $\dot{A}_0$ is lacking in L, so the canonical conjugate to A_0 will vanish.

The second complication is typical for gauge theories such as electromagnetism and we shall deal with it first.

Consider the equation of motion which follows from varying the action with respect to A_0,

$$0 = \delta S = \int d^4x\,(\partial_\mu F^{\mu 0} + J^0)\delta A_0, \tag{1.57}$$

which gives Gauss's law, or Coulomb's law

$$\begin{aligned} 0 = \frac{\delta S}{\delta A_0} &= \partial_m F^{m0} + J^0 = -\nabla\cdot\mathbf{E} + J^0 \\ &= -\partial_m(-\partial_m A^0 - \partial_0 A^m) + J^0. && (1.58) \end{aligned}$$

We can now use the gauge invariance of the theory and impose the Coulomb gauge condition $\partial_m A_m = 0$, which has the result that the time derivative drops out of (1.58), $\partial_m \partial_0 A_m = 0$, such that (1.58) takes the form

$$-\Delta A^0 = J^0. \tag{1.59}$$

Since this equation does not contain time derivatives it is not a dynamical equation anymore, but an equation of constraint at every instant in time. With suitable boundary conditions the potential A^0 is completely determined in terms of J^0. For infinite space

$$A^0(\mathbf{x}, t) = \int d^3y \, \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} J^0(\mathbf{y}, t), \tag{1.60}$$

where we used the fact that the Coulomb potential is a Green function for the laplacian Δ:

$$-\Delta \, \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} = \delta(\mathbf{x}-\mathbf{y}). \tag{1.61}$$

Hence, in this sense A^0 is not a dynamical variable in the Coulomb gauge. We shall use the Coulomb gauge for the canonical formalism and continue to write A^0, for simplicity, keeping in mind that it is a given function of J^0.

In the Coulomb gauge we can rewrite the lagrangian in the form

$$L = \int d^3x \, [\frac{1}{2}\dot{A}_m \dot{A}_m - \frac{1}{2} A_m(-\Delta) A_m + J_m A_m] - E_C, \tag{1.62}$$

$$E_C = \int d^3x \, (-\frac{1}{2}\partial_m A^0 \partial_m A^0 + J^0 A^0) = \int d^3x \, \frac{1}{2} J^0 A^0. \tag{1.63}$$

We used $\partial_m A_m = 0$, $\Delta A^0 = -J^0$ and made partial integrations of ∂_m assuming boundary conditions such that surface terms vanish. The quantity E_C is the Coulomb energy; using (1.60) this can be written as

$$E_C = \frac{1}{2} \int d^3x \, J^0(\mathbf{x}, t) \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} J^0(\mathbf{y}, t). \tag{1.64}$$

The lagrangian is now in the form $L(q, \dot{q})$ with $q_k(t) \to A_m(\mathbf{x}, t)$.

We now have to deal with the continuous nature of the index $\mathbf{x}$ and the constraint $\partial_m A_m(\mathbf{x}, t) = 0$. This can be done by expanding the potentials into a discrete set of basis functions $f^m_\alpha(\mathbf{x})$ satisfying $\partial_m f^m_\alpha(\mathbf{x}) = 0$. Let us enclose the system in a cubic box $-L/2 \leq x_m \leq L/2$ with periodic boundary conditions. For a large enough box its finiteness and the type of boundary conditions should not matter. Periodic boundary conditions are convenient because with it all boundary terms in partial integrations vanish (the box has no boundary) and they are natural for eigenstates of the momentum operator (cf. next section). We can then use the discrete set of eigenfunctions of the laplacian Δ to construct the

$f^m_\alpha(\mathbf{x})$. The real eigenfunctions of the laplacian correspond to products of the standing waves $\cos(k_1x_1)\cos(k_2x_2)\cos(k_3x_3)$, $\sin(k_1x_1)\cos(k_2x_2)\cos(k_3x_3)$, $\ldots$, $\sin(k_1x_1)\sin(k_2x_2)\sin(k_3x_3)$, with $k_m = 2\pi n_m/L$, $n_m = 0, 1, 2, \ldots$, and the eigenvalues are given by $-\Delta \to \omega^2 = \mathbf{k}^2$. Out of these eigenfunctions the real $f^m_\alpha(\mathbf{x})$ can be constructed satisfying $\partial_m f^m_\alpha(\mathbf{x}) = 0$. The details of this are tedious and not needed in the following and we shall just record their properties:

$$-\Delta f^m_\alpha(\mathbf{x}) = \omega^2_\alpha f^m_\alpha(\mathbf{x}), \quad \partial_m f^m_\alpha(\mathbf{x}) = 0, \tag{1.65}$$

$$\int d^3x\, f^m_\alpha(\mathbf{x})^* f^m_\beta(\mathbf{x}) = \delta_{\alpha\beta}, \tag{1.66}$$

$$\sum_\alpha f^m_\alpha(\mathbf{x}) f^n_\alpha(\mathbf{y})^* = P^T_{mn}(\mathbf{x}, \mathbf{y}). \tag{1.67}$$

We have written these equations in general complex form and in the next section we shall give an explicit set of complex basis functions, which are easier to construct. For the moment have to keep in mind that the $f^m_\alpha(\mathbf{x})$ are real. The object $P^T_{mn}(\mathbf{x}, \mathbf{y})$ is a projector on the space of 'transverse' vector functions, i.e. a projector: $P^2 = P$, or

$$\int d^3y\, P^T_{kl}(\mathbf{x}, \mathbf{y})\, P^T_{lm}(\mathbf{y}, \mathbf{z}) = P^T_{km}(\mathbf{x}, \mathbf{z}), \tag{1.68}$$

which is transverse, $\partial_m P^T_{mn}(\mathbf{x}, \mathbf{y}) = 0$. It is the identity operator for vector functions satisfying $\partial_m A_m(\mathbf{x}) = 0$,

$$\int d^3y\, P^T_{mn}(\mathbf{x}, \mathbf{y}) A_n(\mathbf{y}) = A_m(\mathbf{x}). \tag{1.69}$$

An explicit expression for P^T will be given in the next section (cf. (1.100)). In the summation $\sum_\alpha$ we exclude the 'zero mode' $\mathbf{k} = (0, 0, 0)$ (this would be automatic with Dirichlet boundary conditions). This means that we exclude here potentials A_m which are constant in space. Such potentials complicate the (otherwise interesting) mathematics and we usually do not need them in physical applications.

In terms of these basis functions we can now expand the potentials in normal modes,

$$A_m(\mathbf{x}, t) = \sum_\alpha q_\alpha(t) f^m_\alpha(\mathbf{x}), \tag{1.70}$$

$$q_\alpha(t) = \int d^3x\, f^m_\alpha(\mathbf{x}) A_m(\mathbf{x}, t), \tag{1.71}$$

and in terms of the new coordinates q_α the lagrangian takes the form, for $J^\mu = 0$,

$$L = \sum_\alpha (\frac{1}{2}\dot{q}_\alpha \dot{q}_\alpha - \frac{1}{2}\omega^2_\alpha q_\alpha q_\alpha). \tag{1.72}$$

This shows that the electromagnetic field is equivalent to an inifinite set of harmonic oscillators, with unit mass and frequencies ω_α. The canonical description is now an obvious generalization of the case of one harmonic oscillator,

$$\begin{aligned}
p_\alpha &= \partial L/\partial \dot{q}_\alpha = \dot{q}_\alpha, && (1.73)\\
H &= \sum_\alpha \left(\frac{1}{2} p_\alpha p_\alpha + \frac{1}{2}\omega_\alpha^2 q_\alpha q_\alpha\right), && (1.74)\\
(q_\alpha, p_\beta) &= \delta_{\alpha\beta}, \quad (q_\alpha, q_\beta) = (p_\alpha, p_\beta) = 0, && (1.75)\\
\dot{p}_\alpha &= (p_\alpha, H), \quad \dot{q}_\alpha = (q_\alpha, H) = p_\alpha. && (1.76)
\end{aligned}$$

Evidently the canonical conjugate to the field $A_m(\mathbf{x})$ is

$$\begin{aligned}
\Pi_m(\mathbf{x}) &= (A_m(\mathbf{x}), H) = \dot{A}_m(\mathbf{x}) \\
&= \sum_\alpha p_\alpha f_\alpha^m(\mathbf{x}). && (1.77)
\end{aligned}$$

The system is quantized by imposing canonical commutation relations between the p's and q's,

$$[q_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}, \quad [q_\alpha, q_\beta] = [p_\alpha, p_\beta] = 0. \tag{1.78}$$

1.4 Fock space

Since we have a system of harmonic oscillators it is useful to work with creation and annihilation operators

$$a_\alpha = \frac{1}{\sqrt{2\hbar\omega_\alpha}}(\omega_\alpha q_\alpha + ip_\alpha), \quad a_\alpha^\dagger = \frac{1}{\sqrt{2\hbar\omega_\alpha}}(\omega_\alpha q_\alpha - ip_\alpha). \tag{1.79}$$

The Hilbert space resulting from an infinite number of creation and annihilation operators is called Fock space. It has a no-quantum state $|0\rangle$ defined by

$$a_\alpha|0\rangle = 0, \tag{1.80}$$

and normalized basis vectors

$$|\{n_\alpha\}\rangle = \prod_\alpha \frac{(a_\alpha^\dagger)^{n_\alpha}}{\sqrt{n_\alpha!}}|0\rangle \tag{1.81}$$

where only a finite number of occupation numbers n_α are supposed to be nonzero. It is generally simpler to work with unnormalized basis vectors of the form

$$|\alpha_1 \cdots \alpha_n\rangle = a_{\alpha_1}^\dagger \cdots a_{\alpha_n}^\dagger |0\rangle, \quad n = 0, 1, 2, \ldots, \tag{1.82}$$

in terms of which the orthogonality and completeness relations read

$$\langle\alpha_1\cdots\alpha_n|\beta_1\cdots\beta_m\rangle = \delta_{nm}\sum_P \delta_{\alpha_1,\beta_{P1}}\cdots\delta_{\alpha_n,\beta_{Pn}}, \tag{1.83}$$

$$\sum_n \frac{1}{n!}\sum_{\alpha_1,\cdots,\alpha_n} |\alpha_1\cdots\alpha_n\rangle\langle\alpha_1\cdots\alpha_n| = 1. \tag{1.84}$$

Here $\sum_P$ is a summation over all permutations of the indices $1,\ldots,n$. These formulas remain valid with the appropriate modifications ($\sum\to\int$, Kronecker-$\delta\to$ Dirac-δ) in case the index α is continuous, e.g. $\alpha\to\mathbf{k},\lambda$, with $\mathbf{k}$ a momentum label and λ a spin label.

1.5 Energy-momentum eigenstates

The quantized electromagnetic field is now an operator in Hilbert space. The commutation relations between the p_α and q_α imply the following relations between A_m and Π_m,

$$[A_m(\mathbf{x}),\Pi_n(\mathbf{y})] = i\hbar P^T_{mn}(\mathbf{x},\mathbf{y}), \quad [A_m(\mathbf{x}),A_n(\mathbf{y})] = [\Pi_m(\mathbf{x}),\Pi_n(\mathbf{y})] = 0. \tag{1.85}$$

For example,

$$\begin{aligned}[A_m(\mathbf{x}),\Pi_n(\mathbf{y})] &= \sum_{\alpha\beta}[q_\alpha,p_\beta]\, f^m_\alpha(\mathbf{x})f^n_\beta(\mathbf{y})^* = i\hbar\sum_\alpha f^m_\alpha(\mathbf{x})f^n_\alpha(\mathbf{y})^* \\ &= P^T_{mn}(\mathbf{x}.\mathbf{y}).\end{aligned} \tag{1.86}$$

To guide our physical interpretation we shall use the energy momentum P^μ of the field, which is now also an operator, and determine its eigenstates and eigenvalues. In the Coulomb gauge A_0 vanishes when $J^\mu=0$, cf. (1.60). Then

$$\begin{aligned}T^{00} &= \frac{1}{2}(E_mE_m + B_mB_m) \\ &= \frac{1}{2}(\dot A_m\dot A_m + \partial_nA_m\partial_nA_m - \partial_nA_m\partial_mA_n),\end{aligned} \tag{1.87}$$

$$T^{0n} = \epsilon_{nmp}E_mB_p = -\dot A_m\partial_nA_m + \dot A_m\partial_mA_n, \tag{1.88}$$

giving

$$P^0 = \int d^3x\, T^{00} = \int d^3x\,[\frac{1}{2}\Pi_m\Pi_m + \frac{1}{2}A_m(-\Delta)A_m], \tag{1.89}$$

$$P^n = \int d^3x\, T^{0n} = \int d^3x\,(-\Pi_m\partial_nA_m), \tag{1.90}$$

where we used the Coulomb gauge condition $\partial_mA_m=0$ and $\dot A_m=\Pi_m$. Notice that there is no operator ordering ambiguity in P_m: we can also write Π_m to the right of A_m, the difference involves the derivative of the commutator, $\partial_m \delta(\mathbf{x} - \mathbf{y})|_{\mathbf{x}=\mathbf{y}} = 0$. Using the normal mode expansion we find

$$\begin{aligned} P^0 &= \sum_\alpha (\frac{1}{2} p_\alpha p_\alpha + \frac{1}{2} \omega_\alpha^2 q_\alpha q_\alpha) \\ &= H. \end{aligned} \tag{1.91}$$

The momentum operator is less easy to express in terms of the normal modes because the real mode functions $f_\alpha^m(\mathbf{x})$ are not eigenfunctions of ∂_n. Therefore we now introduce a different set $f_\alpha^m(\mathbf{x})$ which are eigenfunctions of ∂_n and Δ, and satisfy $\partial_m f_\alpha^m(\mathbf{x}) = 0$. They are complex and have the form

$$f_{\mathbf{k},\lambda}^m(\mathbf{x}) = e^m(\mathbf{k},\lambda) e^{i\mathbf{kx}}, \quad k_m = n_m 2\pi/L, \quad n_m = 0, \pm 1, \pm 2, \ldots, \tag{1.92}$$

These are clearly eigenfunctions of ∂_n and Δ. Recall that the n_m have to be integers to satisfy periodic boundary conditions in a box of size L^3. To satisfy $\partial_m f^m = 0$, the $e^m(\mathbf{k},\lambda)$ have to be orthogonal to $\mathbf{k}$ (hence the terminology 'transverse'),

$$\mathbf{k} \cdot \mathbf{e}(\mathbf{k},\lambda) = 0. \tag{1.93}$$

For example for $\mathbf{k} = (0,0,|\mathbf{k}|) = |\mathbf{k}|\hat{3}$, $\mathbf{e}(\mathbf{k},1) = (1,0,0) = \hat{1}$, $\mathbf{e}(\mathbf{k},2) = (0,1,0) = \hat{2}$, and in general $\mathbf{e}(\mathbf{k},\lambda)$ may be obtained from this by a rotation, a standard rotation that takes $(0,0,|\mathbf{k}|)$ into $\mathbf{k}$. Another set well known from classical electrodynamics consists of the right and left handed polarization vectors

$$\mathbf{e}(\mathbf{k},\pm) = \mp \frac{1}{\sqrt{2}} [\mathbf{e}(\mathbf{k},1) \pm i\mathbf{e}(\mathbf{k},2)]. \tag{1.94}$$

The polarization vectors satisfy

$$e_m(\mathbf{k},\lambda)^* e_m(\mathbf{k},\lambda') = \delta_{\lambda\lambda'}, \tag{1.95}$$

$$\sum_\lambda e_m(\mathbf{k},\lambda) e_n(\mathbf{k},\lambda)^* = \left(\delta_{mn} - \frac{k_m k_n}{\mathbf{k}^2}\right) \equiv P_{mn}^T(\mathbf{k}). \tag{1.96}$$

The basis functions are orthogonal and complete in the sense (1.67), with

$$\alpha \to (\mathbf{k},\lambda), \tag{1.97}$$

$$\delta_{\alpha\alpha'} \to \delta_{\lambda\lambda'} V \delta_{\mathbf{k},\mathbf{k}'}, \tag{1.98}$$

$$\sum_\alpha \to \frac{1}{V} \sum_{\mathbf{k},\lambda}, \tag{1.99}$$

$$P_{mn}^T(\mathbf{x},\mathbf{y}) = \frac{1}{V} \sum_{\mathbf{k}} e^{-i\mathbf{kx}+i\mathbf{ky}} \left(\delta_{mn} - \frac{k_m k_n}{\mathbf{k}^2}\right),. \tag{1.100}$$

where $V = L^3$ is the volume and the zero mode $\mathbf{k} = \mathbf{0}$ is absent again.

We now expand the A_m and Π_m in terms of these basis functions as follows,

$$
\begin{aligned}
A_m(\mathbf{x}) &= \frac{\sqrt{\hbar}}{V}\sum_{\mathbf{k},\lambda}\frac{1}{2k^0}[e^{i\mathbf{kx}}e^m(\mathbf{k},\lambda)a(\mathbf{k},\lambda)+e^{-i\mathbf{kx}}e^m(\mathbf{k},\lambda)^*a^\dagger(\mathbf{k},\lambda)], \quad (1.101)\\
\Pi_m(\mathbf{x}) &= \frac{\sqrt{\hbar}}{V}\sum_{\mathbf{k},\lambda}\frac{1}{2k^0}[-ik^0e^{i\mathbf{kx}}e^m(\mathbf{k},\lambda)a(\mathbf{k},\lambda)+ik^0e^{-i\mathbf{kx}}e^m(\mathbf{k},\lambda)^*a^\dagger(\mathbf{k},\lambda)],
\end{aligned}
$$

where

$$k^0 = |\mathbf{k}|. \tag{1.102}$$

The somewhat strange looking normalization convention involving $1/2k^0$ will prove useful in the following. The above expansions define $a(\mathbf{k},\lambda)$ and $a(\mathbf{k},\lambda)^\dagger$. The form of (1.101) is guided by the inverse of (1.79),

$$q_\alpha = \frac{\sqrt{\hbar}}{2\omega_\alpha}\sqrt{2\omega_\alpha}\,(a_\alpha + a_\alpha^\dagger), \tag{1.103}$$

$$p_\alpha = \frac{\sqrt{\hbar}}{2\omega_\alpha}\sqrt{2\omega_\alpha}\,(-i\omega_\alpha a_\alpha + i\omega_\alpha a_\alpha^\dagger), \tag{1.104}$$

and (1.70,1.77). The relations (1.101) may be inverted as follows. We write

$$a_m(\mathbf{k}) = \sum_\lambda e_m(\mathbf{k},\lambda)a(\mathbf{k},\lambda), \tag{1.105}$$

$$a(\mathbf{k},\lambda) = e_m(\mathbf{k},\lambda)^*a_m(\mathbf{k}). \tag{1.106}$$

Then

$$\int d^3x\, e^{-i\mathbf{kx}}A_m(\mathbf{x}) = \frac{\sqrt{\hbar}}{2k^0}[a_m(\mathbf{k})+a_m^\dagger(-\mathbf{k})], \tag{1.107}$$

$$\int d^3x\, e^{-i\mathbf{kx}}\Pi_m(\mathbf{x}) = \frac{\sqrt{\hbar}}{2}[-ia_m(\mathbf{k})+ia_m^\dagger(-\mathbf{k})], \tag{1.108}$$

giving

$$\sqrt{\hbar}\,a_m(\mathbf{k}) = \int d^3x\, e^{-i\mathbf{kx}}[k^0A_m(\mathbf{x})+i\Pi_m(\mathbf{x})], \tag{1.109}$$

$$\sqrt{\hbar}\,a_m^\dagger(-\mathbf{k}) = \int d^3x\, e^{-i\mathbf{kx}}[k^0A_m(\mathbf{x})-i\Pi_m(\mathbf{x})]. \tag{1.110}$$

The commutation relations between $a_m(\mathbf{k})$ and $a_m^\dagger(\mathbf{k})$ can now be calculated from (1.85) to be

$$
\begin{aligned}
[a_m(\mathbf{k}), a_n^\dagger(\mathbf{l})] &= P^T_{mn}(\mathbf{k})\,2k^0V\,\delta_{\mathbf{k},\mathbf{l}},\\
[a_m(\mathbf{k}), a_n(\mathbf{l})] &= [a_m^\dagger(\mathbf{k}), a_n^\dagger(\mathbf{l})] = 0. \quad (1.111)
\end{aligned}
$$

For example,

$$
\begin{aligned}
[a_m(\mathbf{k}), a_n^\dagger(\mathbf{l})] &= \frac{1}{\hbar} \int d^3x d^3y \, e^{-i\mathbf{kx}+i\mathbf{ly}} \, [k^0 A_m(\mathbf{x}) + i\Pi_m(\mathbf{x}), l^0 A_m(\mathbf{y}) - i\Pi_m(\mathbf{y})] \\
&= (k^0 + l^0) \int d^3x d^3y \, e^{-i\mathbf{kx}+i\mathbf{ly}} \, P^T_{mn}(\mathbf{x}, \mathbf{y}) \\
&= (k^0 + l^0) P^T_{mn}(\mathbf{l}) \int d^3y \, e^{i(\mathbf{l}-\mathbf{k})\mathbf{y}} \\
&= 2k^0 P^T_{mn}(\mathbf{k}) \, V \delta_{\mathbf{k},\mathbf{l}}. \qquad (1.112)
\end{aligned}
$$

It follows that

$$
[a(\mathbf{k}, \lambda), a^\dagger(\mathbf{k}', \lambda')] = 2k^0 V \delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'}, \qquad (1.113)
$$

$$
[a(\mathbf{k}, \lambda), a(\mathbf{l}, \lambda')] = [a^\dagger(\mathbf{k}, \lambda), a^\dagger(\mathbf{l}, \lambda')] = 0. \qquad (1.114)
$$

Hence, the new a and $a^\dagger$ satisfy the commutation relations of creation and annihilation operators of an infinite set of harmonic oscillators labeled by $(\mathbf{k}, \lambda)$.

Expressing the hamiltonian (1.89) and momentum operator (1.90) in terms of the creation and annihilation operators we find (cf. Problems)

$$
P^0 = \frac{\hbar}{V} \sum_{\mathbf{k},\lambda} \frac{1}{2k^0} a^\dagger(\mathbf{k}, \lambda) a(\mathbf{k}, \lambda) k^0 + E_0, \qquad (1.115)
$$

$$
P_m = \frac{\hbar}{V} \sum_{\mathbf{k},\lambda} \frac{1}{2k^0} a^\dagger(\mathbf{k}, \lambda) a(\mathbf{k}, \lambda) k_m, \qquad (1.116)
$$

$$
E_0 = \hbar \sum_{\mathbf{k},\lambda} \frac{1}{2} k^0. \qquad (1.117)
$$

By analogy to the ordinary harmonic oscillator we recognize the number operator $a^\dagger(\mathbf{k}, \lambda) a(\mathbf{k}, \lambda)$ for each mode $(\mathbf{k}, \lambda)$. The ground state (state with lowest energy) is the no-quantum state $|0\rangle$ defined by

$$
a(\mathbf{k}, \lambda)|0\rangle = 0, \qquad (1.118)
$$

with

$$
P^0|0\rangle = E_0|0\rangle, \quad \mathbf{P}|0\rangle = 0. \qquad (1.119)
$$

The excited states are given by

$$
|k, \lambda\rangle = a^\dagger(\mathbf{k}, \lambda)|0\rangle, \qquad (1.120)
$$

$$
|k_1\lambda_1, k_2\lambda_2\rangle = a^\dagger(\mathbf{k}_1, \lambda_1) a^\dagger(\mathbf{k}_2, \lambda_2)|0\rangle, \qquad (1.121)
$$

$$
|k_1\lambda_1, k_2\lambda_2, k_3\lambda_3\rangle = a^\dagger(\mathbf{k}_1, \lambda_1) a^\dagger(\mathbf{k}_2, \lambda_2) a^\dagger(\mathbf{k}_3, \lambda_3)|0\rangle, \qquad (1.122)
$$

etc., with

$$
[P^\mu - \delta_{\mu,0} E_0]|k_1\lambda_1 \ldots k_n\lambda_n\rangle = \hbar(k_1^\mu + \ldots + k_n^\mu)|k_1\lambda_1 \ldots k_n\lambda_n\rangle, \qquad (1.123)
$$

The four-momenta k^μ represent zero mass, $k^\mu k_\mu = 0$. The excited states are the photons. The symmetry of the basis vectors $|k_1\lambda_1 \cdots k_n\lambda_n\rangle$ under interchange of of labels $((\mathbf{k}_i\lambda_i) \leftrightarrow (\mathbf{k}_j\lambda_j)$ has the consequence that photons follow Bose-Einstein statistics.

The ground state energy E_0 is the sum of the ground state energies of the individual harmonic oscillators. This sum diverges because of the infinite number of modes. This infinity is a first embarrassement one encounters in quantum field theory, which is due to a cavalier handling of the infinite number of degrees of freedom in a field. One way to avoid the problem is to start out with a finite number of degrees of freedom and study the limit of letting this number approach infinity. For instance, we can simply cut off the number of modes by restricting $|\mathbf{k}| < K$. Within the canonical formalism with its different handling of time and space and its this can lead to non-Lorentz covariant and even non-gauge invariant expressions. Another way is to restrict the spacetime continuum to a hypercubic lattice with lattice distance a and study the limit $a \to 0$. The lattice is of course also not Lorentz covariant but it has usually sufficient remnant symmetry to avoid noncovariance in the continuum limit. Using such regularizations would force use keeping track of many more details right from the beginning. Here we follow instead the usual introductory path and work 'formally', i.e. with ill defined mathematical expressions, and deal with the inifinities when they arise 'along the way'. This approach is sufficient when we treat interacting quantum fields by perturbation theory. For nonperturbative calculations an ab inito regularization such as the lattice is often necessary.

The problem is physical as well as mathematical. The inclusion of arbitrarily large wave vectors $\mathbf{k}$ corresponds to arbitrarily small wavelengths in space and we do not know the physics at arbitrarily short distances. Similarly, continuous time suggests that we can predict what happens in arbitrary short time intervals, which is questionable.

At this point we could appeal to the idea that only energy differences have physical relevance in our model and subtract the ground state energy from P^0. Such a subtraction should be done with care as we may be throwing away a baby with the bath water. There may be a volume dependence in the ground state energy which is physically relevant. An example of this is the Casimir effect. We shall do the subtraction in the infinite volume limit $L \to \infty$.

In the infinite volume limit the ground state represents the vacuum. In this limit the wave vectors become practically continuous, in the sense that for a continuous function $F(\mathbf{k})$,

$$\frac{1}{V}\sum_{\mathbf{k}} F(\mathbf{k}) \to \int \frac{d^3k}{(2\pi)^3}\, F(\mathbf{k}) = \prod_{m=1}^{3}[\int_{-\infty}^{\infty} \frac{dk_m}{2\pi}]F(\mathbf{k}). \qquad (1.124)$$

Furthermore, in the sense of generalized functions

$$V\delta_{\mathbf{k},\mathbf{l}} \to (2\pi)^3\delta(\mathbf{k}-\mathbf{l}). \qquad (1.125)$$

Hence

$$\langle k,\lambda|k',\lambda'\rangle \to 2k^0(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}, \tag{1.126}$$

and the energy density of the ground state takes the form

$$\mathcal{E}_0 \equiv \frac{E_0}{V} \to \hbar \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}\,(k^0)^2. \tag{1.127}$$

The momentum space volume element (integration measure)

$$d\omega_k \equiv \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}, \quad k^0 = \sqrt{\mu^2+\mathbf{k}^2} \tag{1.128}$$

(written for general mass $m^2 = \hbar^2\mu^2$), is Lorentz invariant: under a Lorentz transformation

$$\begin{aligned} k'^\mu &= \Lambda^\mu{}_\nu k^\nu, & (1.129)\\ k'_{||} &= \gamma k_{||} + \gamma\beta k^0, \quad k'_\perp = k_\perp, & (1.130)\\ k'^0 &= \gamma k^0 + \gamma\beta k_{||}, & (1.131)\\ \beta &= v/c, \quad \gamma = 1/\sqrt{1-\beta^2}. & (1.132)\end{aligned}$$

we have $dk'_{||} = (k'^0/k^0)\,dk_{||}$, and

$$d\omega_{\Lambda k} = d\omega_k. \tag{1.133}$$

1.6 Cosmological constant and the Casimir effect

The energy density of the vacuum ($\sum_\lambda = 2$)

$$\mathcal{E}_0 = \hbar\sum_\lambda \int d\omega_k k^0 k^0 \tag{1.134}$$

avoids the volume divergence $V\to\infty$ of E_0 but it is still divergent for large $|\mathbf{k}|$. It has the form of the 00 component of a tensor, which is the vacuum expectation value of the stress-energy (energy-momentum) tensor,

$$\langle 0|T^{\mu\nu}(x)|0\rangle = \hbar\sum_\lambda \int d\omega_k k^\mu k^\nu \tag{1.135}$$

(cf. Problems). Since it is invariant under Lorentz transformations we expect the form

$$\langle 0|T^{\mu\nu}|0\rangle = -\tau_1 g^{\mu\nu}. \tag{1.136}$$

On the other hand, the energy-momentum tensor of the classical electromagnetic field is traceless,

$$T^{\mu}_{\ \mu} = 0, \tag{1.137}$$

which appears to be respected by (1.135) since $k^\mu k_\mu = 0$. This would imply that $\tau_1 = 0$. However, this is in conflict with the fact that $\mathcal{E}_0$ is clearly positive. Such paradoxes are typical when dealing with ill defined divergent expressions and we should regularize the divergent integral. Since we have not developed the tools yet for a covariant regularization, let us just assume the form (1.136), with infinite τ_1.

A term of the form $-\tau g^{\mu\nu}$ in the energy-momentum tensor is not excluded on physical grounds. We have taken it for granted that we could use the $T^{\mu\nu}$ familiar from classical electrodynamics. There is a way to derive the energy momentum tensor from the lagrangian density $\mathcal{L}$ by the socalled Noether procedure. One then finds that a constant $-\tau$ in $\mathcal{L}$ leads to a term $-\tau g^{\mu\nu}$ in $T^{\mu\nu}$. However the real physical significance of $T^{\mu\nu}$ follows when we consider classical general relativity, where energy-momentum is the source of gravity. In this theory the metric tensor is a dynamical variable and the action for $g_{\mu\nu}$ coupled to the electromagnetic potentials A_μ has the form $S = S_g + S_{gA}$, with S_g the Einstein-Hilbert action and S_{gA} the action for the electromagnetic field in the spacetime described by $g_{\mu\nu}$. We only need S_{gA}, which is just the action we had before generalized to variable metric tensor,

$$S_{gA} = -\int d^4x\, \sqrt{-\det g}\, (\frac{1}{4} g^{\kappa\mu} g^{\lambda\nu} F_{\kappa\lambda} F_{\mu\nu} + \tau), \tag{1.138}$$

where $\sqrt{-\det g}$ is included to obtain a volume element $d^4x\,\sqrt{-\det g}$ which is invariant under general coordinate transformations. We have included in S_{gA} the cosmological constant τ (to be more precise, the conventional cosmological constant $\Lambda = 8\pi G\tau$, with G Newton's constant). The energy momentum tensor enters in the field equation for $g_{\mu\nu}$ and is identified from

$$\delta_g S_{gA} = \int d^4x\, \sqrt{-\det g}\, \frac{1}{2} T^{\mu\nu}\, \delta g_{\mu\nu}, \tag{1.139}$$

were g is the matrix $g_{\mu\nu}$. This gives

$$T^{\mu\nu} = F^{\mu\alpha} F^{\nu}_{\ \alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - \tau g^{\mu\nu}. \tag{1.140}$$

Returning to Minkowski space we just use $g_{\mu\nu} \to \mathrm{diag}(-1, 1, 1, 1)$.

The vacuum expectation value $-\tau_1 g^{\mu\nu}$ of $T^{\mu\nu}$ appears in every expectation value of the energy momentum tensor and we now see that effectively the cosmological constant is given by the sum $\tau + \tau_1$. This means that we can absorb the infinite τ_1 in the cosmological constant by redefining the parameter we started out with in (1.138) as τ_0 and choosing τ_0 such that the effective cosmological constant $\tau = \tau_0 + \tau_1$ has the physical value (which is of course finite). Such a procedure is called 'renormalization', τ_0 is called the 'bare' parameter and τ the 'dressed' (by the interaction with the electromagnetic field) parameter, or more frequently, τ is called the renormalized parameter. Writing $\tau_0 = \tau + \delta\tau$ we can say that $\delta\tau$ counteracts the infinite τ_1 and for this reason the $\delta\tau$ part of the action is called a counterterm. The renormalized cosmological constant is not known very well except that in natural units it is very small. For all practical purposes in quantum field theory without cosmological considerations involving gravity we can set the renormalized $\tau = 0$.

Having set the vacuum energy density equal to zero we can now ask meaningful questions about the energy of the ground state in a finite volume. A famous example is the Casimir effect. Consider two parallel plates of a conductor a distance a apart, with a much smaller than the linear size L of the plates. The presence of the plates is taken into account by imposing perfect boundary conditions corresponding to a perfect conductor. This shifts the ground state energy inside and outside the plates relative to the vacuum, and the result is (see e.g. Itzykson and Zuber sect. 3-2-4)

$$\Delta E_0 = \frac{-\hbar\pi^2 L^2}{720 a^3}. \tag{1.141}$$

It corresponds to a tiny attractive force which has been verified by experiment.

1.7 Photons

We have seen that the mass of the photon is zero,

$$P_\mu P^\mu |k,\lambda\rangle = \hbar^2(\mathbf{k}^2 - k_0^2)|k,\lambda\rangle = 0. \tag{1.142}$$

The spin of the photon can only be understood properly after a closer look at Lorentz invariance, which we defer to a later chapter. For now we remark that the states $|k,\lambda\rangle$ transform just like the polarization vectors $e^\mu(k,\lambda) \equiv (\mathbf{e}(\mathbf{k},\lambda), 0)$, modulo terms $\propto k^\mu$ which correspond to gauge transformations. We can use this to determine the possible helicities of the photon. The helicity is defined as the eigenvalue of the angular momentum operator $\mathbf{J}$ in the direction of motion,

$$\frac{\mathbf{k}}{|\mathbf{k}|} \cdot \mathbf{J}|k,\lambda\rangle = \hbar\lambda|k,\lambda\rangle. \tag{1.143}$$

To determine the helicities we take the momentum along the 3-axis and consider the behavior of the polarization vectors $\mathbf{e}(|\mathbf{k}|\hat{3},\lambda)$ under rotations $\exp(-i\omega J_3)$ about this axis. Such rotations have the form

$$\begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{-i\omega S_3}, \tag{1.144}$$

with S_3 the third component of the spin matrix, the spin component of the angular momentum operator J_3. In the vector representation the spin matrices S_1, S_2 and S_3 are represented by

$$(S_l)_{mn} = -i\epsilon_{lmn}, \tag{1.145}$$

which satisfy $[S_k, S_l] = i\epsilon_{klm}S_m$ and $\mathbf{S}^2 = s(s+1) = 1(1+1) = 2$. The right and left handed polarization vectors were constructed such that they are eigenvectors of S_3 for the special momentum $\mathbf{k} = |\mathbf{k}|\hat{3}$, in which case $\mathbf{e}(\mathbf{k},1) = \hat{1}$ and $\mathbf{e}(\mathbf{k},2) = \hat{2}$:

$$(S_3)_{mn}\, e_n(\mathbf{k},\pm) = \pm e_n(\mathbf{k},\pm), \quad \mathbf{k} = |\mathbf{k}|\hat{3}, \tag{1.146}$$

with the usual phase relations $((S_1+iS_2)_{mn}e_n(\mathbf{k},-) = \sqrt{2}e_m(\mathbf{k},3) = \sqrt{2}\hat{3}$, $(S_1 + iS_2)_{mn}e_n(\mathbf{k},3) = \sqrt{2}e_m(\mathbf{k},+))$. The eigenvector $\hat{3}$ with eigenvalue $J_3 = 0$ does not occur among the polarization vectors.

The photons have helicity ± 1 but there is no helicity zero state, as might be expected from the vector representation in which the eigenvalues of S_3 are +1, 0, -1. The helicity zero polarization vector would be the longitudinal mode $\mathbf{e}(\mathbf{k},3) \propto \mathbf{k}$, which is equivalent to a gauge transformation and therefore unphysical. It was eliminated by the Coulomb gauge condition.

A general one photon state has the form of a wave packet

$$|\varphi\rangle = \sum_\lambda \int d\omega_k\, \varphi(k,\lambda)|k,\lambda\rangle, \tag{1.147}$$

with $\varphi(\mathbf{k},\lambda)$ a momentum space wave function which can be normalized to 1,

$$\langle\varphi|\varphi\rangle = \sum_\lambda \int d\omega_k\, \varphi(k,\lambda)^*\varphi(k,\lambda) = 1. \tag{1.148}$$

It is natural to define a spacetime dependent vector potential by

$$\varphi_\mu(x) = (\varphi_m(x), 0), \tag{1.149}$$

$$\varphi_m(x) = \int d\omega_k\, e^{ikx} \sum_\lambda \varphi(k,\lambda)e_m(\mathbf{k},\lambda), \tag{1.150}$$

which is a solution of Maxwell's equations in vacuum. Intuitively we may think that the photon can be found where $\varphi_\mu(x)$ is maximal or at least nonzero. However, localizability is not an appropriate concept for massless particles as there is no nonrelativistic limit where we can apply the usual formalism of nonrelativistic quantum mechanics. The quantization of the electromagnetic field did not lead naturally to a position operator. There is also no satisfactory gauge invariant and covariant probability current $j^\mu(x)$ which is conserved, $\partial_\mu j^\mu(x) = 0$.

Another way to locate a photon is by 'measuring' its energy momentum tensor, and determine e.g. where the energy density is maximal:

$$\langle\varphi|T^{\mu\nu}(x)|\varphi\rangle = \varphi^{\mu\alpha}(x)^*\varphi^\nu{}_\alpha(x) - \frac{1}{4}g^{\mu\nu}\varphi^{\alpha\beta}(x)^*\varphi_{\alpha\beta}(x) + c.c.\,, \tag{1.151}$$

with $\varphi_{\mu\nu} = \partial_\mu\varphi_\nu - \partial_\nu\varphi_\mu$ (cf. Problems).

1.8 Time evolution

In the Heisenberg picture the states are time independent and the operators carry the time dependence according to the Heisenberg equations of motion, for example

$$\frac{d}{dt}A_m(\mathbf{x},t) = -\frac{i}{\hbar}[A_m(\mathbf{x},t), H(t)], \tag{1.152}$$

with the initial condition

$$A_m(\mathbf{x},0) = A_m(\mathbf{x}) \tag{1.153}$$

at time $t = 0$. When the external source J_μ vanishes the hamiltonian is time independent

$$H = \hbar \sum_\lambda \int d\omega_k k^0 a(\mathbf{k},\lambda)^\dagger a(\mathbf{k},\lambda), \tag{1.154}$$

and the equations for the field are easily integrated in momentum space,

$$\begin{aligned} \frac{d}{dt}a(\mathbf{k},\lambda;t) &= -\frac{i}{\hbar}[a(\mathbf{k},\lambda;t), H] = -ik^0 a(\mathbf{k},\lambda;t), &(1.155)\\ a(\mathbf{k},\lambda;t) &= e^{-ik^0 t}\, a(\mathbf{k},\lambda;0) = e^{-ik^0 t}\, a(\mathbf{k},\lambda). &(1.156)\end{aligned}$$

The resulting potentials

$$A_m(\mathbf{x},t) = \sqrt{\hbar}\sum_\lambda \int d\omega_k [e^{i\mathbf{k}\mathbf{x}-ik^0 t} e_m(\mathbf{k},\lambda) a(\mathbf{k},\lambda) + h.c.] \tag{1.157}$$

satisfy the the Coulomb gauge field equations $\Box A_m = 0$. Note that $\dot{A}_m(\mathbf{x},0) = \Pi_m(\mathbf{x})$. When the source J_μ is not zero the hamiltonian has the form

$$H^t_{\rm tot} = H + H^t_J, \tag{1.158}$$

where H is the source free part (1.154) and

$$H^t_J = -\int d^3x\, J_m(\mathbf{x},t)A_m(\mathbf{x}) + E_C(t), \tag{1.159}$$

with $E_C(t)$ the Coulomb energy.

It is convenient to use the interaction picture in which the 'interaction' refers to the external source. The interaction picture is somewhat in between the Schrödinger and the Heisenberg picture and we shall now review how this works. Let the hamiltonian be given in the form

$$H^t = H_0 + H^t_1, \tag{1.160}$$

where we allow for an explicit time dependence in H^t_1 which is inherited by the total hamiltonian H^t, the explicit time dependence is indicated by the superscript t. We asume that H_0 has no explicit time dependence. In the interaction picture the operators evolve in time according to H_0 and the states according to the residual interactions from H_1^t. The three pictures, Schrödinger, Heisenberg and interaction, coincide at time $t = 0$. The time evolution operator is a solution of

$$\frac{d}{dt}U(t) = -\frac{i}{\hbar}H^t U(t), \quad U(0) = 1. \tag{1.161}$$

The evolution operator corresponding to H_0 is given by

$$\frac{d}{dt}U_0(t) = -\frac{i}{\hbar}H_0 U_0(t), \quad U_0(0) = 1, \tag{1.162}$$

which has the usual solution

$$U_0(t) = e^{-iH_0 t/\hbar}. \tag{1.163}$$

The evolution operator in the interaction picture is defined as

$$U_{\text{int}}(t) = U_0(t)^\dagger U(t). \tag{1.164}$$

In the Schrödinger picture

$$|\psi, t\rangle_{\text{S}} = U(t)|\psi, 0\rangle_{\text{S}}, \tag{1.165}$$
$$O_S(t) = O_S(0) \equiv O, \tag{1.166}$$

where $O = O(A, \Pi)$ is any operator without explicit time dependence. In the Heisenberg picture

$$|\psi, t\rangle_{\text{H}} = |\psi, 0\rangle_{\text{H}} = |\psi, 0\rangle_{\text{S}} \equiv |\psi\rangle, \tag{1.167}$$
$$O_H(t) = U^\dagger(t) O U(t), \tag{1.168}$$

while in the interaction picture the time evolution is devided between states and operators,

$$|\psi, t\rangle_{\text{int}} = U_{\text{int}}(t)|\psi\rangle, \tag{1.169}$$
$$O_{\text{int}}(t) = U_0(t)^\dagger O U_0(t). \tag{1.170}$$

Expectation values are the same in all three pictures,

$$\langle\psi, t|_{\text{S}} O|\psi, t\rangle_{\text{S}} = \langle\psi|O_H(t)|\psi\rangle = \langle\psi, t|_{\text{int}} O_{\text{int}}(t)|\psi, t\rangle_{\text{int}}. \tag{1.171}$$

The evolution operator $U_{\text{int}}(t)$ is a solution of the equation

$$\begin{aligned}\frac{d}{dt}U_{\text{int}}(t) &= -\frac{i}{\hbar}[-U_0^\dagger(t)H_0 U(t) + U_0^\dagger(t)(H_0 + H_1^t)U(t)] \\ &= -\frac{i}{\hbar}U_0^\dagger(t)H_1^t U(t) = -\frac{i}{\hbar}U_0^\dagger(t)H_1^t U_0(t)U_{\text{int}}(t) \\ &\equiv -\frac{i}{\hbar}H_1(t)U_{\text{int}}(t),\end{aligned} \tag{1.172}$$

with initial condition $U_{\rm int}(0) = 1$. Here

$$H_1(t) \equiv H_1^t(t) = U_0^\dagger(t) H_1^t U_0(t) \tag{1.173}$$

has the normal time dependence of an operator in the interaction picture, in addition to its explicit time dependence. The evolution operator starting at any time t_0, not just at $t = 0$,

$$U_{\rm int}(t, t_0) = U_{\rm int}(t) U_{\rm int}^\dagger(t_0) \tag{1.174}$$

satisfies the same differential equation (1.172), with initial condition

$$U_{\rm int}(t, t_0) = 1, \quad t = t_0. \tag{1.175}$$

Furthermore it satisfies the composition relation

$$U_{\rm int}(t, t_0) = U_{\rm int}(t, t_1) U_{\rm int}(t_1, t_0). \tag{1.176}$$

For small time difference

$$U_{\rm int}(t, t') \approx 1 - \frac{i}{\hbar}(t - t') H_1(t') \approx \exp[-\frac{i}{\hbar} \int_{t'}^t dt_1 H_1(t_1)]. \tag{1.177}$$

For large time difference the exponential form is exact if H_1 commutes with itself at unequal times, which is generally not true. We can use the relation (1.176) to obtain a useful series expression for $U_{\rm int}(t, t_0)$. We devide the time interval (t, t_0) into N segments (t_j, t_{j-1}), $j = 1, \ldots, N$, of length $a = (t - t_0)/N$, $t_N = t$, and write

$$\begin{aligned} U_{\rm int}(t, t_0) &= U_{\rm int}(t, t_{N-1}) U_{\rm int}(t_{N-1}, t_{N-2}) \cdots U_{\rm int}(t_1, t_0) \\ &\approx [1 - \frac{iaH_1(t_{N-1})}{\hbar}] \cdots [1 - \frac{iaH_1(t_0)}{\hbar}]. \end{aligned} \tag{1.178}$$

Expanding in powers of H_1 and taking the limit $N \to \infty$ leads to

$$\begin{aligned} U_{\rm int}(t, t_0) &= \sum_n \frac{(-i/\hbar)^n}{n!} \int_{t_0}^t dt_1 \cdots dt_n \, T \, H_1(t_1) \cdots H_1(t_n) \\ &\equiv T \exp[\frac{-i}{\hbar} \int_{t_0}^t dt' \, H_1(t')] \end{aligned} \tag{1.179}$$

(the combinatorics is the same as that of $(1 + \frac{x}{N})^N = 1 + N\frac{x}{N} + \frac{1}{2}N(N-1)(\frac{x}{N})^2 + \cdots \to \exp x$ as $N \to \infty$). Here T is the time ordering 'operator', the instruction to order the operators $H_1(t_j)$ from right to left according to increasing time. For instance,

$$\begin{aligned} T H_1(t) H_1(t') &= H_1(t) H_1(t'), \quad t > t', \\ &= H_1(t') H_1(t), \quad t < t'. \end{aligned} \tag{1.180}$$

We shall use the interaction picture with the identification $H_0 \to H$, the source-free H of eq. (1.154), and $H_1^t \to H_J^t$ of eq. (1.159). It then follows from (1.173) and (1.157) that

$$\begin{aligned} H_1(t) &\to -\int d^3x\, J_m(\mathbf{x},t) e^{iHt/\hbar} A_m(\mathbf{x}) e^{-iHt/\hbar} + E_C(t) \\ &= -\int d^3x\, J_m(\mathbf{x},t) A_m(\mathbf{x},t) + E_C(t) \quad (1.181)\\ &\equiv H_J(t). \quad (1.182) \end{aligned}$$

Since the Coulomb term is a c-number at this stage we can separate its effect in the evolution operator into a phase, such that

$$\begin{aligned} U_{\rm int}(t_1,t_2) &= \exp[\frac{-i}{\hbar}\int_{t_2}^{t_1} dt\, E_C(t)]\, T\exp[\frac{i}{\hbar}\int_{t_2}^{t_1} d^4x\, J_m(x)A_m(x)] \quad (1.183)\\ &\equiv U_J(t_1,t_2). \quad (1.184) \end{aligned}$$

1.9 Classical field

Intuitively we expect that the classical electromagnetic field can be understood as the expectation value of the quantum field in suitable states. For a one photon state $|\varphi\rangle$, $\langle\varphi|A_\mu(x)|\varphi\rangle = 0$. Of course, we should expect classical behavior only for states with large quantum numbers, i.e. large numbers of photons. However, since $A_m(x)$ changes the number of photons, it expectation value in a state with a definite number of photons is zero.

Let us assume the situation in which the external source $J^\mu(x)$ is zero initially and switched on slowly at some time t_-, and let $|0\rangle$ be the vacuum for $t < t_-$. Consider the state $|0,t\rangle$ which evolves out of the vacuum under the influence of the external source. We shall show that the classical field $A_\mu^{(c)}$ may be identified as

$$A_m^{(c)}(\mathbf{x},t) \equiv \langle 0,t|A_m(\mathbf{x},t)|0,t\rangle. \quad (1.185)$$

In the interaction picture $|0,t\rangle$ is given by

$$\begin{aligned} |0,t\rangle &= U_J(t,-\infty)|0\rangle, \quad (1.186)\\ U_J(t,-\infty) &= T\exp[\frac{-i}{\hbar}\int_{-\infty}^{t} dt'\, H_J(t')], \quad (1.187)\\ H_J(t) &= -\int d^3x\, A_m(\mathbf{x},t)J_m(\mathbf{x},t) + E_C(t). \quad (1.188) \end{aligned}$$

Since H_J is linear in the creation and annihilation operators $|0,t\rangle$ has the form of a 'coherent state'. The operator $A_m(x)$ evolves according to the hamiltonian with $J=0$. Since A_0 is already a c-number in the Coulomb gauge $A_0^{(c)} = A_0$ and we need to evaluate

$$A_m^{(c)}(\mathbf{x},t) = \langle 0|U_J(t,-\infty)^\dagger\, A_m(\mathbf{x},t) U_J(t,-\infty)|0\rangle. \tag{1.189}$$

Note that the phase factor associated with the Coulomb energy E_C cancels in this expression. Differentiating with respect to time we get

$$\begin{aligned}
\partial_t A_m^{(c)}(\mathbf{x},t) &= \langle 0|U_J(t,-\infty)^\dagger\, \frac{i}{\hbar}\,[H_J(t), A_m(\mathbf{x},t)] U_J(t,-\infty)|0\rangle \\
&\quad + \langle 0|U_J(t,-\infty)^\dagger\, \Pi_m(\mathbf{x},t) U_J(t,-\infty)|0\rangle \\
&= \langle 0|U_J(t,-\infty)^\dagger\, \Pi_m(\mathbf{x},t) U_J(t,-\infty)|0\rangle, \qquad (1.190)
\end{aligned}$$

since $A_m(\mathbf{x},t)$ commutes with $A_n(\mathbf{y},t)$ at equal times. A second differentiation gives (cf. Problems)

$$\begin{aligned}
\partial_t^2 A_m^{(c)}(\mathbf{x},t) &= \langle 0|U_J(t,-\infty)^\dagger\, \frac{i}{\hbar}\,[H_J(t), \Pi_m(\mathbf{x},t)] U_J(t,-\infty)|0\rangle \\
&\quad + \langle 0|U_J(t,-\infty)^\dagger\, \dot{\Pi}_m(\mathbf{x},t) U_J(t,-\infty)|0\rangle \\
&= \langle 0,t|\frac{-i}{\hbar}\int d^3x'\,[A_n(\mathbf{x}',t), \Pi_m(\mathbf{x},t)]\, J_n(\mathbf{x}',t)|0,t\rangle \\
&\quad + \langle 0,t|\frac{i}{\hbar}[H, \Pi_m(\mathbf{x},t)]|0,t\rangle \\
&= \langle 0,t|[J_m^T(\mathbf{x},t) + \Delta A_m(\mathbf{x},t)]|0,t\rangle \qquad (1.191) \\
&= J_m^T(\mathbf{x},t) + \Delta A_m^{(c)}(\mathbf{x},t), \qquad (1.192)
\end{aligned}$$

where

$$J_m^T(\mathbf{x},t) = \int d^3x'\, P_{mn}^T(\mathbf{x},\mathbf{x}') J_n(\mathbf{x}',t). \tag{1.193}$$

This can be rewritten as

$$\Box A_m^{(c)}(x) = -J_m^T(x), \tag{1.194}$$

which is just the classical equation for $A_m^{(c)}$ in the Coulomb gauge, since

$$\partial_\nu F^{(c)\nu\mu} = \Box A^{(c)\mu} - \partial^\mu \partial_\nu A^{(c)\nu} = -J^\mu, \tag{1.195}$$

leads to

$$\begin{aligned}
0 &= \Box A_m^{(c)}(\mathbf{x},t) - \partial_m \partial_0 A^{(c)0}(\mathbf{x},t) + J_m(\mathbf{x},t) \\
&= \Box A_m^{(c)}(\mathbf{x},t) - \partial_m \int d^3x'\, \frac{\partial_0 J^0(\mathbf{x}',t)}{4\pi|\mathbf{x}-\mathbf{x}'|} + J_m(\mathbf{x},t) \\
&= \Box A_m^{(c)}(\mathbf{x},t) + \partial_m \partial_n \int d^3x'\, \frac{J_n(\mathbf{x}',t)}{4\pi|\mathbf{x}-\mathbf{x}'|} + J_m(\mathbf{x},t) \qquad (1.196) \\
&= \Box A_m^{(c)}(\mathbf{x},t) + J_m^T(\mathbf{x},t), \qquad (1.197)
\end{aligned}$$

where we used $\partial_0 J^0 = -\partial_n J_n$ and (1.313). The boundary conditions in time follow from

$$A_m^{(c)}(\mathbf{x},t) = \langle 0,t|A_m(\mathbf{x},t)|0,t\rangle \to \langle 0|A_m(\mathbf{x},t)|0\rangle = 0, \quad \text{for } t\to-\infty, \tag{1.198}$$

which are the usual retarded boundary conditions.

1.10 Vacuum persistence amplitude

The amplitude for the vacuum to remain unchanged under the influence of the source (the vacuum persistence amplitude) is given by

$$\langle 0|U_J(\infty,-\infty)|0\rangle \equiv Z(J), \tag{1.199}$$

and $|Z(J)|^2$ is the corresponding probability. This amplitude plays an important role in the following.

Expanding in J we have

$$\begin{aligned} Z(J) &= \langle 0|0\rangle + \frac{i}{\hbar}\int d^4x\, J_m(x)\langle 0|A_m(x)|0\rangle \\ &+ \frac{i^2}{2!\,\hbar^2}\int d^4x\, d^4y\, J_m(x)J_n(y)\langle 0|TA_m(x)A_n(y)|0\rangle \\ &- \frac{i}{2\,\hbar}\int d^4x\, d^4y\, J^0(x)J^0(y)\frac{\delta(x^0-y^0)}{4\pi|\mathbf{x}-\mathbf{y}|} + O(J^4). \end{aligned} \tag{1.200}$$

The first term is 1, the second term is zero. We shall evaluate the third term by inserting intermediate states,

$$1 = \sum_n \frac{1}{n!}\sum_{\lambda_1\cdots\lambda_n}\int d\omega_{k_1}\cdots d\omega_{k_n}\,|k_1\lambda_1\cdots k_n\lambda_n\rangle\langle k_1\lambda_1\cdots k_n\lambda_n|. \tag{1.201}$$

Since the free field $A_m(x)$ is linear in the creation and annihilation operators only the one particle intermediate states contribute (this is only true for free fields),

$$\begin{aligned} \langle 0|TA_m(x)A_n(y)|0\rangle &= \sum_\lambda \int d\omega_k[\theta(x^0-y^0)\langle 0|A_m(x)|k\lambda\rangle\langle k\lambda|A_n(y)|0\rangle \\ &\quad + \theta(y^0-x^0)\langle 0|A_n(y)|k\lambda\rangle\langle k\lambda|A_m(x)|0\rangle]. \end{aligned} \tag{1.202}$$

Using

$$\begin{aligned} \langle 0|A_m(x)|k\lambda\rangle &= \sqrt{\hbar}e_m(\mathbf{k},\lambda)e^{ikx}, \\ \langle k\lambda|A_n(y)|0\rangle &= \sqrt{\hbar}e_n(\mathbf{k},\lambda)^*e^{-iky}, \end{aligned} \tag{1.203}$$

and

$$\sum_\lambda e_m(\mathbf{k},\lambda)e_n(\mathbf{k},\lambda)^* = \delta_{mn} - \frac{k_m k_n}{\mathbf{k}^2} = P^T_{mn}(\mathbf{k}), \tag{1.204}$$

Figure 1.1: Contours in the complex k_0 plane for $t > 0$ (a) and $t < 0$ (b).

this gives

$$\begin{aligned}\langle 0|TA_m(x)A_n(y)|0\rangle &= \hbar\int d\omega_k P^T_{mn}(\mathbf{k})\,[\theta(x^0-y^0)e^{ik(x-y)} \\ &\quad + \theta(y^0-x^0)e^{-ik(x-y)}] \qquad (1.205)\\ &= \hbar\int d\omega_k\, P^T_{mn}(\mathbf{k})\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})-ik^0|x^0-y^0|}, \qquad (1.206)\end{aligned}$$

where $k^0 = |\mathbf{k}|$, we changed variables $\mathbf{k} \to -\mathbf{k}$ in the second term and used

$$\theta(t)e^{-ik^0t} + \theta(-t)e^{ik^0t} = e^{-ik^0|t|}. \qquad (1.207)$$

To evaluate this further we use the identity (written for general mass $\hbar\mu$)

$$-i\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\,\frac{e^{+ik_0t}}{\mu^2+\mathbf{k}^2-k_0^2-i\epsilon} = \frac{e^{-i\sqrt{\mu^2+\mathbf{k}^2}\,|t|}}{2\sqrt{\mu^2+\mathbf{k}^2}}, \qquad (1.208)$$

in which $\epsilon \to +0$ (and the integration variable k_0 should not be confused with $k^0 = -k_0 = |\mathbf{k}|$ in (1.207). This identity can be checked by contour integration, cf. fig. 1.1. The poles of $1/(\mu^2+\mathbf{k}^2-k_0^2-i\epsilon)$ are at $\pm(\sqrt{\mu^2+\mathbf{k}^2}-i\epsilon)$ with residues $\mp 1/(2\sqrt{\mu^2+\mathbf{k}^2})$. For $t > 0$ the contour can be closed along a circle in the upperhalf plane with radius $\to\infty$, and then only the pole at $k_0 = -(\sqrt{\mu^2+\mathbf{k}^2} - i\epsilon)$ contributes; for $t < 0$ the contour can be closed in the lower half plane and only the pole at $k_0 = +(\sqrt{\mu^2+\mathbf{k}^2}-i\epsilon)$ contributes. It follows that ($k^2 = \mathbf{k}^2 - k_0^2$)

$$\begin{aligned}\langle 0|TA_m(x)A_n(y)|0\rangle &= -i\hbar\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\,\frac{P^T_{mn}(\mathbf{k})}{k^2-i\epsilon}, \qquad (1.209)\\ &\equiv -i\hbar G_C^{mn}(x-y). \qquad (1.210)\end{aligned}$$

Combining this with the $J^0 J^0$ term the vacuum persistence amplitude can be written as

$$Z(J) = 1 + \frac{i}{2\hbar} \int d^4x\, d^4y\, J_\mu(x) J_\nu(y) G_C^{\mu\nu}(x-y) + O(J^4), \tag{1.211}$$

with $G_C^{\mu\nu}(x-y)$ given by

$$G_C^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{N_C^{\mu\nu}(k)}{k^2 - i\epsilon}, \tag{1.212}$$

where

$$\begin{aligned} N_C^{\mu\nu}(k) &= -\frac{k^2}{\mathbf{k}^2}, \quad (\mu,\nu) = (0,0), && (1.213)\\ &= 0, \quad (\mu,\nu) = (m,0) \text{ or } (0,n), && (1.214)\\ &= P_{mn}^T(k), \quad (\mu,\nu) = (m,n). && (1.215) \end{aligned}$$

We used

$$\frac{\delta(x^0 - y^0)}{4\pi|\mathbf{x} - \mathbf{y}|} = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{1}{\mathbf{k}^2}, \tag{1.216}$$

for $(\mu,\nu) = (0,0)$. The object $G_C^{\mu\nu}(x-y)$ is called the propagator (in Coulomb gauge, as indicated by the subscript C; later we shall encounter propagators in other gauges).

The amplitude $Z(J)$ looks noncovariant but it is Lorenz invariant. This is most easily shown in 'momentum space', i.e. expressing $Z(J)$ in terms of the Fourier variables k_μ. Inserting (1.212) into (1.211) gives

$$Z(J) = 1 + \frac{i}{2\hbar} \int \frac{d^4k}{(2\pi)^4}\, J_\mu(-k)\, \frac{N_C^{\mu\nu}(k)}{k^2 - i\epsilon}\, J_\nu(k) + O(J^4), \tag{1.217}$$

where

$$J^\mu(k) = \int d^4x\, e^{-ikx}\, J^\mu(x). \tag{1.218}$$

Next we note that $N_C^{\mu\nu}(k)$ can be expressed in the form

$$N_C^{\mu\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu + (kn)(k^\mu n^\nu + n^\mu k^\nu)}{k^2 + (kn)^2}, \tag{1.219}$$

where n is the time like unit vector

$$n^\mu = \delta_{\mu,0}, \quad n^2 = -1. \tag{1.220}$$

This shows that $N_C^{\mu\nu}(k)$ is equal to $g^{\mu\nu}$ up to terms in volving k^μ, k^ν or both. The terms $\propto k^\mu$, k^ν in the propagator are called gauge terms, since they depend on the choice of gauge. Using current conservation $\partial_\mu J^\mu = 0$, or in momentum space $k_\mu J^\mu(k) = 0$, we see that the gauge terms do not contribute and we can express the vacuum amplitude as a Lorentz scalar,

$$Z(J) = 1 + \frac{i}{2\hbar} \int \frac{d^4k}{(2\pi)^4} J_\mu(-k) \frac{g^{\mu\nu}}{k^2 - i\epsilon} J_\nu(k) + O(J^4). \tag{1.221}$$

1.11 Propagator

The propagator is a Green function, it is the inverse of the Maxwell wave operator (cf. (1.39))

$$K_{\mu\nu} = -\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu, \tag{1.222}$$

in the sense that a solution of

$$K_{\mu\nu} A^\nu(x) = J_\mu(x), \quad \partial_\mu J^\mu(x) = 0, \tag{1.223}$$

is given by

$$A^\mu(x) = \int d^4y \, G_C^{\mu\nu}(x-y) J_\nu(y). \tag{1.224}$$

The differential operator $K_{\mu\nu}$ has zero eigenvalues since any vector potential of the form $A_\mu = \partial_\mu \omega$ ('pure gauge') gives zero, $K_{\mu\nu}\partial^\nu \omega = -\partial^2 \partial^\nu \omega + \partial^2 \partial^\nu \omega = 0$. Therefore K has no inverse on a general function space. Imposing a gauge condition such as the Coulomb gauge there are solutions to (1.223). It is essential that $\partial_\mu J^\mu(x) = 0$ since the left hand side has also zero divergence. The solution is still not unique unless we impose boundary conditions in time, e.g. retarded boundary conditions for which $A(x) \to 0$ as $x^0 \to -\infty$. In our case we have so-called Feynman boundary conditions in time.

Feynman (also called 'causal') boundary conditions are as follows:

$$\begin{aligned} A^\mu(x) &= \text{superposition of } e^{-ik^0 x^0} \text{ for } x^0 \to +\infty & (1.225)\\ &= \text{superposition of } e^{+ik^0 x^0} \text{ for } x^0 \to -\infty, & (1.226) \end{aligned}$$

where $k^0 > 0$. In momentum space,

$$\begin{aligned} K_{\mu\nu} &\to k^2 g_{\mu\nu} - k_\mu k_\nu \equiv K_{\mu\nu}(k), & (1.227)\\ G_C^{\mu\nu}(k) &= \frac{N_C^{\mu\nu}(k)}{k^2 - i\epsilon}, & (1.228)\\ K_{\mu\nu}(k) G_C^{\nu\rho}(k) &= \frac{k^2}{k^2 - i\epsilon}\left(\delta_\mu^\rho - \frac{[k_\mu + (kn)n_\mu]k^\rho}{k^2 + (kn)^2}\right), & (1.229) \end{aligned}$$

where we used (1.212), (1.219). Note that $K_{\mu\nu}(k)k^\nu = 0$. Since $k^2/(k^2 - i\epsilon) = 1$, it follows that

$$K_{\mu\nu} G^{\nu\rho}(x-y) = \delta_\mu^\rho \, \delta^4(x-y) + \text{terms} \propto \partial^\rho. \tag{1.230}$$

The terms $\propto \partial^\rho$ vanish when integrated with J_ρ. As can be seen from (1.205,1.210) the propagator satisfies the Feynman boundary conditions:

$$\begin{aligned} G_C^{\mu\nu}(x-y) &= i \int d\omega_k \, e^{ik(x-y)} P_C^{\mu\nu}(k), \quad x^0 > y^0,\\ &= i \int d\omega_k \, e^{-ik(x-y)} P_C^{\mu\nu}(k), \quad x^0 < y^0, & (1.231) \end{aligned}$$

$$\begin{aligned} P_C^{\mu\nu}(k) &= \sum_\lambda e^\mu(k,\lambda)e^\nu(k,\lambda)^*, \quad e^\mu(k,\lambda) \equiv (\mathbf{e}(\mathbf{k},\lambda),0) & (1.232)\\ &= g^{\mu\nu} - \frac{k^\mu k^\nu + (kn)(k^\mu n^\nu + n^\mu k^\nu)}{(kn)^2}, & (1.233)\\ &= N_C^{\mu\nu}(k), \quad k^2 = 0, & (1.234)\end{aligned}$$

(the Coulomb part of the propagator contributes only for $x^0 = y^0$).

1.12 Vacuum amplitude to all orders in J

We shall now calculate $Z(J)$ to all orders in J. Since $U_J(\infty,-\infty)$ contains a phase factor coming from the Coulomb energy we first separate this factor from the amplitude,

$$\begin{aligned} Z(J) &= \exp[-\frac{i}{\hbar}\int dt\, E_C(t)]\, Z'(J), & (1.235)\\ Z'(J) &= \langle 0|U'_J(\infty,-\infty)|0\rangle, & (1.236)\\ U'_J(t_1,t_2) &= T\exp[\frac{i}{\hbar}\int_{t_2}^{t_1} d^4x\, J_m(x)A_m(x)], & (1.237)\end{aligned}$$

where $Z'(J)$ depends only on J_m. We functionally differentiate $Z'(J)$ (cf. Appendix),

$$\begin{aligned} \delta Z'(J) &= \frac{i}{\hbar}\int d^4x\, \langle 0|U'_J(\infty,x^0)A_m(x)U'_J(x^0,-\infty)|0\rangle \delta J_m(x), & (1.238)\\ \frac{\delta Z'(J)}{\delta J_m(x)} &= \frac{i}{\hbar}\langle 0|U'_J(\infty,x^0)A_m(x)U'_J(x^0,-\infty)|0\rangle, & (1.239)\\ &\equiv \frac{i}{\hbar}A_m^{(c)}(x)Z'(J), & (1.240)\\ A_m^{(c)}(x) &= \frac{\langle 0|U'_J(\infty,x^0)A_m(x)U'_J(x^0,-\infty)|0\rangle}{\langle 0|U'_J(\infty,-\infty)|0\rangle}. & (1.241)\end{aligned}$$

Here $A_m^{(c)}(x)$ is a c-number field, like a classical field. In the same way as for the classical field $\langle 0,x^0|A_m(x)|0,x^0\rangle$ in the previous section we can derive the equation of motion

$$\Box A_m^{(c)}(x) = -J_m^T(x). \qquad (1.242)$$

However, here the boundary conditions in time are different (not retarded): for $x^0 \to -\infty$,

$$\begin{aligned} A_m^{(c)}(x) &\to \frac{\langle 0|U'_J(\infty,-\infty)A_m(x)|0\rangle}{\langle 0|U'_J(\infty,-\infty)|0\rangle}\\ &= \sum_\lambda \int d\omega_k\, e_m(\mathbf{k},\lambda)^* e^{-ikx}\frac{\langle 0|U'_J(\infty,-\infty)a_m^\dagger(\mathbf{k},\lambda)|0\rangle}{\langle 0|U'_J(\infty,-\infty)|0\rangle}, & (1.243)\end{aligned}$$

while for $x^0 \to +\infty$

$$
\begin{aligned}
A_m^{(c)}(x) &\to \frac{\langle 0|A_m(x)U'_J(\infty,-\infty)|0\rangle}{\langle 0|U'_J(\infty,-\infty)|0\rangle} \\
&= \sum_\lambda \int d\omega_k\, e_m(\mathbf{k},\lambda)e^{ikx}\frac{\langle 0|a_m(\mathbf{k},\lambda)U'_J(\infty,-\infty)|0\rangle}{\langle 0|U'_J(\infty,-\infty)|0\rangle}. \qquad (1.244)
\end{aligned}
$$

This implies that for $x^0 \to -\infty$ the field $A_m^{(c)}$ contains only socalled negative frequencies $\propto \exp(+ik^0x^0)$ while for $x^0 \to +\infty$ it contains only positive frequencies $\propto \exp(-ik^0x^0)$ $(k^0 > 0)$. These are just the Feynman or 'causal' boundary conditions, and $A_m^{(c)}(x)$ is given by

$$A_m^{(c)}(x) = \int d^4x\, G_C^{mn}(x-y)J_n(y). \qquad (1.245)$$

Hence, $Z'(J)$ satisfies the following equation incorporating the boundary conditions in time,

$$\frac{\hbar}{i}\frac{\delta Z'(J)}{\delta J_m(x)} = [\int d^4x\, G_C^{mn}(x-y)J_n(y)]Z'(J). \qquad (1.246)$$

We need the solution of this equation with the boundary condition $Z'(J) = 1$ for $J = 0$. The solution is given by

$$Z'(J) = \exp[\frac{i}{2\hbar}\int d^4x\, d^4y\, J_m(x)G_C^{mn}(x-y)J_n(y)]. \qquad (1.247)$$

Taking into account the contribution from the Coulomb energy we have for the complete amplitude (cf. (1.200), (1.210))

$$Z(J) = \exp[\frac{i}{2\hbar}\int d^4x\, d^4y\, J_\mu(x)J_\nu(y)G_C^{\mu\nu}(x-y)],, \qquad (1.248)$$

which reproduces the previous $O(J^2)$ result (1.211).

1.13 Effective action and Feynman propagator

We can reexpress this result as follows,

$$Z(J) = \exp[\frac{i}{\hbar}(S(A) + \int d^4x\, J_\mu A^\mu)], \qquad (1.249)$$

where we redefined S by writing the source contribution separately,

$$S(A) = -\int d^4x\, \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\int d^4x\, \frac{1}{2}A^\mu K_{\mu\nu}A^\nu, \qquad (1.250)$$

and for notational convenience we drop the label (c) on $A_\mu^{(c)}$ in this section, i.e. A_μ is not an operator field but a classical field. This field is to be calculated from

$$0 = \frac{\delta S}{\delta A^\mu} + J_\mu = -K_{\mu\nu} A^\nu + J_\mu, \tag{1.251}$$

with Feynman boundary conditions in time. The solution is

$$A^\mu(x) = \int d^4y \, G_C^{\mu\nu}(x-y) J_\nu(y). \tag{1.252}$$

Substitution in (1.249) using (1.251) gives back (1.248),

$$\begin{aligned} S(A) + \int d^4x J_\mu(x) A^\mu(x) &= \frac{1}{2} \int d^4x J_\mu(x) A^\mu(x) \\ &= \frac{1}{2} \int d^4x \, d^4y \, J_\mu(x) G_C^{\mu\nu}(x-y) J_\nu(y)]. \end{aligned} \tag{1.253}$$

We can also use the covariant Lorentz gauge

$$\partial_\mu A_\mu = 0, \tag{1.254}$$

and the corresponding Green function is the Lorentz gauge (often called Landau gauge) propagator

$$G_L^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \, e^{ik(x-y)} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - i\epsilon} \right) \frac{1}{k^2 - i\epsilon}. \tag{1.255}$$

For a conserved current $\partial_\mu J^\mu = 0$ the vacuum amplitudes are identical.

We can also leave out all k^μ, k^ν terms from $G^{\mu\nu}$ and use Feynman's propagator

$$G_F^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\epsilon} \tag{1.256}$$

in the expression for the vacuum amplitude. This is usually referred to as 'using the Feynman gauge'. However, $G_F^{\mu\nu}$ cannot be obtained by a gauge condition in the usual sense, but by modifying the action. Consider the action

$$S(A) = -\int d^4x \, [\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2]. \tag{1.257}$$

This action leads to the equations of motion

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi} \partial^\nu \partial_\mu A^\mu + J^\nu = 0, \tag{1.258}$$

or

$$(-\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu - \frac{1}{\xi} \partial_\mu \partial_\nu) A^\nu = J_\mu. \tag{1.259}$$

The added term depending on the coefficient ξ breaks gauge invariance and the wave operator has no zero eigenvalue anymore (with retarded or Feynman boundary conditions). The propagator can now be defined as the inverse of the wave operator,

$$(-\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu - \frac{1}{\xi}\partial_\mu \partial_\nu) G^{\nu\rho}(x-y) = \delta_\mu^\rho \delta(x-y). \tag{1.260}$$

The solution with Feynman boundary conditions reads in momentum space

$$G^{\mu\nu}(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - i\epsilon} \right) \frac{1}{k^2 - i\epsilon} + \xi \frac{k^\mu k^\nu}{(k^2 - i\epsilon)^2}. \tag{1.261}$$

For $\xi = 1$ this is the Feynman propagator (1.256).

Another way to see that the vacuum amplitude is unchanged with this modified action, is taking the divergence of the equation of motion, which gives

$$\Box \partial_\mu A^\mu = 0. \tag{1.262}$$

The solution of this with Feynman boundary conditions is

$$\partial_\mu A^\mu = 0, \tag{1.263}$$

so the term $\propto \xi^{-1}$ in the action in the expression for the vacuum amplitude vanishes.

An alternative quantization procedure for the electromagnetic field is based on the modified action with $\xi = 1$, the Gupta-Bleuler method. This leads to additional unphysical 'photons' called ghosts, and indefinite (positive and negative metric) in 'Hilbert space'. One then has to show that these undesirable features do not matter in physical quantities. The advantage of this method is that it leads to manifestly Lorentz covariant expressions.

We shall see later that also in the general situation with interacting fields the vacuum amplitude can be expressed in terms of an action, the effective action. In our simple case the effective action is just $S(A)$.

1.14 Emission and absorption of photons

Suppose the source has the form $J^\mu(x) = J_1^\mu(x) + J_2^\mu(x)$, such that the spacetime region where J_1 is nonzero lies to the future of the region where J_2 is nonzero, as illustrated in fig. 1.2. Let t_+ be a time after J_1 has acted, t_- a time before J_2 has acted and t_0 a time in between the times where $J_{1,2}$ are nonzero (e.g. $t_0 = 0$). The evolution operator satisfies the relation $U_J(t_+, t_-) = U_J(t_+, t_0) U_J(t_0, t_-)$, and for the above choice of sources we can write

$$U_{J_1 + J_2} = U_{J_1} U_{J_2}, \tag{1.264}$$

Figure 1.2: Spacetime regions where $J_1^\mu(x)$ and $J_2^\mu(x)$ are nonzero.

with

$$\begin{aligned} U_J &\equiv U_J(\infty, -\infty) \\ &= T\,\exp[\frac{i}{\hbar}\int d^4x\, A_m(x) J_m(x) - \frac{i}{\hbar}\int dt E_C(t)]. \end{aligned} \quad (1.265)$$

Introducing intermediate states at time t_0 we have

$$\begin{aligned} Z(J_1+J_2) &= \langle 0|U_{J_1}U_{J_2}|0\rangle && (1.266) \\ &= \sum_n \frac{1}{n!}\sum_{\lambda_1\cdots\lambda_n}\int d\omega_{k_1}\cdots d\omega_{k_n} \\ &\quad \langle 0|U_{J_1}|k_1\lambda_1\cdots k_n\lambda_n\rangle\langle k_1\lambda_1\cdots k_n\lambda_n|U_{J_2}|0\rangle, && (1.267) \\ &\equiv \sum_n \frac{1}{n!}\sum_{\lambda_1\cdots\lambda_n}\int d\omega_{k_1}\cdots d\omega_{k_n} \\ &\quad \langle 0|k_1\lambda_1\cdots k_n\lambda_n\rangle_{J_1}\langle k_1\lambda_1\cdots k_n\lambda_n|0\rangle_{J_2}, && (1.268) \end{aligned}$$

where we introduced the amplitudes for production and absorption of photons by the sources J_2 and J_1,

$$\begin{aligned} \langle k_1\lambda_1\cdots k_n\lambda_n|0\rangle_{J_2} &= \langle k_1\lambda_1\cdots k_n\lambda_n|U_{J_2}|0\rangle, && (1.269) \\ \langle 0|k_1\lambda_1\cdots k_n\lambda_n\rangle_{J_1} &= \langle 0|U_{J_1}|k_1\lambda_1\cdots k_n\lambda_n\rangle. && (1.270) \end{aligned}$$

We can now use the explicit expression for the vacuum amplitude,

$$\begin{aligned} Z(J) &= \exp[\frac{i}{2\hbar}\int d^4x\, d^4y\, J_\mu(x) G^{\mu\nu}(x-y) J_\nu(y)], && (1.271) \\ Z(J_1+J_2) &= Z(J_1)Z(J_2)\exp[\frac{i}{\hbar}\int d^4x\, d^4y\, J_{1\mu}(x) G^{\mu\nu}(x-y) J_{2\nu}(y)], \end{aligned}$$

$$= \quad Z(J_1)Z(J_2)\exp[\frac{1}{\hbar}\sum_\lambda\int d\omega_k\, iJ_1(\mathbf{k},\lambda)^*\, iJ_2(\mathbf{k},\lambda)], \qquad (1.272)$$

where we used (1.232) and the notation

$$J(\mathbf{k},\lambda) = e_m(\mathbf{k},\lambda)^*\int d^4x\, e^{-ikx}J_m(x), \quad k^0 = |\mathbf{k}|. \qquad (1.273)$$

Expanding the $J_1 - J_2$ cross term as a series in J_1 and J_2 an comparing with the right hand side of (1.268) we see that the emission and absorption amplitudes are given by

$$\langle k_1\lambda_1\cdots k_n\lambda_n|0\rangle_{J_2} \quad = \quad Z(J_2)\hbar^{-n/2}\prod_{l=1}^n iJ_2(\mathbf{k}_l,\lambda_l), \qquad (1.274)$$

$$\langle 0|k_1\lambda_1\cdots k_n\lambda_n\rangle_{J_1} \quad = \quad Z(J_1)\hbar^{-n/2}\prod_{l=1}^n iJ_1(\mathbf{k}_l,\lambda_l)^*. \qquad (1.275)$$

For a weak source we have to leading order in J,

$$\langle k,\lambda|0\rangle_{J_2} \quad = \quad iJ_2(\mathbf{k},\lambda)/\sqrt{\hbar}, \qquad (1.276)$$

$$\langle 0|k,\lambda\rangle_{J_1} \quad = \quad iJ_1(\mathbf{k},\lambda)^*/\sqrt{\hbar}, \qquad (1.277)$$

and we see for instance that the momentum space wave function of a singly produced photon is given by

$$\varphi(k,\lambda) = iJ_2(\mathbf{k},\lambda)/\sqrt{\hbar}. \qquad (1.278)$$

1.15 Radiation by a source

The probability P_n for producing n photons in momentum range R by a source J (we drop the subscript 2 on J_2),

$$P_n^R = \frac{1}{n!}\sum_{\lambda_1\cdots\lambda_n}\int_R d\omega_{k_1}\cdots d\omega_{k_n}|\langle k_1\lambda_1\cdots k_n\lambda_n|0\rangle_J|^2 \qquad (1.279)$$

is given by

$$P_n^R = P_0(J)\,\frac{\hbar^{-n}}{n!}[\sum_\lambda\int_R d\omega_k|J(\mathbf{k},\lambda)|^2]^n, \qquad (1.280)$$

where P_0 is the probability that no photon is radiated,

$$P_0(J) = |Z(J)|^2 = \exp[-\frac{1}{\hbar}\mathrm{Im}\int d^4x\, d^4y\, J_{2\mu}(x)G_C^{\mu\nu}(x-y)J_{2\nu}(y)]. \qquad (1.281)$$

One way to evaluate this expression is in momentum space,

$$\begin{aligned} P_0(J) &= \exp[-\frac{1}{\hbar}\mathrm{Im}\int \frac{d^4k}{(2\pi)^4} J_{2\mu}(-k) G_C^{\mu\nu}(k) J_{2\nu}(k)], & (1.282) \\ &= \exp[-\frac{1}{\hbar}\mathrm{Im}\int \frac{d^4k}{(2\pi)^4} J_{2\mu}(-k) g^{\mu\nu} J_{2\nu}(k)\, \frac{1}{k^2 - i\epsilon}], & (1.283) \end{aligned}$$

using the representation (written for general mass μ)

$$\frac{1}{\mu^2 + k^2 - i\epsilon} = P\,\frac{1}{\mu^2 + k^2} + i\pi\delta(\mu^2 + k^2), \tag{1.284}$$

where P denotes the principal value and

$$\delta(\mu^2 + k^2) = \frac{\delta(k^0 - \sqrt{\mu^2 + \mathbf{k}^2})}{2\sqrt{\mu^2 + \mathbf{k}^2}} + \frac{\delta(k^0 + \sqrt{\mu^2 + \mathbf{k}^2})}{2\sqrt{\mu^2 + \mathbf{k}^2}}, \tag{1.285}$$

with the corollary for an arbitrary function $f(k)$,

$$\int \frac{d^4k}{(2\pi)^4}\, \pi\delta(\mu^2 + k^2) f(k) = \int d\omega_k\, \frac{1}{2}[f(\mathbf{k}, k^0) + f(\mathbf{k}, -k^0)]. \tag{1.286}$$

In our case $f(k) = -J_0(k)J_0(-k) + J_m(k)J_m(-k)$ and using the change of variables $\mathbf{k} \to -\mathbf{k}$ the vacuum persistence probability can be written as

$$P_0(J) = \exp[-\frac{1}{\hbar}\sum_\lambda \int d\omega_k |J(\mathbf{k}, \lambda)|^2. \tag{1.287}$$

Note that the Coulomb term in $Z(J)$ is a phase factor and does not contribute in P_0.

If the region R is chosen to be all of momentum space, then P_n follows a Poisson distribution,

$$P_n = \frac{e^{-\bar{n}}\bar{n}^n}{n!}, \tag{1.288}$$

where

$$\bar{n} = \frac{1}{\hbar}\sum_\lambda \int d\omega_k |J(\mathbf{k}, \lambda)|^2 = \sum_{n=0}^{\infty} n P_n. \tag{1.289}$$

If R is the region $R = \{\mathbf{k}; |\mathbf{k}| < \Delta\}$, then the total probability for emission into R is given by

$$\begin{aligned} \sum_n P_n^R &= P_0 \exp[\frac{1}{\hbar}\sum_\lambda \int_R d\omega_k |J(\mathbf{k}, \lambda)|^2] & (1.290) \\ &= \exp[-\frac{1}{\hbar}\sum_\lambda \int_{|\mathbf{k}|>\Delta} \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|}\, |J(\mathbf{k}, \lambda)|^2]. & (1.291) \end{aligned}$$

For a current for which $\sum_\lambda |J(\mathbf{k},\lambda)|^2 = O(\mathbf{k}^{-2})$ as $\mathbf{k} \to 0$, the vacuum probability P_0 and more generally the probability to emit any finite number of photons vanishes, because in the expression (1.287) for P_0 the integral $\int_{|\mathbf{k}|<\Delta} d\omega_k$ diverges at $\mathbf{k} = 0$ (a so-called infrared divergence). Such currents are realistic, they typically occur in Bremsstrahlung (hence the name 'infrared catastrophe' for the infrared divergence). However, the expression (1.291) for the probability to emit any number of photons is still finite. In particular, this is the relevant expression if we do not observe any photon with energy greater than Δ and do not try to measure photons with energy smaller than Δ. More information can be found in Itzykson and Zuber sects. 1-3-2 and 4-1-2, Bjorken and Drell sect. 17.10.

1.16 Locality

We started from an action S which has nice invariance properties and is local: it has the form $S = \int d^4x\, \mathcal{L}(x)$ where $\mathcal{L}(x)$ is a Lorentz scalar which depends on the fields at x and in the immediate neighborhood of x (through the derivatives). This leads to covariant and local classical equations equations of motion. No signals can travel faster than the velocity of light. Upon quantization we have ended up with non-Lorentz and nongauge invariant expressions which furthermore look terribly nonlocal: the projector

$$P^T_{mn}(\mathbf{x}-\mathbf{y}) = \delta_{mn}\delta(\mathbf{x}-\mathbf{y}) + \partial_m\partial_n \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \tag{1.292}$$

drops off very slowly for large separation $|\mathbf{x}-\mathbf{y}|$. This noncovariance and nonlocality is due to the choice of gauge, the Coulomb gauge. The advantage of the Coulomb gauge is that it focuses on the physical degrees of freedom of the electromagnetic field, rather than gauge degrees of freedom, and leads quickly to results in the simple situation we are dealing with, in which the field is coupled only to an external current. We shall discuss Lorentz invariance more fully in a later chapter and content ourselves for the moment with the fact that gauge invariant quantities turned out to be Lorentz invariant. For instance, the vacuum amplitude is Lorentz invariant.

An important expression of locality and Lorentz invariance is the following. Two observables $O_{1,2}$ associated with compact spacetime regions $R_{1,2}$ ('local observables') commute, when all points $x_1 \in R_1$ are spacelike to all points $x_2 \in R_2$. In the standard lore of quantum mechanics observables correspond to measurements, and measurements in spacelike separated regions should not be able to influence each other. Observables have to be gauge invariant. An example is given by the field strength $F_{\mu\nu}(x)$. Locality is expressed by

$$[F_{\kappa\lambda}(x), F_{\mu\nu}(y)] = 0, \quad (x-y)^2 > 0. \tag{1.293}$$

This is indeed the case as will now be shown for the case of vanishing external current.

Using the expansion (recall $e^\mu(k,\lambda) = (\mathbf{e}(\mathbf{k},\lambda), 0)$)

$$A^\mu(\mathbf{x},t) = \sqrt{\hbar}\sum_\lambda \int d\omega_k [e^{ikx}\, e^\mu(k,\lambda) a(\mathbf{k},\lambda) + h.c.], \tag{1.294}$$

gives

$$[A^\lambda(x), A^\nu(y)] = \hbar \int d\omega_k\, (e^{ik(x-y)} - e^{-ik(x-y)}) P_C^{\lambda\nu}(k), \tag{1.295}$$

where (recall $n^\mu = \delta_{\mu,0}$)

$$\begin{aligned} P_C^{\mu\nu} &= \sum_\lambda e^\mu(k,\lambda) e^\nu(k,\lambda)^* \\ &= g^{\mu\nu} - \frac{k^\mu k^\nu + (kn)(k^\mu n^\nu + n^\mu k^\nu)}{(kn)^2}. \end{aligned} \tag{1.296}$$

Working out the derivatives in $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ we get

$$\begin{aligned} [F^{\kappa\lambda}(x), F^{\mu\nu}(y)] &= \hbar \int d\omega_k\, \{e^{ik(x-y)}[k^\kappa k^\mu P_C^{\lambda\nu}(k) - k^\lambda k^\mu P_C^{\kappa\nu}(k) \\ &\quad - k^\kappa k^\nu P_C^{\lambda\mu}(k) + k^\lambda k^\nu P_C^{\kappa\mu}(k)] - (k \to -k)\}. \end{aligned} \tag{1.297}$$

Now the operation of the curl in $F^{\kappa\lambda}$ projects to zero any 'longitudinal' part $\propto k^\lambda$ in $P_C^{\lambda\nu}$, such that only the $g^{\lambda\nu}$ part of $P_C^{\lambda\nu}$ contributes. In position space we can then write

$$\begin{aligned} [F_{\kappa\lambda}(x), F_{\mu\nu}(y)] &= -\hbar(\partial_\kappa\partial_\mu g_{\lambda\nu} - \partial_\lambda\partial_\mu g_{\kappa\nu} \\ &\quad - \partial_\kappa\partial_\nu g_{\lambda\mu} + \partial_\lambda\partial_\nu g_{\kappa\mu})\, i\Delta(x-y), \end{aligned} \tag{1.298}$$

$$\Delta(x-y) = -i \int d\omega_k\, (e^{ik(x-y)} - e^{-ik(x-y)}). \tag{1.299}$$

The (generalized) function $\Delta(x)$ has the following properties:

- $\Delta(x)$ is Lorentz invariant, $\Delta(\Lambda x) = \Delta(x)$,
- $\Delta(x) = 0$ for $x^0 = 0$, $\mathbf{x} \neq 0$.

Since $x = (\mathbf{x}, 0)$ is spacelike and $\Delta(x)$ is Lorentz invariant it follows that it vanishes for general spacelike distances,

$$\Delta(x-y) = 0, \quad (x-y)^2 > 0. \tag{1.300}$$

It is also interesting to note that $\Delta(x)$ is the solution of $\partial^2\Delta(x) = 0$ with initial conditions $\Delta(x) = 0$, $\partial_0\Delta(x) = -\delta(\mathbf{x})$ at $x^0 = 0$.)

Consequently the field strengths and all local observables that can be made out of these have the locality property (1.293).

1.17 Appendix

Eq. (1.238) is intuitively clear from the representation of the interaction picture evolution operator in terms of time ordered products of H_J. We elaborate here further on this. For clarity we set $\hbar = 1$. We have

$$Z'(J) = \langle 0|T\, e^{i\int_{-\infty}^{\infty} d^4x\, J_m(x) A_m(x)}|0\rangle. \tag{1.301}$$

The time ordered product

$$T\, A_{m_1}(x_1)\cdots A_{m_n}(x_n) \tag{1.302}$$

is completely symmetric in the interchange of labels $x_i, m_i \leftrightarrow x_j, m_j$. Hence, $Z'(J)$ is given by

$$Z' = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n\, \langle 0|T\, A_{m_1}(x_1)\cdots A_{m_n}(x_n)|0\rangle\, J_{m_1}(x_1)\cdots J_{m_n}(x_n). \tag{1.303}$$

Then

$$\begin{aligned} \delta\, Z' \;=\;& \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \cdots d^4x_n\, \langle 0|T\, A_{m_1}(x_1)\cdots A_{m_n}(x_n)|0\rangle \\ & J_{m_1}(x_1)\cdots J_{m_{n-1}}(x_{n-1}) \\ & n\, \delta J_{m_n}(x_{m_n}). \end{aligned} \tag{1.304}$$

Relabeling $n-1 \to n$ and using the combinatorics of $e^{a+b} = e^a\, e^b$ we can rewrite this in various ways

$$\begin{aligned} \delta\, Z' \;=\;& \int d^4x\, i\delta J_m(x) \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4y_1 \cdots d^4y_n \\ & \langle 0|T\, A_m(x)\, A_{k_1}(y_1)\cdots A_{k_n}(y_n)|0\rangle\, J_{k_1}(y_1)\cdots J_{k_n}(y_n) \end{aligned} \tag{1.305}$$

$$\begin{aligned} =\;& \int d^4x\, i\delta J_m(x)\, \langle 0|T\, [e^{i\int_{x^0}^{\infty} d^4y\, J_k(y) A_k(y)}\, A_m(x) \\ & e^{i\int_{-\infty}^{x^0} d^4z\, J_l(z) A_l(z)}]|0\rangle \end{aligned} \tag{1.306}$$

$$=\; \int d^4x\, i\delta J_m(x)\, \langle 0|U_J'(\infty, x^0)\, A_m(x)\, U_J'(x^0, -\infty)|0\rangle. \tag{1.307}$$

From (1.305) we get furthermore the useful formula

$$\frac{\delta Z'}{i\delta J_m(x)} = \langle 0|T\, A_m(x)\, e^{i\int_{-\infty}^{\infty} d^4y\, J_n(y) A_n(y)}|0\rangle, \tag{1.308}$$

and repeating the differentiations,

$$\frac{\delta^n Z'}{i\delta J_{m_1}(x_1)\cdots i\delta J_{m_n}(x_n)} = \langle 0|T\, A_{m_1}(x_1)\cdots A_{m_n}(x_n)\, e^{i\int_{-\infty}^{\infty} d^4y\, J_n(y) A_n(y)}|0\rangle. \tag{1.309}$$

1.18 Problems

1. The identity
$$\epsilon^{\kappa\lambda\mu\nu}\,\partial_\lambda F_{\mu\nu} = 2\epsilon^{\kappa\lambda\mu\nu}\,\partial_\lambda\partial_\mu A_\nu = 0 \tag{1.310}$$
implies the homogeneous Maxwell equations
$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{1.311}$$
Use the homogeneous and inhomogenous Maxwell equations to derive the divergence relation for the energy-momentum tensor (1.48).

2. The formulas (1.84) also apply to a finite number of degrees of freedom. Check explicitly the case $n = 2$ (e.g. for two degrees of freedom $\alpha = 1, 2$).

3. Verify
$$\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\, \frac{1}{\mathbf{k}^2}, \tag{1.312}$$
by applying the laplacian Δ to left and right hand side and using (1.61).

4. Verify that in the infinite volume limit the formula (1.100) for the transverse projector goes over in
$$\begin{aligned} P^T_{mn}(\mathbf{x},\mathbf{y}) &= \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left(\delta_{mn} - \frac{k_m k_n}{\mathbf{k}^2}\right) \\ &= \delta_{mn}\delta(\mathbf{x}-\mathbf{y}) + \partial_m\partial_n \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}. \end{aligned} \tag{1.313}$$
Notice that $P^T_{mn}(\mathbf{x},\mathbf{y}) = P^T_{nm}(\mathbf{y},\mathbf{x})$.

5. Verify the other commutation relations in (1.111).

6. To obtain the expressions (1.115) for the hamiltonian, we insert (1.101) into (1.89), using (1.105):
$$\begin{aligned} H &= \frac{1}{V^2}\sum_{\mathbf{k},\mathbf{l}} \frac{1}{4k^0 l^0} \int d^3x\, \frac{1}{2}\,[\\ &\quad (-ik^0 e^{i\mathbf{k}\mathbf{x}}\, a_m(\mathbf{k}) + ik^0 e^{-i\mathbf{k}\mathbf{x}}\, a_m(\mathbf{k})^\dagger)(-il^0 e^{i\mathbf{l}\mathbf{x}}\, a_m(\mathbf{l}) + il^0 e^{-i\mathbf{l}\mathbf{x}}\, a_m(\mathbf{l})^\dagger) \\ &\quad + (e^{i\mathbf{k}\mathbf{x}}\, a_m(\mathbf{k}) + e^{-i\mathbf{k}\mathbf{x}}\, a_m(\mathbf{k})^\dagger)\, \mathbf{l}^2\, (e^{i\mathbf{l}\mathbf{x}}\, a_m(\mathbf{l}) + e^{-i\mathbf{l}\mathbf{x}}\, a_m(\mathbf{l})^\dagger)]. \end{aligned} \tag{1.314}$$
The integration sets $\mathbf{l} = \pm\mathbf{k}$ and the aa and $a^\dagger a^\dagger$ terms cancel ($k^0 = |\mathbf{k}|$), leaving
$$H = \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2k^0}\, k^0\, \frac{1}{2}\,[a_m(\mathbf{k})a_m(\mathbf{k})^\dagger + a_m(\mathbf{k})^\dagger a_m(\mathbf{k})]. \tag{1.315}$$

We can now convert to $a(\mathbf{k},\lambda)$ or use the commutation relation (1.111) directly with (sum over m) $P^T_{mm}(\mathbf{k}) = 2 = \sum_\lambda$ to put $a^\dagger$ to the left of a,

$$a_m(\mathbf{k})a_m(\mathbf{k})^\dagger = a_m(\mathbf{k})^\dagger a_m(\mathbf{k}) + 2k^0 V \sum_\lambda . \tag{1.316}$$

This gives (1.89) after converting to $a(\mathbf{k},\lambda)$. This calculation of the hamiltonian is basically the same as for the one dimensional harmonic oscillator. The calculation of the momentum operator (1.90) proceeds in similar fashion,

$$\begin{aligned}
\mathbf{P} &= -\frac{1}{V^2}\sum_{\mathbf{k},\mathbf{l}} \frac{1}{4k^0 l^0} \int d^3x\, [\\
&\quad (-ik^0 e^{i\mathbf{k}\mathbf{x}}\, a_m(\mathbf{k}) + ik^0 e^{-i\mathbf{k}\mathbf{x}}\, a_m(\mathbf{k})^\dagger)\, i\mathbf{l}\, (e^{i\mathbf{l}\mathbf{x}}\, a_m(\mathbf{l}) - e^{-i\mathbf{l}\mathbf{x}}\, a_m(\mathbf{l})^\dagger) \\
&= \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2k^0}\,\frac{1}{2}\mathbf{k}\,[a_m(\mathbf{k})a_m(-\mathbf{k}) + a_m(\mathbf{k})^\dagger a_m(-\mathbf{k})^\dagger \\
&\quad + a_m(\mathbf{k})a_m(\mathbf{k})^\dagger + a_m(\mathbf{k})^\dagger a_m(\mathbf{k})], \qquad (1.317) \\
&= \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{2k^0}\,\mathbf{k}\, a_m(\mathbf{k})^\dagger a_m(\mathbf{k}) \qquad (1.318)
\end{aligned}$$

($\mathbf{k}$ is odd under $\mathbf{k} \to -\mathbf{k}$, such that accompanying factors even under $\mathbf{k} \to -\mathbf{k}$ do not contribute). Expressing the result in terms of $a(\mathbf{k},\lambda)$ gives (1.90).

7. Derive (1.135) by normal ordering, i.e. interchanging the creation and annihilation operators (using their commutation relations) such that all a's stand to the right of all $a^\dagger$'s, and the fact that any a gives zero on $|0\rangle$ and and any $a^\dagger$ gives zero on $\langle 0|$.

8. Calculate $\langle\varphi|A_m(x)A_n(y)|\varphi\rangle$, for example by inserting intermediate states, and verify the expression (1.151) for the expectation value $\langle\varphi|T^{\mu\nu}(x)|\varphi\rangle$.

9. Recall that $(g^{-1})_{\mu\nu} = g^{\mu\nu}$ and verify

$$gg^{-1} = 1 \to \delta g g^{-1} + g\delta g^{-1} = 0 \Longrightarrow \delta g^{-1} = -g^{-1}\delta g\, g^{-1}, \tag{1.319}$$

or

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta}\, \delta g_{\alpha\beta}, \tag{1.320}$$

and

$$\begin{aligned}
\det g &= \frac{1}{4!}\epsilon^{\mu_1\cdots\mu_4}\epsilon^{\nu_1\cdots\nu_4} g_{\mu_1\nu_1}\ldots g_{\mu_4\nu_4}, \qquad (1.321) \\
\delta \det g &= \frac{1}{3!}\epsilon^{\mu_1\cdots\mu_4}\epsilon^{\nu_1\cdots\nu_4} g_{\mu_1\nu_1}\ldots g_{\mu_3\nu_3}\,\delta g_{\mu_4\nu_4} \\
&= (\det g)(g^{-1})_{\mu_4\nu_4}\,\delta g_{\mu_4\nu_4} = (\det g) g^{\mu\nu}\,\delta g_{\mu\nu}, \qquad (1.322)
\end{aligned}$$

and

$$\delta\sqrt{-\det g} = \frac{1}{2}\sqrt{-\det g}\, g^{\mu\nu}\delta g_{\mu\nu}, \tag{1.323}$$

and finally (1.140).

10. To verify the step leading to (1.191), evaluate the commutator $[H, \Pi_m(\mathbf{x}, t)]$ using the form (1.89) for H; recall $[ab, c] = a[b, c] + [a, c]b$. This also verifies that the operator field equations

$$\Box A_m(x) = 0 \tag{1.324}$$

follow from the canonical quantization procedure.

11. Consider Green functions of the operator $-\Box + \mu^2$,

$$(-\Box + \mu^2)G(x - y) = \delta(x - y). \tag{1.325}$$

The Feynman propagator

$$G(x - y) = \int \frac{d^4\, k}{(2\pi)^4}\, e^{ik(x-y)} \frac{1}{\mu^2 + k^2 - i\epsilon} \tag{1.326}$$

is a solution to (1.325) with Feynman boundary conditions. Using contour integration verify that

$$\frac{1}{\mu^2 + k^2 - i\epsilon} \to \frac{1}{\mu^2 + \mathbf{k}^2 - (k_0 + i\epsilon)^2} \tag{1.327}$$

corresponds to retarded boundary conditions.

Chapter 2

Interactions with scalar fields

We introduce scalar fields in this chapter and coupled these to the electromagnetic field. The resulting field theory describes the interaction of photons with charged particles of spin zero. The vacuum amplitude is elegantly summarized in terms of the effective action and the formalism leads naturally to Feynman diagrams. The equations of motion are translated into the Dyson-Schwinger equations for the effective action, and the perturbative solution of these equations leads to the diagrammatic loop expansion, the semiclassical expansion in powers of $\hbar$. From the vacuum amplitude we obtain the scattering amplitude and the diagrams give an intuitive picture of scattering in terms of virtual particle exchange.

From now on we use units in which Planck's constant $\hbar = 1$ and the velocity of light $c = 1$. Then the dimensions of various quantities are like [mass] = [energy] = [momentum] = $[A_\mu]$ = [length^{-1}] = [time^{-1}]. The action is dimensionless. To convert to ordinary units we use appropriate powers of $\hbar$ and c. A particularly useful combination is $\hbar c = 197.3 \approx 200$ MeV fm, where fm (femto meter or Fermi) denotes the unit of length 10^{-13} cm. For example a mass m of 200 MeV corresponds to a length $1/m$ of about 1 fm. The the unit of electromagnetic charge $e \approx 0.30$, which follows from the fine structure constant $\alpha = e^2/(4\pi) \approx 1/137$.

2.1 Free scalar field

We have seen that the quanta of the electromagnetic field can be interpreted as particles, the photons, which can occur in two spin states corresponding to the two independent polarization vectors $\mathbf{e}(\mathbf{k}, \lambda)$. It is now natural to look for other field systems for the description of other kinds of particles. The simplest is the scalar field, the quantization of which leads to spinless particles. Having gone through the quantization of the more complicated case of the electromagnetic field, the corresponding formulas for the scalar field are a pleasant simplification. We urge the reader to go through the formulas in the preceding chapter, drop the polarization vectors and vector index m on A^m, and obtain the corresponding formulas for the scalar field. We summarize here some of the relevant formulas.

The action for the free scalar field $\varphi(x)$ is given by

$$\begin{aligned} S &= \int d^4x\, \mathcal{L}(x) = \int dx^0\, L, && (2.1)\\ \mathcal{L}(x) &= -\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \frac{1}{2}m^2\varphi(x)^2 - \tau_0, && (2.2)\\ L(\varphi,\dot\varphi) &= \int d^3x\,[\frac{1}{2}\,\dot\varphi^2 - \frac{1}{2}\,(\nabla\varphi)^2 - \frac{1}{2}\,m^2\varphi^2 - \tau_0], && (2.3) \end{aligned}$$

where we have included a bare cosmological constant τ_0. The equation of motion including an external scalar source $J(x)$ is given by

$$\begin{aligned} 0 &= \frac{\delta}{\delta\varphi(x)}\,(S + \int d^4x'\, J(x')\varphi(x')) \\ &= (\Box - m^2)\varphi(x) + J(x). && (2.4) \end{aligned}$$

For $J = 0$ this equation is known as the Klein-Gordon equation. The energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + g^{\mu\nu}\mathcal{L} \tag{2.5}$$

is conserved for vanishing source $J = 0$ as a consequence of the equation of motion, $\partial_\mu T^{\mu\nu} = 0$.

The canonical conjugate of the field $\varphi(x)$ is denoted by $\pi(x)$ and can be found by making a mode expansion, as done for the electromagnetic field. It can also be defined by generalizing the partial derivative (1.8) to a functional derivative,

$$\pi(\mathbf{x}) = \frac{\delta}{\delta\dot\varphi(\mathbf{x})}L(\varphi,\dot\varphi) = \dot\varphi(\mathbf{x}), \tag{2.6}$$

where the time dependence is left implicit. The hamiltonian with source J takes the form

$$H_{\rm tot} = H - \int d^3x\, J\varphi, \tag{2.7}$$

where

$$\begin{aligned} H &= \int d^3x\, \pi\partial_0\varphi - L = \int d^3x\, T^{00} && (2.8)\\ &= \int d^3x\,[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \tau_0] && (2.9) \end{aligned}$$

is the hamiltonian for $J = 0$.

After quantization the canonical commutation relations at time $x^0 = 0$ are given by

$$\begin{aligned} [\varphi(\mathbf{x}), \pi(\mathbf{y})] &= i\delta(\mathbf{x} - \mathbf{y}), && (2.10)\\ [\pi(\mathbf{x}), \pi(\mathbf{y})] &= [\varphi(\mathbf{x}), \varphi(\mathbf{y})] = 0. && (2.11) \end{aligned}$$

The creation and annihilation operators appear in the canonical variables according to

$$\begin{aligned}
\varphi(\mathbf{x}) &= \int d\omega_p \, [e^{i\mathbf{p}\mathbf{x}} a(\mathbf{p}) + e^{-i\mathbf{p}\mathbf{x}} a(\mathbf{p})^\dagger], & (2.12)\\
\pi(\mathbf{x}) &= \int d\omega_p \, [-ip^0 e^{i\mathbf{p}\mathbf{x}} a(\mathbf{p}) + ip^0 e^{-i\mathbf{p}\mathbf{x}} a(\mathbf{p})^\dagger], & (2.13)\\
d\omega_p &= \frac{d^3p}{(2\pi)^3 2p^0}, \quad p^0 = \sqrt{m^2 + \mathbf{p}^2}, & (2.14)
\end{aligned}$$

and satisfy the commutation relations

$$\begin{aligned}
[a(\mathbf{p}), a(\mathbf{p}')^\dagger] &= 2p^0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}'), & (2.15)\\
[a(\mathbf{p}), a(\mathbf{p}')] &= [a(\mathbf{p})^\dagger, a(\mathbf{p}')^\dagger] = 0. & (2.16)
\end{aligned}$$

The energy momentum operator of the source free field can be written as

$$P^\mu = \int d\omega_p \, a(\mathbf{p})^\dagger a(\mathbf{p}) \, p^\mu \tag{2.17}$$

where we adjusted the bare cosmological constant τ_0 such that the renormalized cosmological constant is zero and the energy of the vacuum is zero. The creation operators create spin zero particles out of the vacuum $|0\rangle$ with four-momentum p,

$$P^\mu |p\rangle = p^\mu |p\rangle, \quad |p\rangle = a(\mathbf{p})^\dagger |0\rangle, \tag{2.18}$$

and similar for multi particle states. The mass of the particles is m, as can be seen from $p_\mu p^\mu = -m^2$.

The vacuum amplitude is given by

$$\begin{aligned}
Z(J) &= \exp[i\frac{1}{2} \int d^4x d^4y \, J(x) G(x-y) J(y)], & (2.19)\\
&= \exp[iS(\varphi^{(c)}) + i \int d^4x \, J(x) \varphi^{(c)}(x)], & (2.20)
\end{aligned}$$

with the propagator

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{ip(x-y)} \, \frac{1}{m^2 + p^2 - i\epsilon} \tag{2.21}$$

implementing Feynman boundary conditions in time for the classical field

$$\varphi^{(c)}(x) = \int d^4y \, G(x-y) J(y). \tag{2.22}$$

Finally, the amplitudes for emission and absorption of particles by the source are given by

$$\langle p|0\rangle^J = iJ(p), \quad \langle 0|p\rangle^J = iJ(p)^*, \tag{2.23}$$

to leading order in J.

2.2 Yukawa potential

The Coulomb potential decribes the interaction energy of two static (time independent) charge distributions. The analogue in scalar field theory is the Yukawa potential. The static classical field is the solution of the equation

$$(-\Delta + m^2)\varphi^{(c)}(\mathbf{x}) = J(\mathbf{x}), \tag{2.24}$$

which can be solved with the help of the static Green function

$$\begin{aligned}
\varphi^{(c)}(\mathbf{x}) &= \int d^3y\, G_{\text{stat}}(\mathbf{x}-\mathbf{y})J(\mathbf{y}), & (2.25)\\
(-\Delta + m^2)G_{\text{stat}}(\mathbf{x}-\mathbf{y}) &= \delta(\mathbf{x}-\mathbf{y}), & (2.26)\\
G_{\text{stat}}(\mathbf{x}-\mathbf{y}) &= \int \frac{d^3p}{(2\pi)^3}\,\frac{e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{m^2+\mathbf{p}^2}\\
&= \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}. & (2.27)
\end{aligned}$$

The last line follows by using spherical coordinates with $\mathbf{p}(\mathbf{x}-\mathbf{y}) = pr\cos\theta$, integrating first over angles,

$$\begin{aligned}
G(\mathbf{x}-\mathbf{y}) &= \frac{1}{(2\pi)^2}\int_0^\infty dp\,\frac{p^2}{m^2+p^2}\,\frac{2\sin pr}{pr}, & (2.28)\\
&= \frac{1}{4\pi r}\text{Re}\int_{-\infty}^{\infty}\frac{dp}{2\pi i}\,\frac{2p\,e^{ipr}}{m^2+p^2}, & (2.29)
\end{aligned}$$

and then over p using contour integration by closing the contour in the upper half of the complex p-plane. With a source of the form $J(\mathbf{x}) = J_1(\mathbf{x}) + J_2(\mathbf{x})$, substitution of $\varphi^{(c)}$ into (2.7) gives the energy

$$\begin{aligned}
E &= E_{11} + E_{22} + 2E_{12}, & (2.30)\\
E_{ij} &= -\frac{1}{2}\int d^3x d^3y\, J_i(\mathbf{x})\frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}J_j(\mathbf{y}). & (2.31)
\end{aligned}$$

Notice that the interaction energy $2E_{12}$ is negative when both J_1 and J_2 are positive.

The expression

$$\frac{e^{-mr}}{4\pi r} \tag{2.32}$$

is known as the Yukawa potential. It has the form of a screened Coulomb potential with screening length $1/m$. For distances $r \gg 1/m$ the interaction becomes negligible and $1/m$ is a measure of the range of the interaction. For $m \to 0$ we get the infinite range Coulomb potential. Other common names for $1/m$ are: the Compton wave length and the correlation length (by analogy with Statistical

Physics). The parameter m in the free scalar field action plays the dual role of particle mass and interaction range. Yukawa introduced the scalar field in the thirties to explain the nuclear forces. After some initial confusion (see e.g. Pais for a historical account) the spinless particles corresponding to this field were identified with the pions. The pion mass $m_\pi \approx 140$ MeV corresponds to an interaction range of $1/m_\pi \approx 200/140 = 1.4$ fermi.

2.3 Complex scalar field

Two fields $\varphi_\alpha(x)$, $\alpha = 1, 2$, describe two types of spinless particles. If they have the same mass, then the action

$$S = -\int d^4x\,(\frac{1}{2}\partial_\mu\varphi_\alpha\partial^\mu\varphi_\alpha + \frac{1}{2}m^2\varphi_\alpha\varphi_\alpha) \tag{2.33}$$

(where a summation is implied over repeated α) has a continous symmetry: it is invariant under SO(2) transformations, orthogonal rotations in two dimensions,

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \varphi'(x) = R\varphi(x). \tag{2.34}$$

We may think of φ_α as a vector in 'internal' space (' α-space' – as opposed to ordinary spacetime), which gets rotated by the matrix R. Writing

$$R = e^{-i\omega q} = \cos\omega - iq\sin\omega = 1 - i\omega q + \cdots, \tag{2.35}$$

$$q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad q^T = -q, \quad R^T = R^{-1}, \tag{2.36}$$

we see that q is the generator of these transformations.

The invariance of S under the continous SO(2) symmetry implies a local conservation law (Noether's theorem), which can be derived as follows. We make a small variation of φ that has the form of a spacetime dependent symmetry transformation,

$$\delta\varphi_\alpha(x) = -iq_{\alpha\beta}\varphi_\beta(x)\,\delta\omega(x). \tag{2.37}$$

If φ satisfies the equations of motion, then the action is stationary (we assume here $J = 0$):

$$\begin{aligned} 0 = \delta S &= -\int d^4x\,[\partial^\mu\varphi_\alpha\,(-iq_{\alpha\beta})\,\partial_\mu(\varphi_\beta\delta\omega) + m^2\varphi_\alpha\,(-iq_{\alpha\beta})\,\varphi_\beta\delta\omega] \\ &= \int d^4x\,\partial^\mu\varphi_\alpha\,iq_{\alpha\beta}\,\varphi_\beta\,\partial_\mu\delta\omega \qquad (2.38) \\ &\equiv \int d^4x\,j^\mu\partial_\mu\delta\omega = -\int d^4x\,(\partial_\mu j^\mu)\,\delta\omega. \qquad (2.39) \end{aligned}$$

Since the variations $\delta\omega$ are arbitrary we have the conservation of a current,

$$j^\mu = i\partial^\mu \varphi_\alpha\, q_{\alpha\beta}\, \varphi_\beta, \quad \partial_\mu j^\mu = 0. \tag{2.40}$$

The charge Q corresponding to the current is conserved,

$$Q = \int d^3x\, j^0(x), \quad \partial_0 Q = 0. \tag{2.41}$$

In the quantum theory $\varphi_\alpha(x)$ and $\pi_\alpha(x)$ satisfy the equal time commutation relations

$$[\varphi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)] = i\delta_{\alpha\beta}\, \delta(\mathbf{x}-\mathbf{y}), \tag{2.42}$$

with the other commutators vanishing. Then Q becomes an operator in Hilbert space,

$$Q = \int d^3x\, (-i\partial_0\varphi_\alpha\, q_{\alpha\beta}\, \varphi_\beta) = -i\int d^3x\, \pi_\alpha\, q_{\alpha\beta}\, \varphi_\beta, \tag{2.43}$$

and it is a generator of SO(2) transformations in the following sense

$$\begin{aligned} [Q, \varphi_\gamma(y)] &= -q_{\gamma\delta}\, \varphi_\delta(y), & (2.44)\\ e^{i\omega Q}\, \varphi_\alpha\, e^{-i\omega Q} &= \left(e^{-i\omega q}\right)_{\alpha\beta} \varphi_\beta = R_{\alpha\beta}\, \varphi_\beta. & (2.45) \end{aligned}$$

The second line can be checked by differentiating with respect to ω and integrating again,

$$\begin{aligned} F_\alpha(\omega) &= e^{i\omega Q}\, \varphi_\alpha\, e^{-i\omega Q}, \quad F_\alpha(0) = \varphi_\alpha, & (2.46)\\ \frac{d}{d\omega}F_\alpha(\omega) &= -iq_{\alpha\beta}\, F_\beta(\omega) \Rightarrow F_\alpha(\omega) = \left(e^{-i\omega q}\right)_{\alpha\beta} F_\beta(0). & (2.47) \end{aligned}$$

The eigenvectors of q define a basis in internal space,

$$e_1(\pm) = \frac{1}{\sqrt{2}}, \quad e_2(\pm) = \pm\frac{i}{\sqrt{2}}, \quad q_{\alpha\beta}\, e_\beta(\pm) = \pm e_\alpha(\pm), \tag{2.48}$$

and we can expand the classical φ_α in terms of charge eigenfields φ and φ^*,

$$\varphi_\alpha = \varphi\, e_\alpha(+) + \varphi^*\, e_\alpha(-), \quad \varphi = e_\alpha(+)^*\, \varphi_\alpha, \quad \varphi^* = e_\alpha(-)^*\, \varphi_\alpha. \tag{2.49}$$

In terms of the complex field φ the action takes the form

$$S = -\int d^4x\, (\partial_\mu\varphi^*\partial^\mu\varphi + m^2\varphi^*\varphi). \tag{2.50}$$

In the complex formalism we treat φ and φ^* as independent variables. For example, the equation of motion for φ is obtained by varying φ^* only,

$$\begin{aligned} 0 &= -\int d^4x\, (-\partial_\mu\partial^\mu\varphi + m^2\varphi)\, \delta\varphi^*, & (2.51)\\ 0 &= \frac{\delta S}{\delta\varphi^*(x)} = (\Box - m^2)\varphi(x). & (2.52) \end{aligned}$$

The SO(2) transformations now take the U(1) form (U(1) = group of unitary transformations in 1 dimension)

$$\varphi' = e^{-i\omega}\,\varphi, \quad \varphi'^* = e^{i\omega}\,\varphi^*, \tag{2.53}$$

under which the action (2.50) is clearly invariant.

In the quantum theory

$$\pi \equiv \pi_\varphi = \partial_0\varphi^\dagger, \quad \pi^\dagger \equiv \pi_{\varphi^*} = \partial_0\varphi, \tag{2.54}$$

and we have

$$[\varphi(\mathbf{x},t), \pi(\mathbf{y},t)] = [\varphi^\dagger(\mathbf{x},t), \pi^\dagger(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y}), \tag{2.55}$$

with the other commutators vanishing. The current takes the form

$$\begin{aligned} j^\mu &= i\partial^\mu[\varphi\, e_\alpha(+) + \varphi^\dagger\, e_\alpha(-)]\, q_{\alpha\beta}\, [\varphi\, e_\beta(+) + \varphi^\dagger\, e_\beta(-)] \\ &= -i\partial^\mu\varphi\,\varphi^\dagger + i\partial^\mu\varphi^\dagger\,\varphi, \end{aligned} \tag{2.56}$$

where we have been careful about the ordering of operators, using the real formulation as a starting point. In the real formulation there is no ordering ambiguity in the sense that e.g. for the charge density,

$$j^0(x) = -iq_{\alpha\beta}\,\pi_\alpha(x)\,\varphi_\beta(x) = -iq_{\alpha\beta}\,\varphi_\beta(x)\,\pi_\alpha(x), \tag{2.57}$$

where we used $\delta(\mathbf{0})q_{\alpha\beta}\,\delta_{\alpha\beta} = \delta(\mathbf{0})\mathrm{Tr}\,q = 0$. The commutation relations of Q with φ and $\varphi^\dagger$ read

$$[Q, \varphi(x)] = -\varphi(x), \quad [Q, \varphi(x)^\dagger] = \varphi(x)^\dagger. \tag{2.58}$$

In more detail we have

$$\varphi(x) = \frac{1}{\sqrt{2}}\,[\varphi_1(x) - i\varphi_2(x)], \tag{2.59}$$

and we can write

$$\begin{aligned} \varphi(x) &= \int d\omega_p\,[e^{ipx}\,a(p,+) + e^{-ipx}\,a(p,-)^\dagger], &(2.60)\\ a(p,\pm) &= \frac{1}{\sqrt{2}}\,[a_1(p) \mp ia_2(p)], &(2.61)\\ Q &= \int d\omega_p\,[a(p,+)^\dagger a(p,+) - a(p,-)^\dagger a(p,-)]. &(2.62) \end{aligned}$$

We see that Q counts the number of '$+$' quanta minus the number of '$-$' quanta. By convention we call the '$+$' quanta particles and the '$-$' quanta antiparticles, i.e. the one particle charge eigenstates $|p\pm\rangle$ of Q are interpreted as particles ($Q = +1$) and antiparticles ($Q = -1$),

$$\begin{aligned} a(p,\pm)^\dagger|0\rangle &\equiv |p\pm\rangle, &(2.63)\\ Q|p\pm\rangle &= \pm|p\pm\rangle. &(2.64) \end{aligned}$$

2.4 Coupling to the electromagnetic field

The complex scalar field system has a global U(1)≃SO(2) invariance, by which we mean that the angle ω in the transformation

$$\varphi'(x) = e^{i\omega}\varphi(x), \quad \varphi'(x)^* = e^{-i\omega}\,\varphi'(x)^*, \tag{2.65}$$

does not depend on the spacetime coordinate x. We can interpret these transformations also as passive transformations of the coordinate system in internal space. It is natural to ask if the reference system that picks out the real and imaginary parts of φ, or equivalently its 1 and 2 components, has to be globally defined. For example do we have to choose it the same here in Amsterdam now as on the Moon five years later? It is possible to allow for arbitrary local transformations of the internal coordinate frame, with an action invariant under U(1) transformations with angle $\omega(x)$ depending on spacetime. To achieve this we need to compensate the noninvariance of the derivative terms in the lagrangian,

$$\mathcal{L}(x) = -\partial_\mu\varphi(x)^*\partial^\mu\varphi(x) - m^2\varphi(x)^*\varphi(x), \tag{2.66}$$

because under a local transformation

$$\varphi'(x) = e^{i\omega(x)}\,\varphi(x), \tag{2.67}$$

the term $m^2\varphi(x)^*\varphi(x)$ is invariant but the derivative transforms in an inhomogenous and noncovariant way

$$\partial_\mu\varphi'(x) = \partial_\mu[e^{i\omega(x)}\,\varphi(x)] = e^{i\omega(x)}\,[\partial_\mu\varphi(x) + i\partial_\mu\omega(x)\varphi(x)]. \tag{2.68}$$

Instead, a covariant derivative $D_\mu\varphi$ transforming as

$$D'_\mu(x)\varphi'(x) = e^{i\omega(x)}\,D_\mu(x)\varphi(x) \tag{2.69}$$

would allow for the construction of an invariant lagrangian

$$\mathcal{L}(x) = -[D_\mu(x)\varphi(x)]^* D^\mu(x)\varphi(x) - m^2\varphi(x)^*\varphi(x), \tag{2.70}$$

The well known construction of the covariant derivative uses the invariance of the electromagnetic field system under the gauge transformation

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\omega(x), \tag{2.71}$$

where e is an arbitrary constant. The form

$$D_\mu(x) = \partial_\mu - ieA_\mu(x), \tag{2.72}$$

has the required property: under the combined gauge transformation (2.67), (2.71),

$$\begin{aligned} D'_\mu(x)\,\varphi'(x) &= [\partial_\mu - ieA'_\mu(x)]\,\varphi'(x) \\ &= [\partial_\mu - ieA_\mu(x) - i\partial_\mu\omega(x)]\,e^{i\omega(x)}\,\varphi(x) \\ &= e^{i\omega(x)}\,[\partial_\mu - ieA_\mu(x)]\,\varphi(x) \\ &= e^{i\omega(x)}\,D_\mu(x)\,\varphi(x). \end{aligned} \tag{2.73}$$

A derivative involves the comparison of fields at infinitesimally close points in spacetime. The electromagnetic potentials play the role of a connection, which is used in comparing ('connecting') the orientations of the internal spaces at these infinitesimally close points.

The classical action for the combined electromagnetic and scalar field system is now given by

$$S = S_A + S_{A\varphi}, \tag{2.74}$$

$$S_A = -\int d^4x\,\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.75}$$

$$S_{A\varphi} = -\int d^4x\,[(D_\mu\,\varphi)^*\,D^\mu\,\varphi + m^2\varphi^*\varphi] \tag{2.76}$$

$$\begin{aligned} &= -\int d^4x\,[\partial_\mu\varphi^*\partial^\mu\varphi + m^2\varphi^*\varphi \\ &\quad + e(i\varphi^*\,\partial_\mu\varphi - i\partial_\mu\varphi^*\,\varphi)A^\mu + e^2\varphi^*\varphi\,A_\mu A^\mu]. \end{aligned} \tag{2.77}$$

In the formalism using real fields the action $S_{A\varphi}$ reads

$$S_{A\varphi} = -\int d^4x\,[\frac{1}{2}(D_\mu\varphi)^T\,D^\mu\varphi + \frac{1}{2}m^2\varphi^T\varphi], \tag{2.78}$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad D_\mu\varphi = (\partial_\mu - ieqA_\mu)\,\varphi. \tag{2.79}$$

We see in (2.77) the appearence of terms of cubic and quartic order in the fields. These are called interaction terms, since free field systems (including external sources) have only terms at most quadratic in the fields. The parameter e is called a coupling constant, since it governs the strength of the interactions. We can use this theory for the description of charged pions $\pi^\pm$ in an external electromagnetic potential A_μ. Then it can be shown that e is the elementary unit of charge, as suggested by the notation. This identification will be made on the basis of a scattering experiment (cf. (2.251)).

The theory is still invariant under the global U(1) transformation (2.65) implying the gauge invariant conserved current

$$j^\mu = i(D^\mu\varphi)^*\varphi - i\varphi^*D^\mu\varphi. \tag{2.80}$$

It is furthermore invariant under the discrete transformation

$$\varphi' = \varphi^*, \quad \varphi'^* = \varphi, \quad A'_\mu = -A_\mu, \tag{2.81}$$

which changes the sign of the charge $Q = \int d^3x\, j^0$ and is therefore called charge conjugation. In the real field formalism this transformation is a reflection in internal space,

$$\varphi'_1 = \varphi_1, \quad \varphi'_2 = -\varphi_2, \quad \text{or} \quad \varphi' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \varphi. \tag{2.82}$$

As far as φ is concerned this transformation completes SO(2) into O(2), the orthogonal group in two dimensions including reflections.

The quantization of the complete coupled φ–A field theory in the Coulomb gauge is straightforward but cumbersome. We shall not go through the details (see e.g. Bjorken & Drell II for quantization in the Coulomb gauge), but list a number of noteworthy points:

1. Similar to the case of the cosmological constant, it turns out that the parameters we start out with in the formulation of the theory – the bare parameters – are not equal to the parameters we measure – the renormalized parameters. We therefore make the replacement in the action

   $$e \to e_0, \quad m^2 \to \mu_0^2. \tag{2.83}$$

   Furthermore, it turns out that we need a gauge invariant bare self coupling of the form $\lambda_0(\varphi^*\varphi)^2$ in order to be able to cancel a type of infinities. The renormalized and parameters e, m and λ are then functions of the bare e_0, μ_0^2 and λ_0 and the choice of regularization.

   So the quantum theory will be based on the A–φ action

   $$S_{A\varphi} = -\int d^4x\, \{[(\partial^\mu - ie_0A^\mu)\varphi]^* \,(\partial_\mu - ie_0A_\mu)\varphi + \mu_0^2\varphi^*\varphi + \lambda_0(\varphi^*\varphi)^2 + \tau_0\}, \tag{2.84}$$

   where we have put the bare cosmological constant in $S_{A\varphi}$; the action S_A remains unchanged.

2. The canonical conjugate of φ involves A_0,

   $$\pi = \frac{\delta L}{\delta \partial_0 \varphi} = [D_0\varphi]^* = \partial_0\varphi^* + ie_0A_0\varphi^*, \tag{2.85}$$

   and similar for π^*. In the canonical formalism we have to express $\partial_0\varphi$ and $\partial_0\varphi^*$ in terms of π and π^*.

3. The canonical equal time commutation relations in the quantum theory are unchanged, e.g. at $t = 0$,

$$
\begin{aligned}
[\varphi(\mathbf{y}), \pi(\mathbf{x})] &= [\varphi(\mathbf{y})^\dagger, \pi(\mathbf{x})^\dagger] = i\delta(\mathbf{x} - \mathbf{y}), && (2.86)\\
[A_m(\mathbf{x}), \Pi_n(\mathbf{y})] &= iP^T_{mn}(\mathbf{x} - \mathbf{y}), && (2.87)\\
[\pi(\mathbf{x}), A_m(\mathbf{y})] &= [\varphi(\mathbf{x}), A_m(\mathbf{y})] = [\pi(\mathbf{x}), \Pi_m(\mathbf{y})] = [\varphi(\mathbf{x}), \Pi_m(\mathbf{y})] \\
&= \cdots = 0. && (2.88)
\end{aligned}
$$

   The equal time commutators between canonical scalar field and electromagnetic variables vanish according to the canonical rules.

4. The current in Maxwell's equations contributed by the scalar field is given by

$$e_0 j^\mu = ie_0[(D^\mu\varphi)^\dagger \varphi - \varphi^\dagger D_\mu \varphi]. \tag{2.89}$$

   The corresponding charge density

$$e_0 j^0 = -ie_0(\pi\varphi - \pi^\dagger\varphi^\dagger) \tag{2.90}$$

   is now an operator and therefore also

$$A_0(\mathbf{x}, t) = e_0 \int d^3y \, \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \, j^0(\mathbf{y}, t), \tag{2.91}$$

   and the Coulomb energy

$$H_C(t) = e_0^2 \frac{1}{2} \int d^3x d^3y \, \frac{j^0(\mathbf{x}, t)\, j^0(\mathbf{y}, t)}{4\pi|\mathbf{x} - \mathbf{y}|}. \tag{2.92}$$

5. Charge conjugation interchanges particles and antiparticles, as can be seen from (2.81), (2.60).

2.5 Equation for the vacuum amplitude in φ^4 theory

In the following we shall illustrate some derivations with a system that is simpler than scalar electrodynamics, the φ^4 theory. Its classical action is given by

$$S(\varphi) = -\int d^4x \, (\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4 + \tau), \tag{2.93}$$

where φ is a real scalar field. The hamiltonian

$$H = \int d^3x \, (\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4 + \tau), \tag{2.94}$$

Figure 2.1: Classical energy density of the φ^4 theory, for $\mu^2 > 0$ (a) and for $\mu^2 < 0$ (b).

can be seen as an infinite number of coupled anharmonic oscillators. There is also an analogy with the three dimensional Ising model, in which φ is an average over Ising spins in a small volume d^3x. The classical ground state has $\pi^2 = (\nabla\varphi)^2 = 0$ and minimal energy density

$$U = \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4 + \tau. \tag{2.95}$$

The function U is sketched in fig. 2.1. Since we assume the energy to be bounded from below, $\lambda > 0$. For $\mu^2 > 0$, the ground state is at $\varphi = 0$, while for $\mu^2 < 0$ there are two mimima at

$$\varphi = \pm\sqrt{\frac{-\mu^2}{\lambda}} \tag{2.96}$$

It follows that for negative μ^2 the symmetry $\varphi(x) \to -\varphi(x)$ is broken in the ground state, and one speaks of spontaneous symmetry breaking. This is analogous to the phenomenon of spontaneous magnetization in the Ising model. To single out a definite ground state we can add a small term to the action which breaks the symmetry $\varphi \to \varphi$ explicitly,

$$\Delta S = \int d^4x\, \epsilon\varphi(x). \tag{2.97}$$

We see that ϵ plays the role of a constant external field in the Ising model. In our present terminology ϵ can be interpreted as a constant external source $J(x) = \epsilon$.

In the quantum theory we anticipate renormalization and make the replacements $\mu^2 \to \mu_0^2$, $\lambda \to \lambda_0$, $\tau \to \tau_0$, $\epsilon \to \epsilon_0$. The field equation with an external source J

$$0 = (\partial^2 - \mu_0^2)\varphi - \lambda_0\varphi^3 + J, \tag{2.98}$$

follows from the Heisenberg equations of motion with total hamiltonian $H - \int d^3x\, J\varphi$.

The fields can still be written in terms of creation and annihilation operators at some time such as $t = 0$, e.g.

$$\varphi(\mathbf{x}) = \int d\omega_p\, [e^{i\mathbf{px}}\, a(\mathbf{p}) + e^{-i\mathbf{px}}\, a(\mathbf{p})^\dagger], \tag{2.99}$$

but the time dependence is now given by the nonlinear field equation (2.98) not simply that of a free field,

$$\varphi(x) \neq \int d\omega_p \, [e^{i\mathbf{p}\mathbf{x}-ip^0x^0}\, a(\mathbf{p}) + e^{-i\mathbf{p}\mathbf{x}+ip^0x^0}\, a(\mathbf{p})^\dagger], \quad x^0 \neq 0, \tag{2.100}$$

The hamiltonian H is no longer of the form $\int d\omega_p p^0 a(\mathbf{p})^\dagger a(\mathbf{p})$ but contains terms of fourth order in the creation and annihilation operators, due to the φ^4 term $\lambda_0 \int d^3x\, \varphi^4(\mathbf{x})$. Hence, the vacuum state $|0\rangle$, i.e. the ground state in the limit of infinite volume, is much more complicated than in the free case and and not given by $a(\mathbf{p})|0\rangle = 0$. The state $|\emptyset\rangle$ defined by $a(\mathbf{p})|\emptyset\rangle = 0$ may be called the no-quantum state. Ordinary perturbation theory then suggests that the true vacuum $|0\rangle$ is a superposition of $|\emptyset\rangle$, $a(\mathbf{p})^\dagger|\emptyset\rangle$, $a(\mathbf{p}_1)^\dagger a(\mathbf{p}_2)^\dagger|\emptyset\rangle$, ... One sometimes speaks of $|\emptyset\rangle$ as the bare vacuum and $|0\rangle$ as the dressed vacuum. The above is already true of course in the simple case of the one dimensional anharmonic oscillator with $H = \frac{1}{2m}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{4}\lambda q^4$.

Similarly, the other eigenstates of H may be considered as being dressed by the φ^4 interaction. This holds in particular for the one particle states, which are assumed to be the true eigenstates of P^μ ($P^0 = H$),

$$P^\mu|p\rangle = p^\mu|p\rangle. \tag{2.101}$$

Because $|p\rangle \neq a(\mathbf{p})^\dagger|0\rangle$, it is also not true in general that $\langle p|\varphi(x)|0\rangle = \exp(-ipx)$. However, for covariance reasons we may write

$$\langle p|\varphi(x)|0\rangle = \sqrt{Z_\varphi}\, e^{-ipx}, \tag{2.102}$$

where Z_φ is a constant, traditionally called the wave function renormalization constant.

Although it is of interest to determine the structure of various eigenstates of H in terms of the quanta at $t = 0$, it is cumbersome and detracts from the most immediate physical quantities we wish to calculate, such as scattering amplitudes. Over the years people have learned to concentrate on the vacuum amplitude $Z(J)$ and extract from it the relevant physical quantities.

Let us formulate the ingredients in $Z(J)$. The vacuum $|0\rangle$ is the state with lowest energy, adjusted to zero by and appropriate choice of τ_0,

$$H|0\rangle = 0. \tag{2.103}$$

Recall that H does not contain the source J and that we use the interaction picture to take J into account. The interaction hamiltonian in the interaction picture is given by

$$H_J(x^0) = -\int d^3x\, J(x)\varphi(x). \tag{2.104}$$

The interaction picture field $\varphi(x)$ evolves in time under the influence of H, as if J were zero. The vacuum amplitude is given by

$$\begin{aligned} Z(J) &= \langle 0|U_J(\infty,-\infty)|0\rangle = \langle 0|T\, e^{-i\int_{-\infty}^{\infty} dt\, H_J(t)}|0\rangle, && (2.105)\\ &= \langle 0|T\, e^{i\int d^4x\, J(x)\varphi(x)}|0\rangle. && (2.106)\end{aligned}$$

By differentiating $Z(J)$ with respect to J we 'bring down' the φ's,

$$\frac{\delta Z(J)}{i\delta J(x_1)\cdots i\delta J(x_n)} = \langle 0|T\, e^{i\int d^4x\, J(x)\varphi(x)}\varphi(x_1)\cdots\varphi(x_n)|0\rangle. \qquad (2.107)$$

For $J=0$ we get vacuum expectation values of time ordered products of fields, sometimes called τ-functions,

$$\langle 0|T\,\varphi(x_1)\cdots\varphi(x_n)|0\rangle, \qquad (2.108)$$

from which we can construct many quantities of interest.

We now convert the equation of motion (2.98) into an equation for $Z(J)$, similar to what was done for electromagnetic field in sect. 1.12. For simplicity of notation we denote the classical field $\varphi^{(c)}$ by ϕ,

$$\phi(x) \equiv \frac{1}{Z(J)}\frac{\delta Z(J)}{\delta J(x)} = \frac{\langle 0|U_J(\infty,x^0)\varphi(x)U_J(x^0,-\infty)|0\rangle}{\langle 0|U_J(\infty,-\infty)|0\rangle}. \qquad (2.109)$$

Differentiating twice with respect to x^0 we get

$$\begin{aligned} \partial_0\phi(x) &= Z(J)^{-1}\langle 0|U_J(\infty,x^0)\pi(x)U_J(x^0,-\infty)|0\rangle, && (2.110)\\ \partial_0^2\phi(x) &= Z(J)^{-1}\langle 0|U_J(\infty,x^0)\{J(x)+i[H,\pi(x)]\}U_J(x^0,-\infty)|0\rangle \\ &= Z(J)^{-1}\langle 0|U_J(\infty,x^0)\,[J(x)+\Delta\varphi(x)-\mu_0^2\varphi(x)-\lambda_0\varphi(x)^3] \\ &\quad U_J(x^0,-\infty)|0\rangle \\ &= Z(J)^{-1}\langle 0|T\, e^{i\int d^4y\, J(y)\varphi(y)}\,[J(x)+\Delta\varphi(x)-\mu_0^2\varphi(x)-\lambda_0\varphi(x)^3]|0\rangle \\ &= Z(J)^{-1}\left[J(x)+\Delta\frac{\delta}{i\delta J(x)}-\mu_0^2\frac{\delta}{i\delta J(x)}-\lambda_0\left(\frac{\delta}{i\delta J(x)}\right)^3\right]Z(J), \\ &&& (2.111)\end{aligned}$$

which can be rewritten as

$$\begin{aligned} 0 &= J(x)+Z(J)^{-1}\left[(\partial^2-\mu_0^2)\phi(x)-\lambda_0\phi(x)^3\right]_{\phi(x)\to\frac{\delta}{i\delta J(x)}}Z(J) && (2.112)\\ &= J(x)+Z(J)^{-1}\left[\frac{\delta S(\phi)}{\delta\phi(x)}\right]_{\phi(x)\to\frac{\delta}{i\delta J(x)}}Z(J). && (2.113)\end{aligned}$$

This 'Dyson-Schwinger equation' for $Z(J)$ together with Feynman boundary conditions in time will be our starting point for a calculational scheme.

2.6 Effective action

We now express the vacuum amplitude $Z(J)$ in terms of an effective action $\Gamma(\phi)$. In fact we will first define $\Gamma(\phi)$ in terms of $Z(J)$, then assume $\Gamma(\phi)$ to be given and reexpress $Z(J)$ in terms of $\Gamma(\phi)$. In the next section we will use eq. (2.113) to formulate a method for calculating $\Gamma(\phi)$. It is equal to the classical action up to so-called quantum corrections, $\Gamma(\phi) = S(\phi) + O(\hbar)$.

We start by introducing $W(J)$ defined by

$$Z(J) = e^{iW(J)}. \tag{2.114}$$

Then

$$\phi(x) = \frac{1}{Z}\frac{\delta Z(J)}{i\delta J(x)} = \frac{\delta}{\delta J(x)} W(J). \tag{2.115}$$

In terms of $W(J)$ we define the connected Green functions, also called correlation functions, by

$$G(x_1 \cdots x_n) = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W. \tag{2.116}$$

These Green functions are completely symmetric in their arguments. For $n = 1$, $G(x) = \phi(x)$. Differentiating $Z(J)$ and setting $J = 0$ afterwards gives

$$\phi(x) = \langle 0|\varphi(x)|0\rangle \equiv \phi_0, \tag{2.117}$$

$$(-i)G(xy) = \langle 0|T\,\varphi(x)\varphi(y)|0\rangle - \langle 0|\varphi(x)|0\rangle\langle 0|\varphi(y)|0\rangle, \tag{2.118}$$

We see that $G(xy)$ is the fully dressed (i.e. including all effects of the interactions) propagator, and (2.118) illustrates the name 'correlation function' by analogy with such functions in Statistical Physics. In our example of the φ^4 theory $\langle 0|\varphi(x)|0\rangle$ may be nonzero, depending on the choice of parameters μ_0^2 and λ_0. In case $\langle 0|\varphi(x)|0\rangle \neq 0$ the symmetry $\varphi \to -\varphi$ is spontaneously broken in the vacuum.

In general $J \neq 0$. The field ϕ depends on J, $\phi = \phi(J)$, and we assume that this relation may be inverted, $J = J(\phi)$. In the same fashion $W(J)$ may be considered a function of ϕ, and we now define $\Gamma(\phi)$ by a Legendre transformation,

$$\Gamma(\phi) = W(J) - \int d^4x\, J(x)\phi(x). \tag{2.119}$$

To streamline the derivations below and to bring the equations into a form that also applies to other theories it is now very convenient to follow DeWitt and use a condensed notation: all indices, spacetime and discrete are lumped into an index k,

$$\phi(x) \to \phi^k, \quad x \to k, \tag{2.120}$$

and we use a summation convention for repeated indices, e.g.

$$J_k\phi^k \equiv \int d^4x\, J(x)\phi(x). \tag{2.121}$$

For the case of the coupled scalar–electromagnetic field system the index k also distinguishes various fields,

$$\phi^k \to \varphi^{(c)}(x),\ \varphi^{(c)}(x)^*,\ A_\mu^{(c)}(x). \tag{2.122}$$

Functional differentiation with respect to ϕ^k is denoted by a comma,

$$\frac{\delta\Gamma(\phi)}{\delta\phi(x)} \to \Gamma_{,k}(\phi). \tag{2.123}$$

For example

$$\begin{aligned}
\Gamma(\phi) &= \Gamma(\phi_0) + \Gamma_{,k}(\phi_0)(\phi^k - \phi_0^k) + \frac{1}{2}\Gamma_{,kl}(\phi_0)(\phi^k - \phi_0^k)(\phi^l - \phi_0^l) + \cdots \\
&= \sum_n \frac{1}{n!}\Gamma_{,k_1\cdots k_n}(\phi_0)(\phi^{k_1} - \phi_0^{k_1})\cdots(\phi^{k_n} - \phi_0^{k_n}) && (2.124) \\
&= \sum_n \frac{1}{n!}\int d^4x_1\cdots d^4x_n \\
&\quad \frac{\delta^n\Gamma(\phi_0)}{\delta\phi(x_1)\cdots\delta\phi(x_n)}(\phi(x_1)-\phi_0)\cdots(\phi(x_n)-\phi_0), && (2.125)
\end{aligned}$$

where[1] $\phi_0 = \phi(J=0)$. In the φ^4 model the derivatives of the classical action S around $\phi = 0$ are given by,

$$\begin{aligned}
S_{,k}(0) &= 0, && (2.126) \\
S_{,kk'}(0) &= \left[\frac{\delta S}{\delta\phi(x)\delta\phi(x')}\right]_{\phi=0} \equiv S(x,x';0) \\
&= -\left(-\partial^2 + \mu_0^2 - i\epsilon\right)\delta(x-x'), && (2.127) \\
S_{,k_1k_2k_3}(0) &= \left[\frac{\delta S}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)}\right]_{\phi=0} \equiv S(x_1x_2x_3;0) = 0, && (2.128) \\
S_{,k_1\cdots k_4}(0) &= \left[\frac{\delta S}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)\delta\phi(x_4)}\right]_{\phi=0} \equiv S(x_1\cdots x_4;0) \\
&= -6\lambda_0\,\delta(x_1-x_2)\,\delta(x_1-x_3)\,\delta(x_1-x_4), && (2.129)
\end{aligned}$$

where we have replaced $\mu_0^2 \to \mu_0^2 - i\epsilon$ to enforce the Feynman boundary conditions in time. We have

$$\begin{aligned}
S(\phi) &= \sum_{n=1}^{4}\frac{1}{n!}S_{,k_1\cdots k_n}(0)\phi^{k_1}\cdots\phi^{k_n} && (2.130) \\
&= \sum_{n=1}^{4}\frac{1}{n!}\int d^4x_1\cdots d^4x_n\,S(x_1\cdots x_n;0)\,\phi(x_1)\cdots\phi(x_n). && (2.131)
\end{aligned}$$

[1] In case of spontaneous symmetry breaking in infinite volume $W(J)$ is not differentiable in $J=0$, see e.g. Brown sect. 6.5. We should keep J a little away from zero and hence ϕ in $\Gamma(\phi_0)$ a little away from ϕ_0, such that the differentiations make sense. After all necessary differentiations have been carried out we can let $J \to 0$.

In the condensed notation eqs. (2.116), (2.119) read

$$G^{k_1 \cdots k_n} = \frac{\delta}{\delta J_{k_1}} \cdots \frac{\delta}{\delta J_{k_n}} W, \tag{2.132}$$

and

$$\Gamma(\phi) = W(J) - J_k \phi^k. \tag{2.133}$$

Differentiating (2.133) with respect to J_k gives

$$\frac{\delta \Gamma}{\delta J_l} = \frac{\delta W}{\delta J_l} - \phi^l - J_k \frac{\delta \phi^k}{\delta J_l} = -J_k \frac{\delta \phi^k}{\delta J_l}, \tag{2.134}$$

and using on the left hand side of this equation the chain rule

$$\frac{\delta}{\delta J_l} = \frac{\delta \phi^m}{\delta J_l} \frac{\delta}{\delta \phi^m} = G^{lm} \frac{\delta}{\delta \phi^m}, \tag{2.135}$$

gives

$$G^{lm} \frac{\delta \Gamma}{\delta \phi^m} = -J_k \, G^{kl}, \tag{2.136}$$

or

$$\Gamma_{,p} = -J_p. \tag{2.137}$$

Here we assumed G^{kl} to be nonsingular, i.e. to have an inverse when considered as a continous matrix. This is assured by the Feynman boundary conditions in time, as expressed by the $i\epsilon$ in (2.127). In electrodynamics it requires in addition fixing the gauge or adding the $(\partial_\mu A^\mu)^2$ term to the lagrangian. Eq. (2.137) shows that ϕ is the solution of the stationary action equation $(\delta/\delta\phi^p)(\Gamma + J_k \phi^k) = 0$. Differentiating again, $\delta \Gamma_{,p} / \delta J_l$, using (2.135), gives

$$\Gamma_{,pq} \, G^{ql} = -\delta^l_p, \tag{2.138}$$

which shows that G^{kl} is the inverse of $-\Gamma_{,kl}$. Further differentiation $\delta/\delta\phi^r$ gives

$$\Gamma_{,pqr} \, G^{ql} + \Gamma_{,pq} \, G^{ql}{}_{,r} = 0, \tag{2.139}$$

and contracting with G^{rm} using $G^{mr}(\delta/\delta\phi^r) = \delta/\delta J_m$,

$$\Gamma_{,pqr} \, G^{ql} G^{rm} + \Gamma_{,pq} \, G^{qlm} = 0. \tag{2.140}$$

Contracting these last two equations with G^{pk} using (2.138) gives

$$\begin{aligned} G^{kl}{}_{,r} &= \Gamma_{,pqr} \, G^{pk} G^{ql}, && (2.141) \\ G^{klm} &= \Gamma_{,pqr} \, G^{pk} G^{ql} G^{rm}. && (2.142) \end{aligned}$$

Further differentiation of (2.142) with respect to $\delta/\delta J_n$ gives, using (2.141) and the chain rule (2.135)

$$\begin{aligned} G^{klmn} &= \Gamma_{,pqrs} \, G^{pk} G^{ql} G^{rm} G^{sn} \\ &\quad + \Gamma_{,pqr} \, (G^{ka} G^{pb} G^{nc} \Gamma_{,abc} \, G^{ql} G^{rm} + 2 \text{ perm.}), \end{aligned} \tag{2.143}$$

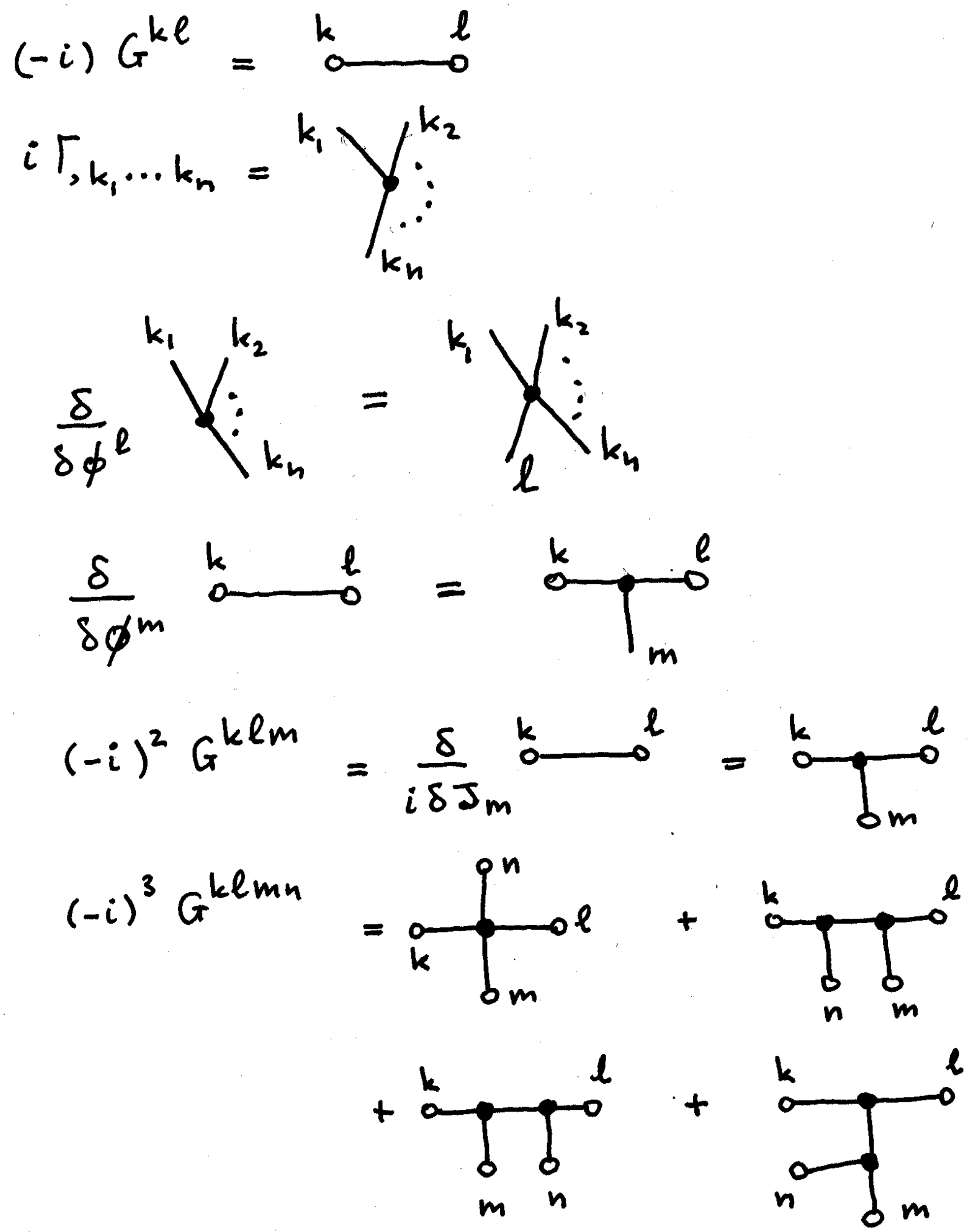

Figure 2.2: Graphical representation. The little $\circ$ at the end of lines indicates the presence of the propagator. Note that the $\circ$ are absent in $\Gamma_{,k_1\cdots k_n}$.

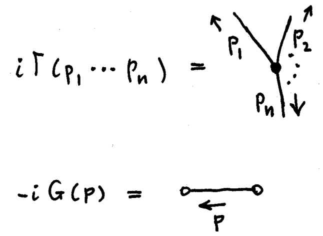

Figure 2.3: Vertex functions and propagator in momentum space.

and so on. The graphical representation given in figure 2.2 clarifies the procedure. We see that the correlation functions can be expressed as a sum of tree diagrams, in which the lines represent the exact (as opposed to free) propagator G^{kl} and the vertices represent the exact $\Gamma_{,p_1 \cdots p_m}$. For this reason the derivatives of Γ are called vertex functions[2]. In this way we obtain $G^{k_1 \cdots k_n}$ in terms of G^{kl} and $\Gamma_{,p_1 \cdots p_m}$. Since G^{kl} is the inverse of $-\Gamma_{,kl}$ all correlation functions are expressed in terms of $\Gamma(\phi)$.

For $J(x) = 0$ the correlation functions become translation invariant, as they are combinations of τ-functions (2.108): $\phi_0(x) = \langle 0|\varphi(x)|0\rangle$ does not depend on x and $G(x_1, \cdots, x_n) = G(x_1 + z, \cdots, x_n + z)$. Also the vertex functions are then translation invariant and the expressions simplify in momentum space. Our conventions are as follows,

$$\begin{aligned} & \int d^4x_1 \cdots d^4x_n \, e^{-i(p_1x_1+\cdots+p_nx_n)} \, \Gamma(x_1 \cdots x_n), \\ \equiv \; & (2\pi)^4 \, \delta^4(p_1 + \cdots + p_n)\Gamma(p_1 \cdots p_n) \end{aligned} \tag{2.144}$$

$$\begin{aligned} & \int d^4x_1 \cdots d^4x_n \, e^{-i(p_1x_1+\cdots+p_nx_n)} \, G(x_1 \cdots x_n), \\ \equiv \; & (2\pi)^4 \, \delta^4(p_1 + \cdots + p_n)G(p_1 \cdots p_n) \end{aligned} \tag{2.145}$$

$$G(p, -p) \equiv G(p), \quad \Gamma(p, -p) \equiv \Gamma(p). \tag{2.146}$$

Note the extraction of a four momentum conserving delta function, which is present because of translation invariance. The corresponding diagrams are given in fig. 2.3. As a consequence, e.g. the three and four point correlation functions are given by

$$G(p_1p_2p_3) = G(p_1)G(p_2)G(p_3)\Gamma(p_1p_2p_3), \tag{2.147}$$

$$\begin{aligned} G(p_1 \cdots p_4) = \; & G(p_1)G(p_2)G(p_3)G(p_4)[\Gamma(p_1p_2p_3p_4) \\ & + \Gamma(p_1, p_2, -p_1 - p_2)G(p_1 + p_2)\Gamma(p_1 + p_2, p_3, p_4) \\ & + \Gamma(p_1, p_3, -p_1 - p_3)G(p_1 + p_3)\Gamma(p_1 + p_3, p_2, p_4) \\ & + \Gamma(p_1, p_4, -p_1 - p_4)G(p_1 + p_4)\Gamma(p_1 + p_4, p_2, p_3)], \end{aligned} \tag{2.148}$$

[2]The factors i and $(-i)$ in fig. 2.2 look artificial at this stage and can be omitted. These factors are introduced for conventional reasons and appear anyway in a later stage.

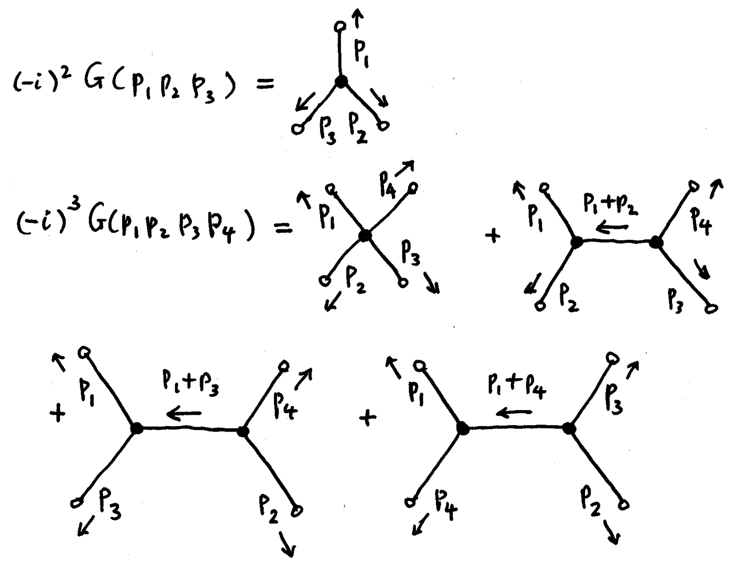

Figure 2.4: Graphical representation of the three and four point correlation functions. As in the previous figures the lines and solid dots denote the exact (fully dressed) propagators and vertex functions.

according to the diagrams in fig. 2.4

2.7 Dyson-Schwinger equations and the loop expansion

It is useful to restore Planck's constant $\hbar$ temporarily. We know already the explicit appearence of $\hbar$ in the vacuum amplitude (see e.g. (1.249a)),

$$\begin{aligned} Z(J) &= e^{\frac{i}{\hbar}W(J)} = e^{\frac{i}{\hbar}[\Gamma(\phi)+J_k\phi^k]}, && (2.149)\\ \phi^k &= e^{-\frac{i}{\hbar}W(J)}\,\frac{\hbar\delta}{i\delta J_k}\,e^{\frac{i}{\hbar}W(J)}, && (2.150)\end{aligned}$$

where $W(J)$ and $\Gamma(\phi)$ may still depend implicitly on $\hbar$. Equation (2.113) for the vacuum amplitude can be written in the condensed notation as

$$\begin{aligned} 0 &= e^{-\frac{i}{\hbar}W(J)}\,[J_k + S_{,k}\,(\frac{\hbar\delta}{i\delta J})]\,e^{\frac{i}{\hbar}W(J)}, && (2.151)\\ &= e^{-\frac{i}{\hbar}W(J)}\left[J_k + S_{,k}\,(0) + S_{,kl}\,(0)\frac{\hbar\delta}{i\delta J_l} + \frac{1}{2}S_{,klm}\,(0)\frac{\hbar\delta}{i\delta J_l}\frac{\hbar\delta}{i\delta J_m}\right.\\ &\quad \left. + \frac{1}{3!}S_{,klmn}\,(0)\frac{\hbar\delta}{i\delta J_l}\frac{\hbar\delta}{i\delta J_m}\frac{\hbar\delta}{i\delta J_n}\right]\,e^{\frac{i}{\hbar}W(J)}. && (2.152)\end{aligned}$$

The classical action S does not depend on $\hbar$.

To evaluate (2.152) we insert $1 = e^{-\frac{i}{\hbar}W}\, e^{\frac{i}{\hbar}W}$ in between the $\delta/\delta J$'s and use the following operator identity

$$
\begin{aligned}
e^{-\frac{i}{\hbar}W(J)} \frac{\hbar\delta}{i\delta J_k}\, e^{\frac{i}{\hbar}W(J)} &= \phi^k + \frac{\hbar\delta}{i\delta J_k} && (2.153)\\
&= \phi^k - i\hbar G^{kl}\frac{\delta}{\delta\phi^l} && (2.154)\\
&\equiv \hat{\phi}^k, && (2.155)
\end{aligned}
$$

where in the second line we used the chain rule (2.135). Then eq. (2.152) can be rewritten as

$$
0 = [J_k + S_{,k}(0) + S_{,kl}\,\hat{\phi}^l + \frac{1}{2}S_{,klm}(0)\hat{\phi}^l\hat{\phi}^m + \frac{1}{3!}S_{,klmn}\,\hat{\phi}^l\hat{\phi}^m\hat{\phi}^n]\;1, \qquad (2.156)
$$

where the differential operator in $[\cdots]$ acts on the number 1. Using (2.141) we have

$$
\begin{aligned}
\hat{\phi}^l\;1 &= \phi^l, && (2.157)\\
\hat{\phi}^l\hat{\phi}^m\;1 &= (\phi^l - i\hbar G^{lp}\frac{\delta}{\delta\phi^p})\phi^m = \phi^l\phi^m - i\hbar G^{lm}, && (2.158)\\
\hat{\phi}^l\hat{\phi}^m\hat{\phi}^n\;1 &= (\phi^l - i\hbar G^{lp}\frac{\delta}{\delta\phi^p})(\phi^m\phi^n - i\hbar G^{mn})\\
&= \phi^l\phi^m\phi^n - i\hbar\phi^l G^{mn} - i\hbar\phi^m G^{nl} - i\hbar\phi^n G^{lm}\\
&\quad + (-i\hbar)^2 G^{lp}G^{mq}G^{nr}\Gamma_{,pqr}\,. && (2.159)
\end{aligned}
$$

Putting things together see that (2.156) can be rewritten as

$$
\begin{aligned}
-J_{,k} &= S_{,k}(\phi) + (-i\hbar)\frac{1}{2}S_{,klm}(\phi)G^{lm}\\
&\quad + (-i\hbar)^2\frac{1}{3!}S_{,klmn}(\phi)G^{lp}G^{mq}G^{nr}\,\Gamma_{,pqr}\,, && (2.160)
\end{aligned}
$$

where the argument ϕ in the derivatives of S is explicit. Suppressing the ϕ dependence as usual and recalling the effective field equation $\Gamma_{,k} = -J, k$ we finally have our desired equation

$$
\begin{aligned}
i\Gamma_{,k} &= iS_{,k} + \hbar\frac{1}{2}\, iS_{,klm}(-iG^{lm})\\
&\quad + \hbar^2\frac{1}{3!}\, iS_{,klmn}(-iG^{lp})(-iG^{mq})(-iG^{nr})\, i\Gamma_{,pqr}\,. && (2.161)
\end{aligned}
$$

This equation is represented graphically in figure 2.5. Differentiating (2.161) repeatedly and letting $J \to 0$ in the end we obtain an infinite hierarchy of coupled equations, for the full propagator G^{kl} and the vertex functions $\Gamma_{k_1\cdots k_n}$: the Dyson-Schwinger equations. This differentiation is most easily done graphically using

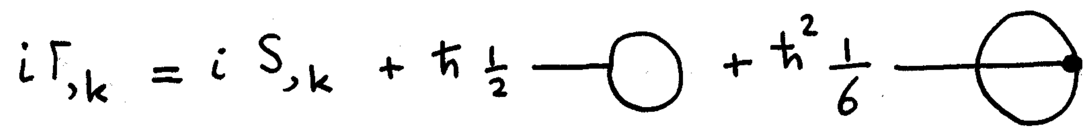

Figure 2.5: Equation for the the effective action. Dots represent fully dressed (exact) vertex functions $\Gamma_{,kl\ldots}$, vertices without the dot represent bare vertex functions $S_{,kl\ldots}$.

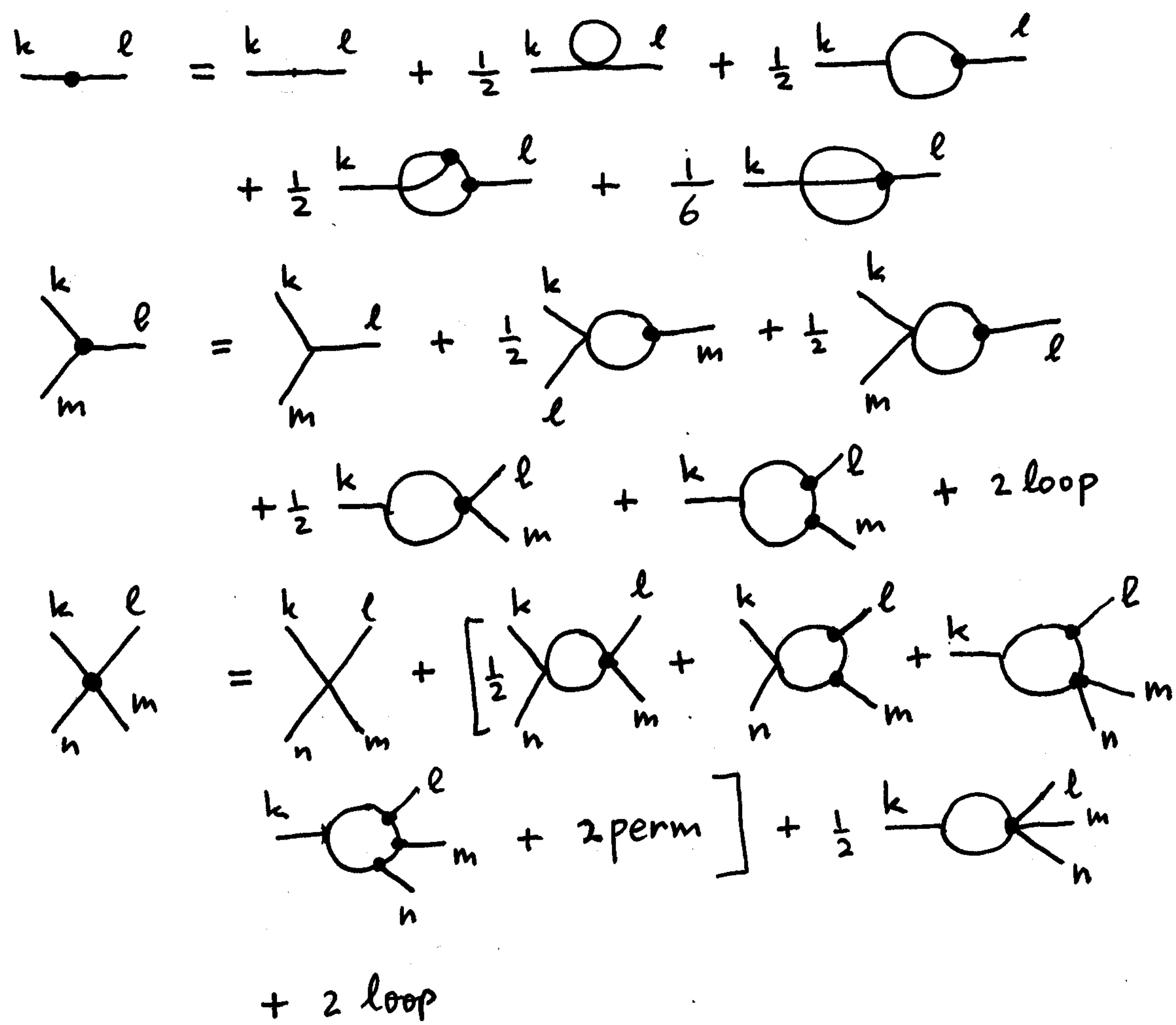

Figure 2.6: Equations for $\Gamma_{,kl}$, $\Gamma_{,klm}$ and $\Gamma_{,klmn}$ ($\hbar = 1$).

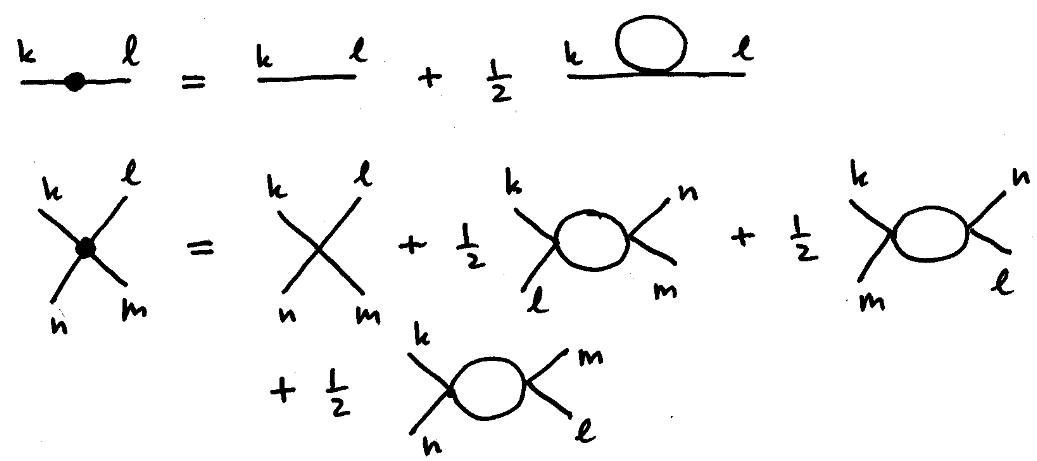

Figure 2.7: One loop approximation for $\Gamma_{,kl}$ and $\Gamma_{,klmn}$ in the φ^4 theory in the symmetric phase. Here the lines denote the bare propagators.

the rules in fig. 2.2. The first few equations following from differentiating are given in fig. 2.6, and so on.

The infinite hiearchy can usually not be solved exactly. One can truncate the hiearchy by setting the n point function $\Gamma_{(n)} = 0$ for n larger than some $n_{\max}$, e.g. $n_{\max} = 4$, and keeping only the one loop terms in the Dyson-Schwinger equations. The error in such truncations is difficult to assess a priori and in electrodynamics the procedure has problems with gauge invariance. Comparison with numerical simulations using the lattice regularization have shown however that the truncation approach may give reasonable results. A systematic approximation is obtained by iteration, by inserting the left hand side into the right hand side, repeatedly. This leads to an expansion of $\Gamma_{(n)}$ in powers of $\hbar$ (which is gauge covariant). The power of $\hbar$ corresponds to the number of loops in the diagrams, hence the name loop expansion. The semiclassical approximation is $\Gamma \approx S$. The one loop approximation is obtained by simply replacing the full propagators and vertex functions on the right hand side of the Dyson-Schwinger equations by the bare ones and dropping the two loop terms. For example, the φ^4 theory has in the symmetric phase only two and four point bare vertex functions (cf. (2.127) – (2.129)), and to one loop order the two and four point vertex functions are given by the diagrams in fig. 2.7.

As announced in the previous section, $\Gamma = S + O(\hbar)$. It is not difficult to see that each power of $\hbar$ is accompanied by a power of the coupling constant λ_0, or some other coupling constant in a more complicated theory with more than one coupling constant. Setting $\hbar = 1$, the semiclassical expansion is an expansion in one of the couplings, keeping ratios of the coupling constants fixed.

2.8 Path integral representation

Equation (2.151) for $Z(J)$ has the form of a linear differential equation wich can be solved by Fourier transformation. We write

$$Z(J) = \int D\varphi\, e^{\frac{i}{\hbar}J_k\varphi^k}\, \tilde{Z}(\varphi), \quad D\varphi \equiv \prod_k d\varphi^k, \tag{2.162}$$

where the integration variables φ^k should not be confused with the quantum operator field. As usual, differentiations become multiplications in Fourier space, and the equation for $Z(J)$ gets transformed as

$$0 = [J_k + S_{,k}\,(\frac{\hbar\delta}{i\delta J})]Z(J) = \int D\varphi\,[J_k + S_{,k}\,(\varphi)]\, e^{\frac{i}{\hbar}J_k\varphi^k}\, \tilde{Z}(\varphi). \tag{2.163}$$

Replacing J_k by $\hbar\delta/i\delta\varphi^k$ acting on the exponential and making a partial integration we discover that the solution is given by

$$Z(J) = const. \int D\varphi\, e^{\frac{i}{\hbar}[S(\varphi)+J_k\varphi^k]}, \tag{2.164}$$

which can easily be checked directly,

$$\begin{aligned}
\int D\varphi\,[J_k + S_{,k}\,(\varphi)]\, e^{\frac{i}{\hbar}[S(\varphi)+J_l\varphi^l]} &= \int D\varphi\,[\frac{\delta}{\varphi^k}(S(\varphi) + J_l\varphi^l)]\, e^{\frac{i}{\hbar}[S(\varphi)+J_l\varphi^l]} \\
&= \frac{\hbar}{i}\int D\varphi\, \frac{\delta}{\delta\varphi^k}\, e^{\frac{i}{\hbar}[S(\varphi)+J_l\varphi^l]} \\
&= 0,
\end{aligned} \tag{2.165}$$

because the surface terms vanish due to the $i\epsilon$ terms in the action, see e.g. eq. (2.127). The integration constant is fixed by the property $Z(J) = 0$,

$$Z(J) = \frac{\int D\varphi\, e^{\frac{i}{\hbar}[S(\varphi)+J_k\varphi^k]}}{\int D\varphi\, e^{\frac{i}{\hbar}S(\varphi)}}. \tag{2.166}$$

Eq. (2.166) is the path integral representation of the vacuum amplitude.

The fact that we are dealing with functional differential equations and corresponding functional Fourier transformation is helpfully hidden in the compact notation, but should of course not be forgotten. For example, the formal continuous product in

$$\int D\varphi = \int \prod_k d\varphi^k \to \prod_x \int_{-\infty}^{\infty} d\varphi(x), \tag{2.167}$$

is mathematically ill defined and needs to be given meaning by a regularization. We could for instance use a discrete mode expansion, place a cutoff on the number of modes, and remove this cutoff in a later stage. An obvious choice is the lattice regularization, in which the x are restricted to the points of a lattice in spacetime. Then the continuum limit needs careful study. This method at once gives a precise and simple definition to quantum field theory and facilitates numerical simulations on computers, which have led to spectacular successes in the nonperturbative field theory, in particular QCD, the theory of the strong interactions.

For $\hbar \to 0$ the stationary phase argument leads to the semiclassical result

$$Z(J) \approx e^{\frac{i}{\hbar}[S(\phi)+J_k\phi^k]}, \tag{2.168}$$

with ϕ^k the solution of

$$S_{,k}(\phi) + J_k = 0. \tag{2.169}$$

The perturbative expansion for $\hbar \to 0$ is a systematic stationary phase expansion, which can be seen as a steepest descent or saddle point expansion by continuing φ to complex values. Although these arguments are formal at this level, such manipulations of path integrals have turned out to provide a powerful tool in quantum field theory.

As a simple example, let us write $S = S_0 + S_1$, where S_0 contains only the quadratic terms in the fields and S_1 the higher order terms. Then $Z(J)$ can be evaluated as, setting $\hbar = 1$ for simplicity,

$$Z(J) = e^{iS_1(\frac{\delta}{i\delta J})}\, Z_0(J), \tag{2.170}$$

with $Z_0(J)$ the free field vacuum amplitude

$$\begin{aligned} Z_0(J) &= \int D\varphi\, e^{i[S_0(\varphi)+J_k\varphi^k]} && (2.171)\\ &= \int D\varphi\, e^{-i\frac{1}{2}\varphi^k G_{0kl}^{-1}\varphi^l + iJ_k\varphi^k}, && (2.172)\end{aligned}$$

were we suppressed the normalizing *const*. This free field path integral is formally just a multiple gaussian integral, which can be solved by making a translation $\varphi^k \to \varphi^k + G_0^{kl}J_l$,

$$Z_0(J) = e^{i\frac{1}{2}J_k G_0^{kl} J_l} \int D\varphi\, e^{-i\frac{1}{2}\varphi^k G_{0kl}^{-1}\varphi^l}. \tag{2.173}$$

The remaining integral is just a constant ($\propto \sqrt{\det G_0}$), which plays no role in the present discussion. We have reproduced the free field form for the vacuum amplitude, and by expansion of $\exp[iS_1(\frac{\delta}{i\delta J})]$ we get an explicit formula for the perturbative expansion of $Z(J)$. This leads to Feynman diagrams, which may be ordered into various connected and irreducible parts, as seen earlier with the effective action technique.

The path integral integral is a beautiful independent formulation of quantum theory and our brief introduction here does not do it sufficient justice.

2.9 Vertices in φ^4 theory

Setting $J = 0$ (or const.) after all necessary differentiations have been carried out, thranslation invariance allows for transfering the equations to momentum space. In the semiclassical approximation $\Gamma = S$ and the field equation $S_{,k}(\phi_0) = -J_k$ is for $J = 0$ an equation for the vacuum expectation value ϕ_0,

$$(\mu_0^2 + \lambda_0 \phi_0^2)\phi_0 = 0. \tag{2.174}$$

Since λ_0 is positive, for $\mu_0^2 > 0$ the solution is $\phi_0 = 0$ and the system is in a symmetric phase (no spontaneous symmetry breaking). The vertex functions $S_{,k_1 \cdots k_n}(0)$ have alsready been given in (2.129), and read in momentum space

$$\begin{aligned} S(p,-p) &= -(\mu_0^2 + p^2 - i\epsilon), && (2.175)\\ S(p_1,p_2,p_3) &= 0, && (2.176)\\ S(p_1,\cdots,p_4) &= -6\lambda_0. && (2.177) \end{aligned}$$

Only the two and four point bare vertices are nonzero. From the propagator $G(p) = -S(p,-p)^{-1}$ we see that the bare particle mass $m_0^2 = \mu_0^2$.

For $\mu_0^2 < 0$ there are three solutions, $\phi_0 = 0$ and

$$\phi_0 = \pm\sqrt{\frac{-\mu_0^2}{\lambda_0}}, \tag{2.178}$$

but as we have seen already in sect. 2.5 the ground state corresponds to one of the $\phi_0 \neq 0$ solutions. The system is in a broken phase. To get a unique ground state we break the symmetry $\phi \to -\phi$ explicitly and do not let $J(x) \to 0$ but in stead let $J(x) \to \epsilon_0$, which produces the term (2.97). We may think of ϵ_0 being infinitesimal or, and this is the case in the application of φ^4-like models to low energy pions physics, ϵ_0 may have some nonzero value determined by experiment. In the broken phase there is also three point vertex. For $\epsilon_0 \to 0$, the vertex functions $S_{,k_1 \cdots k_n}(\phi_0)$ are given by

$$\begin{aligned} S(p,-p) &= -(-2\mu_0^2 + p^2 - i\epsilon), && (2.179)\\ S(p_1,p_2,p_3) &= -6\lambda_0\phi_0, && (2.180)\\ S(p_1,\cdots,p_4) &= -6\lambda_0, && (2.181) \end{aligned}$$

where ϕ_0 has the semiclassical value (2.178). We see that in the broken phase the bare particle mass is given by $m_0^2 = -2\mu_0^2 = 2\lambda_0\phi_0^2$, if we use for ϕ_0 its semiclassical value.

In the semiclassical approximation the we can drop the subscript 0 in all these quantities, $\mu^2 = \mu_0^2$, $\lambda = \lambda_0$, and the particle mass is $m^2 = m_0^2$.

2.10 Vertices in scalar electrodynamics

We add a source for each field to the action (2.84) of scalar electrodynamics,

$$S(\varphi, \varphi^*, A) + \int d^4x\, (\varphi^* J + J^* \varphi + J_\mu A^\mu). \tag{2.182}$$

The source terms for the scalar field break gauge invariance and the resulting correlation functions depend on the gauge. We first fix the gauge and then add the sources. Having fixed the gauge we can also relax the condition of current conservation $\partial_\mu J^\mu = 0$, which allows for unconstrained functional differentiation. The total current is no longer conserved anyhow, because the breaking of gauge invariance by the scalar sources causes $\partial_\mu j^\mu \neq 0$. We have seen before in chapter 1 that the photon Green function depends on the gauge but that the physical results extracted from $Z(J)$ are gauge invariant. As we shall see later also the scattering amplitudes are gauge invariant, and this can be understood from the fact that the sources are removed to infinity in spacetime.

We have canonically quantized the system in the Coulomb gauge so let us first make some remarks about this case. The hamiltonian H after quantization changes in two ways by the addition of the sources: in the Coulomb energy operator (2.92):

$$e_0 j^0(x) \to e_0 j^0(x) + J^0(x), \tag{2.183}$$

and we have to add to H the terms

$$-\int d^3x\, [(\varphi^\dagger(x) J(x) + J^\dagger(x)\varphi(x) + J_m(x) A^m(x)]. \tag{2.184}$$

The interaction hamiltonian H_J in the source-interaction picture is the difference between the total hamiltonian including sources and the source free hamiltonian. We shall not go through the cumbersome derivation of the 'equation of motion equation' for $Z(J)$ from the canonical commutation relations in the Coulomb gauge. It will have the general form (2.151), with S the action in Coulomb gauge. The resulting bare vertices and propagators look ugly, non-Lorentz covariant, and the resulting expressions are awkward to work with.

We therefore move quickly to a general covariant gauge, obtained by adding the term

$$-\int d^4x\, \frac{1}{2\xi_0}\, (\partial_\mu A^\mu)^2 \tag{2.185}$$

to the non-gauge-fixed action, as in sect. 1.13, and after this we add the sources. Although we can no longer use current conservation to show that $\partial_\mu A^\mu = 0$ as a consequence of the equations of motion with sources, as in sect. 1.13, we may expect ξ_0-independence of the physical results. This is an important test for the correctness of the procedure, which we will do for scattering amplitudes in the semiclassical approximation. A proper demonstration of the equivalence of the Coulomb gauge and the covariant gauges lies outside the scope of these lecture notes. We assume now the validity of the 'equation of motion' for $Z(J)$ in the generic form (2.151), and as a consequence all the results concerning the effective action and Dyson-Schwinger equations apply.

The resulting bare vertex functions now follow from the complete action of scalar electrodynamics, which reads in the formalism using complex fields

$$
\begin{aligned}
S &= S_A + S_{A\phi}, &(2.186)\\
S_A &= -\int d^4x\,\frac{1}{2}A^\mu[(-\partial^2 - i\epsilon)g_{\mu\nu} + (1-\xi_0^{-1})\partial_\mu\partial_\nu]A^\nu, &(2.187)\\
S_{A\phi} &= -\int d^4x\,[\phi^*(-\partial^2 + \mu_0^2 - i\epsilon)\phi + ie_0(\phi^*\partial_\mu\phi - \partial_\mu\phi^*\phi)\,A^\mu \\
&\quad + e_0^2\phi^*\phi A_\mu A^\mu + \lambda_0(\phi^*\phi)^2 + \tau_0] &(2.188)
\end{aligned}
$$

We limit ourselves here to the case $\mu_0^2 > 0$, for which there is no spontaneous symmetry breaking. (The case of negative μ_0^2 is very interesting, it describes a relativistic superconductor.) Then $\phi_0^k = 0$ and we have to evaluate the functional derivatives of S at zero fields. The only new aspect is the derivative $\phi^*\phi A$ coupling. Writing these terms as

$$\int d^4u d^4v d^4w\,\phi^*(u)\phi(v)A^\mu(w)S_{\phi^*\phi A^\mu}(u,v,w), \tag{2.189}$$

we see that $S_{\phi^*\phi A^\mu}(u,v,w)$ can be written in the form

$$\int d^4x\,ie_0[\partial_\mu\delta(x-u)\delta(x-v)\delta(x-w) - \partial_\mu\delta(x-v)\delta(x-u)\delta(x-w)], \tag{2.190}$$

in which ∂_μ acts on x. The integration over x can of course be carried out easily but the above form is convenient for transformation to momentum space, where the x integral gives the delta function of conservation of momentum. In momentum space we have then the nonzero vertex functions

$$
\begin{aligned}
S_{A^\mu A^\nu}(k,-k) &= -[(k^2 - i\epsilon)g_{\mu\nu} - (1-\xi_0^{-1})k_\mu k_\nu], &(2.191)\\
S_{\phi^*\phi}(p,-p) &= -(\mu_0^2 + p^2 - i\epsilon), &(2.192)\\
S_{\phi^*\phi A^\mu}(p,q,k) &= e_0(p_\mu - q_\mu), &(2.193)\\
S_{\phi^*\phi A^\mu A^\nu}(p,q,k,l) &= -2e_0^2 g_{\mu\nu}, &(2.194)\\
S_{\phi^*\phi\phi^*\phi}(p_1,q_1,p_2,q_2) &= -4\lambda_0, &(2.195)
\end{aligned}
$$

and the nonzero propagators

$$
\begin{aligned}
G^{\mu\nu}(k) \equiv G^{A^\mu A^\nu}(k,-k) &= \frac{g^{\mu\nu} - (1-\xi_0)k^\mu k^\nu/(k^2 - i\epsilon\xi_0)}{k^2 - i\epsilon}, &(2.196)\\
G(p) \equiv G^{\phi\phi^*}(p,-p) &= \frac{1}{\mu_0^2 + k^2 - i\epsilon}, &(2.197)
\end{aligned}
$$

Figure 2.8: Bare propagators and vertex functions for scalar electrodynamics. The arrow on the scalar field line points towards ϕ^*.

Figure 2.9: Bare propagators and vertex functions for scalar electrodynamics in the formalism using real fields.

as represented in fig. 2.8. Note that $G^{\phi\phi} = G^{\phi^*\phi^*} = 0$ because of global U(1) invariance. In the formalism using real scalar fields in which

$$
\begin{aligned}
S_{A\phi} &= -\int d^4x\,[\frac{1}{2}\phi^\alpha(-\partial^2+\mu_0^2-i\epsilon)\phi^\alpha + ie_0 q_{\alpha\beta}\phi^\alpha\partial_\mu\phi^\beta A^\mu \\
&\quad + e_0^2\frac{1}{2}\phi^\alpha\phi^\alpha A_\mu A^\mu + \frac{1}{4}\lambda_0(\phi^\alpha\phi^\alpha)^2 + \tau_0], \qquad (2.198)
\end{aligned}
$$

the vertex functions involving the scalars are given by

$$
\begin{aligned}
S_{\alpha\beta}(p,-p) &= -\delta_{\alpha\beta}(\mu_0^2+p^2-i\epsilon), \quad G^{\alpha\beta}(p) = \frac{\delta_{\alpha\beta}}{\mu_0^2+p^2-i\epsilon} \qquad (2.199)\\
S_{\alpha\beta\mu}(p,q,k) &= e_0\, q_{\alpha\beta}\,(p_\mu - q_\mu), \qquad (2.200)\\
S_{\alpha\beta\mu\nu}(p,q,k,l) &= -2e_0^2\,\delta_{\alpha\beta}\,g_{\mu\nu}, \qquad (2.201)\\
S_{\alpha\beta\gamma\delta}(p,q,r,s) &= -2\lambda_0\,(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}), \qquad (2.202)
\end{aligned}
$$

and represented in fig. 2.9.

2.11 Particles and poles

We have seen in sect. 2.6 that the vacuum amplitude can be expressed in terms of the full vertex functions and the two point correlation function. Since this function plays a special role we discuss here an important property: the particle–pole connection. For $J = 0$ the correlation function of the φ^4 theory given in (2.118) is translation invariant. Inserting intermediate states and separating the one particle contribution using (2.102), which we repeat here for convenience

$$\langle p|\varphi(x)|0\rangle = \sqrt{Z_\varphi}\, e^{-ipx}, \quad \langle 0|\varphi(x)|p\rangle = \sqrt{Z_\varphi}\, e^{ipx}, \tag{2.203}$$

we get for $x^0 > y^0$

$$\begin{aligned} -iG(x-y) &\equiv -iG(xy) = \langle 0|T\,\varphi(x)\varphi(y)|0\rangle - \langle 0|\varphi|0\rangle^2 && (2.204)\\ &= \int d\omega_p\, \langle 0|\varphi(x)|p\rangle\langle p|\varphi(y)|0\rangle + \text{mpc} && (2.205)\\ &= Z_\varphi \int d\omega_p\, e^{ip(x-y)} + \text{mpc}, && (2.206)\end{aligned}$$

where 'mpc' denotes the multiparticle contribution. Note that the vacuum contribution cancels in the sum over intermediate states. For $x^0 < y^0$ there is a similar expression and combining these in the familiar way we get for general times

$$\begin{aligned} G(x-y) &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, G(p), && (2.207)\\ G(p) &= \frac{Z_\varphi}{m^2+p^2-i\epsilon} + \text{mpc}. && (2.208)\end{aligned}$$

This shows that the one particle intermediate states lead to a pole in the propagator $G(p)$ as a function of p^2 with residue Z_φ. The complete expression including the multi particle contributions is called the spectral representation, or the Källén-Lehmann representation. See for example Brown ch. 6.

For the the photon we have similarly in a covariant gauge

$$\begin{aligned} G^{\mu\nu}(p) &= Z_A\, \frac{g^{\mu\nu} + \text{gauge terms}}{p^2 - i\epsilon} + \text{mpc}, && (2.209)\\ G^{\mu\nu}(x-y) &\overset{x^0\geq y^0}{=} \int d\omega_p\, e^{ip(x-y)}\, Z_A[\sum_\lambda e^\mu(p,\lambda)e^\nu(p,\lambda)^* + \text{gauge terms}] \\ &\quad + \text{mpc}. && (2.210)\end{aligned}$$

In the Coulomb gauge, however, Z_A and Z_φ are not constant but depend on $\mathbf{p}$.

For the charged particles of scalar electrodynamics (2.203) is extended to

$$\begin{aligned} \langle p-|\varphi(x)|0\rangle &= \sqrt{Z_\varphi}\, e^{-ipx}, \quad \langle 0|\varphi(x)^\dagger|p-\rangle = \sqrt{Z_\varphi}\, e^{ipx},\\ \langle p+|\varphi(x)^\dagger|0\rangle &= \sqrt{Z_\varphi}\, e^{-ipx}, \quad \langle 0|\varphi(x)|p+\rangle = \sqrt{Z_\varphi}\, e^{ipx}, && (2.211)\end{aligned}$$

which takes charge conservation into account. For example, $Q|0\rangle = 0$, $Q\varphi(x)|0\rangle = [Q,\varphi(x)]\,|0\rangle = -\varphi(x)|0\rangle$, and it follows that $\varphi(x)|0\rangle$ is orthogonal to $|p+\rangle$ which has positive charge. Intuitively, (2.211) can be understood from the fact the φ creates a bare antiparticle (charge $-$) and annihilates a bare particle (charge $+$), and vice versa for $\varphi^\dagger$, as can be seen in (2.60).

We like to stress the generality of the particle-pole connection. It also applies to composite fields and bound states. For example, in electrodynamics of electrons and protons we may construct a scalar field $\varphi_H(x)$ as composed of an electron field ψ_e and a proton field ψ_p, with the quantum numbers of the ground state of the hydrogen atom. Then we can still introduce a source for this field and the effective action formalism still applies. Since the ground state of the hydrogen atom is a spinless particle, the φ_H-correlation function has a pole at the position of the mass of the hydrogen atom. Another example is Quantum Chromodynamics (QCD), the theory of the strong interactions, in which we can construct composite fields for the protons etc. out of quark and gluon fields. In numerical simulations in QCD the bound state masses are in fact essentially computed from the positions of the poles in suitable composite field correlation functions.

If a correlation function of a field φ has no pole on the real p^2 axis, then generically this means that there is no particle with the quantum numbers of φ. However, it is possible that there is a large 'bump' in $G(p)$ near some m^2, due to a nearby pole in $G(p)$, analytically continued into the complex p^2 plane. This happens for particles which are unstable but long lived on the relevant time scale. Then typically near the pole

$$G(p) \to \frac{Z}{p^2 + m^2 - im\Gamma}, \tag{2.212}$$

with $\Gamma \ll m$. For $t > 0$ this leads to

$$\begin{aligned} G(\mathbf{p}, t) &\equiv \int \frac{dp_0}{2\pi}\, e^{ip_0 t}\, \frac{Z_\varphi}{p^2 + m^2 - im\Gamma} \\ &= i\frac{Z}{2\sqrt{\mathbf{p}^2 + m^2 - im\Gamma}}\, \exp[-it\sqrt{\mathbf{p}^2 + m^2 - im\Gamma}], \end{aligned} \tag{2.213}$$

by closing the p_0 contour in the upper half plane. For $\Gamma \ll m$ we may approximate $\sqrt{\mathbf{p}^2 + m^2 - im\Gamma} = \omega(\mathbf{p}) - im\Gamma/2\omega(\mathbf{p})$, $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$, and

$$G(\mathbf{p}, t) = i\frac{Z}{2\omega}\, e^{-i\omega t}\, e^{-\frac{1}{2}\Gamma(m/\omega)t}, \tag{2.214}$$

showing an exponentially decaying time behavior. The physical interpretation is that Γ^{-1} is the life time of an unstable particle, in its rest frame, with the quantum numbers of the field φ, and Γ is the corresponding decay rate. The factor $\omega(\mathbf{p})/m$ is a relativistic time delay factor for a moving particle. See De Wit & Smith sect. 3.6 and Brown sect. 6.3 for a more detailed explanation.

Figure 2.10: Source arrangement for determining emission and absorption amplitudes.

2.12 Scattering and decay amplitudes

From the vacuum amplitude we determine in this section the amplitudes for scattering of particles and decay of unstable particles. We have introduced the correlation functions $G(x_1 \cdots x_n)$ as the functional derivatives of $W(J)$. From now on we assume all derivatives to be evaluated in the limit $J \to 0$, for which $G(x) = \phi(x) \to \phi_0$. Since by definition $W(0) = 0$, it follows that

$$W(J) = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, G(x_1 \cdots x_n) J(x_1) \cdots J(x_n). \tag{2.215}$$

The diagrams for $G(x_1 \cdots x_n)$ up to $n = 4$ are already given fig, 2.2. Notice that there is a full propagator $G(xy)$ at every external line.

We first reconsider the particle emission and absorbtion amplitudes for the present case of interacting fields, following the same reasoning as for the free electromagnetic field. Consider a source $J(x) = J_1(x) + J_2(x)$ as shown in fig. 2.10. The J_1 – J_2 cross term in the vacuum amplitude appears as

$$\begin{aligned} Z(J) &= 1 + \cdots + \int d^4x d^4y \, iJ_1(x)(-i)G(x-y)iJ_2(y) + \cdots, & (2.216) \\ &= \int d^4x d^4y \, iJ_1(x)[Z_\varphi \int d\omega_p \, e^{ip(x-y)} + \text{mpc}] \, iJ_2(y) + \cdots, & (2.217) \end{aligned}$$

where we used (2.206) since $x^0 > y^0$.

It can be shown that for large time separations $t \equiv x^0 - y^0 \to \pm\infty$ the multiparticle contribution 'mpc' to (2.217) is negligible. Large times mean in this context times $t \gg M^{-1}$, where M is a typical particle mass, e.g. the mass m of our scalar particles. For M of the order of 100 MeV the time scale M^{-1} is of the order of 10^{-23} sec. See e.g. Brown, ch. 6 for a discussion of these points. The case of zero mass (photon) requires a separate study, which is so involved that in practise this complication is blissfully ignored at this stage.

Performing the spacetime integrations in (2.217) the J_1 – J_2 cross term takes

Figure 2.11: Emission and absorption of particles (a) and antiparticles (b).

Figure 2.12: Causal arrangement of sources for two particle scattering

the form

$$Z(J) = 1 + \cdots + \int d\omega_p \, i\sqrt{Z_\varphi} J_1(p)^* \, i\sqrt{Z_\varphi} J_2(p) + \cdots, \tag{2.218}$$

which shows that in the interacting case the emission and absorption amplitudes are given by

$$\langle p|0\rangle^J = i\sqrt{Z_\varphi} J(p), \quad \langle 0|p\rangle^J = i\sqrt{Z_\varphi} J(p)^*, \tag{2.219}$$

differing from the free field case only by the factors $\sqrt{Z_\varphi}$.

The derivation above is easily extended to scalar electrodynamics. For the photons we need to replace Z_φ by Z_A and put in the polarization vectors $e^\mu(p,\lambda)$ as in (1.273), (1.277). The charged scalar fields are coupled to the sources according to $S \to S + \int d^4x \, (J^*\varphi + J\varphi^\dagger)$. Comparing with (2.211) or (2.60) we see that $J(x)$ can only emit particles and absorb antiparticles, and vice versa for $J^*(x)$, as illustrated in fig. 2.11.

Returning to the φ^4 theory, consider next a source of the form $J(x) = J_1(x) + J_2(x) + J_3(x) + J_4(x)$ with the various components arranged in spacetime as shown in figure 2.12. The causal relation between the sources is such that particles emitted by sources 3 and 4 can be absorbed by sources 1 and 2. The sources 1 and 2 and also 3 and 4 are separated by macroscopic spacelike distances. The J_1 – J_4 cross term in the vacuum amplitude is given by $\delta^4 Z/\delta J_1(x_1) \cdots \delta J_4(x_4)$, or

$$Z(J) \;=\; 1 + \cdots + \int d^4x_1 \cdots d^4x_4 \, [(-i)G(x_1x_2)(-i)G(x_3x_4)$$

Figure 2.13: Graphical representation of (2.220).

$$
\begin{aligned}
&+(-i)G(x_1x_3)(-i)G(x_2x_4)+(-i)G(x_1x_4)(-i)G(x_2x_3)\\
&+(-i)^3G(x_1x_2x_3x_4)]\, iJ(x_1)iJ(x_2)iJ(x_3)iJ(x_4)+\cdots, \qquad (2.220)
\end{aligned}
$$

which is represented graphically in fig. 2.13. We have neglected the $G(x_1x_2)G(x_3x_4)$ contribution in this figure because for spacelike $z=x_1-x_2$ or $z=x_3-x_4$ the correlation function $G(z)$ drops rapidly to zero ($\propto \exp(-m|z|)$ as for the Yukawa potential). With an eye on fig. 2.13 the interpretation of (2.220) is clear: there is an amplitude in which the particles produced by J_3 and J_4 travel freely before being absorbed by J_2 and J_1, respectively, a similar amplitude for absorbtion by J_1 and J_2, and an amplitude for the possibility that the particles scatter before being absorbed.

In detail the scattering amplitude can be found as follows. The n-point correlation functions carry two-point functions on their external legs (cf. fig. 2.4). We make these external line two-point functions explicit by writing

$$
G(x_1\cdots x_n)=\int d^4y_1\cdots d^4y_n\, G(x_1y_1)\cdots G(x_ny_n)\, H(y_1\cdots y_n). \qquad (2.221)
$$

In momentum space this can be written as

$$
G(p_1\cdots p_n)=G(p_1)\cdots G(p_n)\, H(p_1\cdots p_n), \qquad (2.222)
$$

or

$$
(-i)^{n-1}G(p_1\cdots p_n)=(-i)G(p_1)\cdots(-i)G(p_n)\, iH(p_1\cdots p_n). \qquad (2.223)
$$

The functions $H(p_1\cdots p_n)$ are sometimes called 'amputated Green functions', connected Green functions with external legs removed. For our case $n=4$ this function is has the generic representation in fig. 2.14. Because of the causal arrangement of the sources the correlation functions at the external lines of the four point function $G(x_1\cdots x_4)$ may be replaced by their large time (ordered) form, $\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\,(-i)G(p) \;\to\; Z_\varphi \int d\omega_p\, e^{ip(x-y)}$, and the scattering term in

Figure 2.14: Diagrams for $H(p_1 \cdots p_4)$.

(2.220) is given by

$$\begin{aligned} Z(J) &= 1 + \cdots + Z_\varphi^4 \int d\omega_{p_1} \cdots d\omega_{p_4} \, iJ_1(p_1)^* \, iJ_2(p_2)^* \, iJ_3(p_3) \, iJ_4(p_4) \\ &\quad (2\pi)^4 \delta^4(p_1 + \cdots + p_4) \, iH(p_1, p_2, -p_3, -p_4) \, + \cdots . \end{aligned} \tag{2.224}$$

Leaving out the emission and absorption amplitudes we identify the amplitude for scattering:

$$i(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \left(\sqrt{Z_\varphi}\right)^4 H(p_1, p_2, -p_3, -p_4). \tag{2.225}$$

We have to keep in mind that the time components p^0 have a small negative imaginary part as follows from the evaluation of $G(z)$ for $z^0 > 0$ (recall fig. 1.1), $p^0 = -p_0 = \sqrt{m^2 + \mathbf{p}^2} - i\epsilon$. This is relevant since H has in general branch point singularities and associated cuts in the complex p^2 plane.

The scattering of particles from an initial state $|i\rangle$ to a final state $|f\rangle$ can be described by the scattering matrix or S-matrix $\langle f|S|i\rangle$. The conservation of probability, $\sum_f |\langle f|S|i\rangle|^2 = 1$, is assured by the unitarity of S, $S^\dagger S = 1$. Separating the possibility of no scattering by writing $S = 1 + iT$, the first few matrix elements of S can be decomposed as

$$\langle p|S|q\rangle = \langle p|q\rangle = 2p^0 (2\pi)^3 \, \delta(\mathbf{p} - \mathbf{q}), \tag{2.226}$$

$$\langle p_1 p_2|S|q\rangle = \langle p|S|q_1 q_2\rangle = 0, \tag{2.227}$$

$$\begin{aligned} \langle p_1 p_2|S|q_1 q_2\rangle &= \langle p_1|q_1\rangle\langle p_2|q_2\rangle + \langle p_1|q_2\rangle\langle p_2|q_1\rangle \\ &\quad + i(2\pi)^4 \, \delta^4(p_1 + p_2 - q_1 - q_2) \, T(p_1, p_2; q_1, q_2). \end{aligned} \tag{2.228}$$

The J_1 – J_4 term in the vacuum amplitude can be written in terms of the scattering matrix as

$$\begin{aligned} Z(J) &= \int d\omega_{p_1} d\omega_{p_2} d\omega_{q_1} d\omega_{q_2} \, \langle 0|p_1\rangle^{J_1} \langle 0|p_2\rangle^{J_2} \\ &\quad \langle p_1 p_2|S|q_1 q_2\rangle \langle q_1|0\rangle^{J_3} \langle q_2|0\rangle^{J_4} + \cdots . \end{aligned} \tag{2.229}$$

Comparison with (2.219), (2.228) and (2.225) shows that

$$T(p_1, p_2; q_1, q_2) = \left(\sqrt{Z_\varphi}\right)^4 H(p_1, p_2, -q_1, -q_2). \tag{2.230}$$

In general, polarization factors for spin (and charge, in the real field formalism) appear naturally. The photon propagator $G^{\mu\nu}(k)$ produces $Z_A g^{\mu\nu} = Z_A[\sum_\lambda e^\mu(k,\lambda)e^\nu(k,\lambda)^* + \text{gauge terms}]$ on external photon lines, e.g.

$$\begin{aligned} & J_\mu(x)G^{\mu\mu'}(x-y)H_{\mu'\cdots\nu'}(y,\cdots,u)G^{\nu'\nu}(u-v)J_\nu(v)\cdots \\ \rightarrow\ & i^2 Z_A^2 J_\lambda(k')^* e^\mu(k',\lambda')H_{\mu\cdots\nu}(k',\cdots,-k)e^\nu(k,\lambda)J_\lambda(k), \end{aligned} \tag{2.231}$$

For example, in scalar electrodynamics the amplitude for scattering of a photon on a scalar particle has the form

$$T(p', k'\lambda'; p, k\lambda) = Z_\varphi Z_A\, e^\mu(k',\lambda')^* H_{\mu\nu}(p', k', -p, -k)e^\nu(k,\lambda). \tag{2.232}$$

The amplitude in (2.227) is zero in φ^4 theory due energy-momentum conservation: $-(p_1+p_2)^2 > -q^2 = m^2$. In a more general setting however we can imagine an incoming particle with mass m to be different from the two outgoing particles. If $m_1 + m_2 < m$, energy-momentum conservation allows the ingoing particle to decay into particles 1 and 2, i.e. the incoming particle is unstable. If we approximate in the external line the unstable particle propagator by a stable particle propagator we can still fit it into our description. The decay amplitude is then given by

$$T(p_1 p_2; q) = \sqrt{Z}\sqrt{Z_1}\sqrt{Z_2}\, H(p_1, p_2, -q). \tag{2.233}$$

Such a stable particle approximation is natural in the semiclassical approximation, in which the propagators are simply the free field propagators.

2.13 Cross section and decay rate

In scattering experiments the typical measurable quantity is the differential cross section. Consider a beam of particles hitting a target, or two colliding beams. The initial particles have momenta p_1 and p_2. The differential cross section $d\sigma$ is, loosely speaking, the number of outcoming particles of a given specification, e.g. n particles with momenta p_3, ..., p_n in a momentum range d^3p_3, ..., d^3p_n, devided by the incoming particle flux. The cross section is related to the scattering amplitude T by the formula

$$\begin{aligned} d\sigma &= \frac{1}{F(p_1,p_2)}\, d\omega_{p_3}\cdots d\omega_{p_n} \\ &\quad (2\pi)^4\delta(p_3+\cdots p_n - p_1 - p_2)\,\overline{|T|^2}, \end{aligned} \tag{2.234}$$

$$F(p_1, p_2) = \frac{1}{4\sqrt{(p_1p_2)^2 - m_1^2 m_2^2}}. \tag{2.235}$$

Here $\overline{|T|^2}$ is the modulus-squared of the scatering amplitude, averaged over the spin polarizations of the initial particles and summed over the final spin polarizations,

$$\overline{|T|^2} = \frac{1}{(2s_1+1)(2s_2+1)} \sum_{\lambda_1 \cdots \lambda_n} |\langle p_3\lambda_3, \cdots, p_n\lambda_n|T|p_1\lambda_1, p_2\lambda_2\rangle|^2, \tag{2.236}$$

Where s_1 and s_2 are the spins of the incoming particles. For the photon $2s+1 \to 2$ as it has only two independent polarizations. More refined information can of course be obtained by analysing the spin dependence of the cross sections. The factor $1/F$ is a Lorentz invariant flux factor. When $\mathbf{p}_1$ and $\mathbf{p}_2$ are collinear it is simply related to the relative velocity of the incoming particles,

$$F = p_1^0 p_2^0 v_{\rm rel}, \quad v_{\rm rel} = \left|\frac{\mathbf{p}_1}{p_1^0} - \frac{\mathbf{p}_2}{p_2^0}\right|. \tag{2.237}$$

For the derivation of the above formulas see Brown sect. 3.4, De Wit & Smith ch. 3, or the 1975/76 lecture notes.

In the case of two particle scattering the differential cross section in the centre of mass frame ($\mathbf{p}_1 + \mathbf{p}_2 = 0$) is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{F}\,\frac{k_f}{16\pi^2 W}\overline{|T|^2} = \frac{1}{64\pi^2 W^2}\,\frac{k_f}{k_i}\,\overline{|T|^2}, \tag{2.238}$$

where W is the total energy and k_i and k_f are the magnitudes of the initial and final three momenta,

$$W = p_1^0 + p_2^0, \quad k_i = |\mathbf{p}_1| = |\mathbf{p}_2|, \quad k_f = |\mathbf{p}_3| = |\mathbf{p}_4|. \tag{2.239}$$

These quantities can be expressed in terms of the Mandelstam variables, s, t and u, cf. fig. 2.15,

$$\begin{aligned} s &= -(p_1+p_2)^2 = -(p_3+p_4)^2, && (2.240)\\ t &= -(p_1-p_3)^2 = -(p_2-p_4)^2, && (2.241)\\ u &= -(p_1-p_4)^2 = -(p_2-p_3)^2, && (2.242)\\ s+t+u &= m_1^2+m_2^2+m_3^2+m_4^2. && (2.243) \end{aligned}$$

In the centre of mass frame $s = W^2$ is the squared total energy. The momentum transfer t between particle 1 and 3 is related to the scattering angle θ,

$$t = (p_1^0 - p_3^0)^2 - (k_i^2 + k_f^2 - 2k_i k_f \cos\theta), \tag{2.244}$$

and similar for u. Using

$$dt = 2k_i k_f d\cos\theta, \tag{2.245}$$

Figure 2.15: Two particle scattering.

and integrating over the azimuthal angle ϕ we have

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s k_i^2}\,\overline{|T|^2}. \tag{2.246}$$

We can express k_i, k_f and $\overline{|T|^2}$ as a function of s and t, which brings $d\sigma/dt$ into a manifestly Lorentz invariant form. Furthermore, the invariant momentum transfer t has more physical significance than the scattering angle in some frame. The differential cross section in the laboratory frame ($\mathbf{p}_2 = 0$) can be obtained by expressing s and t in terms of the lab frame variables.

We conclude this brief summary by giving the formula for the differential decay rate of an unstable particle of momentum p, $p^2 = -m^2$ to n outgoing particles, in the rest frame,

$$d\Gamma = (\prod_i d\omega_{p_i})\,(2\pi)^4\delta(p_1 + \cdots p_n - p)\,\frac{|T|^2}{2m}. \tag{2.247}$$

For two particle decay

$$\frac{d\Gamma}{d\Omega} = \frac{k}{32\pi^2 m^2}\,|T|^2, \tag{2.248}$$

where $k = |\mathbf{p}_1| = |\mathbf{p}_2|$.

2.14 Examples in scalar electrodynamics

We give here some examples in the semiclassical approximation, also known as the tree graph approximation, since the relevant Feynman diagrams in terms of the classical propagators and vertices have a tree structure without any loops. Since the propagators have the free field form, $Z_\varphi = Z_A = 1$. Furthermore, there is no difference between bare and renormalized parameters, $e_0 \to e$ etc. For definiteness we shall call the charged scalar particles $\pi^\pm$. The diagrams for the scattering $\pi^+ + \pi^- \to \pi^+ + \pi^-$ are given in fig. 2.16. Notice the annihilation diagram (second diagram), in which π^+ and π^- annihilate into a 'virtual photon' and subsequently get re-emitted. It is as if the incoming and outgoing $\pi^\pm$ produce

Figure 2.16: Diagrams for $\pi^+(p_1) + \pi^-(q_1) \rightarrow \pi^+(p_2) + \pi^-(q_2)$.

effective sources which emit and absorb the virtual photon. Virtual in this jargon refers to the fact that the effective mass of the incoming state, $-(p_1+q_1)^2 > 0$, and not zero as for the photon. The photon is said to be 'off the mass shell' as its four momentum is timelike. In a similar intuitive language we say that the π^+ and π^- in the first diagram exchange a virtual photon. In this case the four momentum of the virtual photon is spacelike.

The scattering amplitude is given by

$$\begin{aligned} T &= e^2(p_1+p_2)_\mu \frac{g^{\mu\nu}-(1-\xi)k^\mu k^\nu/k^2}{k^2}(-q_1-q_2)_\nu \\ &\quad + (p_2-q_2)_\mu \frac{g^{\mu\nu}-(1-\xi)l^\mu l^\nu/l^2}{l^2}(p_1-q_1)_\nu, \end{aligned} \tag{2.249}$$

where $k = p_2 - p_1 = q_1 - q_2$ and $l = p_1 + q_1 = p_2 + q_2$. The gauge dependent ($\xi$-dependent) part of the photon propagator does not contribute because $k(p_1 + p_2) = (p_1 - p_2)(p_1 + p_2) = -m^2 + m^2 = 0$, and similar for $l(p_1 - q_1)$, which is an expression of conservation of the electromagnetic current (cf. Problems). In terms of the Mandelstam variables (2.243) the amplitude can be written in manifestly Lorentz invariant form

$$T = e^2\left(\frac{u-s}{t} + \frac{u-t}{s}\right). \tag{2.250}$$

Another example is the scattering of π^- off a different positively charged particle with mass M. To describe this we introduce a new scalar field for this particle and couple it also to the electromagnetic field. The vertex functions are identical in form, except for the new mass M, and since the annihilation diagram is absent in this case, T is given by the first term in (2.250) only. We quote the differential crossection in the laboratory frame from De Wit & Smith sect. 4.3,

$$\frac{d\sigma}{d\Omega_{\rm lab}} = \frac{\alpha^2}{4E^2\sin^4\frac{1}{2}\theta}\left(\frac{1+E/M\sin^2\frac{1}{2}\theta}{1+2E/M\sin^2\frac{1}{2}\theta}\right), \tag{2.251}$$

where E is the lab energy of the incoming π^- and $\alpha = e^2/4\pi$ is the fine structure constant. For a heavy target $M \rightarrow \infty$ we get Rutherford's formula.

Figure 2.17: Diagrams for $\gamma(k) + \pi^+(p) \to \gamma(k') + \pi^+(p')$.

Notice that at last we have identified the coupling constant e as the elementary charge unit by comparison with experiment.

We finish here with pion-Compton scattering, $\gamma(k) + \pi^+(p) \to \gamma(k') + \pi^+(p')$. From the diagrams in fig. 2.17 we find the scattering amplitude

$$\begin{aligned} T(k'\lambda', p'; k\lambda, p) &= e^{\mu}(k', \lambda')^* \, H_{\mu\nu}(k', p', -k, -p) \, e^{\nu}(k, \lambda), \qquad (2.252)\\ H_{\mu\nu}(k', p', -k, -p) &= e^2 \left[\frac{(2p' + k')_\mu (2p + k)_\nu}{(p + k)^2 + m^2} + \frac{(2p - k')_\mu (2p' - k)_\nu}{(p - k')^2 + m^2} \right.\\ &\quad \left. -2g_{\mu\nu} \right]. \qquad (2.253) \end{aligned}$$

The tensor $H_{\mu\nu}$ is transverse,

$$k'^{\mu} H_{\mu\nu} = k^{\nu} H_{\mu\nu} = 0, \qquad (2.254)$$

where it is essential that the pions are on-shell, $p^2 = p'^2 = -m^2$. This expresses gauge invariance, the amplitude is unchanged when we substitute e.g. $e^{\mu}(k', \lambda')^* \to e^{\mu}(k', \lambda')^* + \omega k'^{\mu}$. The transversality of the amplitudes is also essential for Lorentz invariance. Averaging over initial polarizations and summing over final polarizations,

$$\begin{aligned} \overline{|T|^2} &= \frac{1}{2} \sum_{\lambda\lambda'} |T|^2 \\ &= \frac{1}{2} \sum_{\lambda'} e^{\mu}(k', \lambda')^* e^{\rho}(k', \lambda') \sum_{\lambda} e^{\nu}(k, \lambda) e^{\sigma}(k, \lambda)^* \, H_{\mu\nu} H^*_{\rho\sigma}, \qquad (2.255) \end{aligned}$$

we use

$$\sum_{\lambda} e^{\nu}(k, \lambda) e^{\sigma}(k, \lambda)^* = g^{\nu\sigma} + \text{gauge terms}, \qquad (2.256)$$

where the gauge terms are terms $\propto k^{\nu}$ or k^{σ}. Then the result can be expressed as

$$\begin{aligned} \overline{|T|^2} &= \frac{1}{2} H_{\mu\nu} H^{\mu\nu *} \qquad (2.257)\\ &= 2e^4 \left[m^4 \left(\frac{1}{pk} - \frac{1}{pk'} \right)^2 - 2m^2 \left(\frac{1}{pk} - \frac{1}{pk'} \right) + 2 \right], \qquad (2.258) \end{aligned}$$

which is manifestly Lorentz invariant. We quote De Witt & Smith sect. 4.4 for the differential cross section in the lab frame,

$$\frac{d\sigma}{d\Omega_{\rm lab}} = \frac{\alpha^2}{2m^2}\,\frac{1+\cos^2\theta}{[1+E/m(1-\cos\theta)]^2}. \tag{2.259}$$

In the low energy limit this reduces to the result for classical electromagnetic radiation,

$$\frac{d\sigma}{d\Omega_{\rm lab}} = \frac{\alpha^2}{2m^2}\,(1+\cos^2\theta). \tag{2.260}$$

Integrating over angles we get the Thompson cross section

$$\sigma = \frac{8\pi\alpha^2}{3m^2}. \tag{2.261}$$

2.15 Appendix

The classical energy (2.31) is also the expectation value of the energy operator in the interaction picture

$$H - \int d^3x\, J(x)\varphi(x), \quad H = \int d\omega_p\, a(\mathbf{p})^\dagger a(\mathbf{p})\, p^0, \tag{2.262}$$

in an appropriate state with classical properties. We have seen such a state before in section 1.9 for the case of the electromagnetic field, the state

$$|0,t\rangle = U_J(t,-\infty)|0\rangle = T\, e^{i\int_{-\infty}^{t} d^4x'\, J(x')\varphi(x')}|0\rangle. \tag{2.263}$$

Consider therefore a source $J(x) = J_1(x) + J_2(x)$ which is static for a very long time and goes to zero in the far past. Under these conditions the classical field is given by (the calculation is as in sect. 1.9)

$$\varphi^{(c)}(x) = \langle 0,t|\varphi(x)|0,t\rangle = \int d^4y\, G_{\rm ret}(x-y)J(y), \tag{2.264}$$

where $G_{\rm ret}(x-y)$ is the retarded Green function, given in momentum space by

$$G_{\rm ret}(p) = \frac{1}{m^2+\mathbf{p}^2-(p_0+i\epsilon)^2}. \tag{2.265}$$

For times much larger than the intial transient period in which the source is switched on we can take the static approximation $J(y) \to J(\mathbf{y})$ and integrate over y^0, which leads to the static Green function,

$$\begin{aligned}\int dy^0\, G_{\rm ret}(x-y) &= \int \frac{d^4p}{(2\pi)^4}\int dy^0\, \frac{e^{ip(x-y)}}{m^2+\mathbf{p}^2-(p_0+i\epsilon)^2} \\ &= G_{\rm stat}(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{2.266}$$

Hence $\varphi^{(c)}(x) \to \varphi^{(c)}(\mathbf{x})$, the static field of (2.25), and $\langle 0,t|\int d^3x\, J(x)\varphi(x)|0,t\rangle$ has the corrsponding value ($x^0 = t$). To evaluate $\langle 0,t|H|0,t\rangle$ we use the fact that $|0,t\rangle$ is an coherent eigenstate of $a(\mathbf{k})$ in the static approximation. This can be seen by differentiating

$$U_J(t,-\infty)^\dagger a(\mathbf{k}) U_J(t,-\infty) \tag{2.267}$$

with respect to t, which gives the c-number

$$i\int d^3x\, e^{-ipx}\, J(x) \equiv i e^{ip^0 t}\, J(\mathbf{p},t). \tag{2.268}$$

Integrating this from $-\infty$ to t with $\exp(ip^0 t)\, J(\mathbf{p},t) \to \exp[(ip^0-\epsilon)t]\, J(\mathbf{p})$, $\epsilon \to +0$, then gives

$$U_J(t,-\infty)^\dagger a(\mathbf{p}) U_J(t,-\infty)|0\rangle = a(\mathbf{p}) + \frac{e^{ip^0 t}\, J(\mathbf{p})}{p^0}, \tag{2.269}$$

hence

$$a(\mathbf{p})|0,t\rangle = e^{ip^0 t}\,\frac{J(\mathbf{p})}{p^0}|0,t\rangle. \tag{2.270}$$

It follows that we may replace the annihilation operator in H in $\langle 0,t|H|0,t\rangle$ by the above eigenvalue when acting on the ket $|0,t\rangle$, and similar for the creation operator when acting on the bra $\langle 0,t|$. This gives back the classical expression for the energy in terms of

$$\varphi^{(c)}(\mathbf{x}) = \int d\omega_p\,[e^{i\mathbf{px}} a^{(c)}(\mathbf{p}) + e^{-i\mathbf{px}} a^{(c)}(\mathbf{p})^*], \quad a^{(c)}(\mathbf{p}) = \frac{J(\mathbf{p})}{p^0}. \tag{2.271}$$

2.16 Problems

1. For the free scalar field, verify that $\partial_\mu T^{\mu\nu} = 0$ as a consequence of the equation of motion.

2. Verify that j^μ given in (2.80) is the Noether current associated with the global U(1) invariance of $S_{A\varphi}$.

3. Derive the equations of motion for scalar electrodynamics and verify that ej^μ is the electromagnetic current in Maxwell's equations.

4. Verify that $\partial_\mu j^\mu = 0$ as a consequence of the equations of motion for the scalar fields. When the total action contains external source terms $\int d^4x\,(J^*\varphi + J\varphi^*)$, obtain $\partial_\mu j^\mu$.

5. Using creation and annihilation operators, calculate the expectation values of the current in the free complex scalar field theory, $\langle 0|j^\mu(x)|0\rangle$ and $\langle p\pm|j^\mu(x)|q\pm\rangle$.

6. In the φ^4 theory verify

$$\begin{aligned}\langle 0|T\varphi(x)\varphi(y)\varphi(z)|0\rangle &= \phi_0^3 + (-i)G(x,y)\phi_0 + (-i)G(y,z)\phi_0 \\ &\quad + (-i)G(z,x)\phi_0 + (-i)^2 G(x,y,z).\end{aligned} \tag{2.272}$$

Similarly, express

$$\langle 0|T\varphi(w)\varphi(x)\varphi(y)\varphi(z)|0\rangle \tag{2.273}$$

in terms of the correlation functions.

7. Consider U defined in (2.95) for the φ^4 theory. Verify that the mass m, as defined by the position of the pole in the propagator, is given in the semiclassical approximation by $m^2 = \partial^2 U/\partial\varphi^2$, evaluated at the ground state value of φ.

8. Let $F(A)$ be a functional of $A^\mu(x)$. From the definition of the functional derivative,

$$\delta F = \int d^4x \, \frac{\delta F}{\delta A^\mu(x)} \, \delta A^\mu(x), \tag{2.274}$$

verify that

$$\frac{\delta A^\mu(x)}{\delta A^\nu(y)} = \delta^\mu_\nu \, \delta^4(x-y). \tag{2.275}$$

For a scalar field $\varphi(x)$ the corresponding relation reads

$$\frac{\delta\varphi(x)}{\delta\varphi(y)} = \delta^4(x-y). \tag{2.276}$$

Using this relation we can calculate

$$S(u,v) \equiv -\frac{\delta}{\delta\varphi(u)}\frac{\delta}{\delta\varphi(v)}\int d^4x \, \frac{1}{2}\,\partial_\mu\varphi(x)\partial^\mu\varphi(x) \tag{2.277}$$

$$= -\int d^4x \, \partial_\mu\delta^4(x-u)\partial^\mu\delta^4(x-v) \tag{2.278}$$

$$= \partial^2\,\delta^4(u-v), \tag{2.279}$$

and its Fourier transform (from (2.278))

$$\int d^4u d^4v \, e^{-ipu-iqv}\, S(u,v) = -\int d^4x \, \partial_\mu(e^{-ipx})\partial^\mu(e^{-iqx}) \tag{2.280}$$

$$= -p^2(2\pi)^4\delta(p+q). \tag{2.281}$$

Derive along similar lines that

$$\begin{aligned} S_{\alpha\beta}(u,v) &\equiv -\frac{\delta}{\delta A^\alpha(u)}\frac{\delta}{\delta A^\beta(v)}\int d^4x \, \frac{1}{4}\, F^{\mu\nu}(x)F_{\mu\nu}(x) \\ &= (g_{\alpha\beta}\partial^2 - \partial_\alpha\partial_\beta)\delta^4(u-v), \end{aligned} \tag{2.282}$$

$$\int d^4u d^4v \, e^{-ipu-iqv}\, S_{\alpha\beta}(u,v) = -(p^2 g_{\alpha\beta} - p_\alpha p_\beta)(2\pi)^4\delta(p+q). \tag{2.283}$$

9. Verify the vertex functions of scalar electrodynamics in the real field formalism as given in eqs. (2.199) – (2.202).

10. The vertex functions can also be read off in momentum space, writing S as (using φ^4 theory as example)

$$\begin{aligned} S &= \sum_n \frac{1}{n!} \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4} (2\pi)^4 \, \delta(p_1 + \cdots + p_n) \\ & \quad S(p_1, \cdots, p_n) \, \phi(-p_1) \cdots \phi(-p_n), \qquad (2.284) \\ \phi(p) &= \int d^4x \, e^{-ipx} \, \phi(x). \qquad (2.285) \end{aligned}$$

Rederive the vertex functions of scalar electrodynamics (in the real and complex formalism) using this method.

11. Rederive eqs. (2.211) from

$$\langle p\alpha|\varphi_\beta(x)|0\rangle = \delta_{\alpha\beta} \sqrt{Z_\varphi} \, e^{-ipx} \qquad (2.286)$$

in the real field formalism. Reason that $\delta_{\alpha\beta}$ in the above equation is a consequence of global SO(2) invariance.

12. In scalar electrodynamics, in the real field formalism, draw the semiclassical diagrams for pion-Compton scattering, indicating the relevant indices, in momentum space, and write the expression for

$$H_{\mu\alpha\nu\beta}(k', p', -k, -p). \qquad (2.287)$$

Then reobtain (2.252)-(2.253) from

$$T(k'\lambda', p', k\lambda, p) = e^\mu(k', \lambda')^* \, e_\alpha(+)^* \, H_{\mu\alpha\nu\beta}(k', p', -k, -p) \, e^\nu(k, \lambda) \, e_\beta(+). \qquad (2.288)$$

13. *Pions and the linear sigma model*

The lagrangian of the linear σ model

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi_\alpha\partial^\mu\varphi_\alpha - \frac{1}{2}\mu^2\varphi_\alpha\varphi_\alpha - \frac{1}{4}\lambda(\varphi_\alpha\varphi_\alpha)^2 \qquad (2.289)$$

($\alpha = 1, 2, 3, 4,\ \lambda > 0$), is invariant under $O(4)$ rotations $\varphi_\alpha \to R_{\alpha\beta}\varphi_\beta$, $R^T = R^{-1}$. For $\mu^2 < 0$ in the semiclassical aproximation, the system undergoes spontaneous symmetry breaking, $\langle 0|\varphi_\alpha|0\rangle \neq 0$. Adding the explicit symmetry breaking term

$$\Delta\mathcal{L} = \epsilon\varphi_4 \qquad (2.290)$$

to $\mathcal{L}$, gives

$$\langle 0|\varphi_\alpha|0\rangle = v\delta_{\alpha,4}. \qquad (2.291)$$

For the descriptions of pions we make the identification $\pi_a = \varphi_a$, $a = 1, 2, 3$, while φ_4 corresponds to the σ particle. The latter may be identified with a very broad spin zero, isospin zero, enhancement in $\pi\pi$ scattering. In the excercises below we compute the width of the sigma particle, the pi-pi scattering amplitude and compare with experiment.

1. Express v, m_π and m_σ in terms of λ, μ^2 and ϵ.

For $m_\sigma > 2m_\pi$ the sigma particle can decay into two pions.

2. Show that the $\sigma\pi^a\pi^b$ vertex function equals $-2\lambda v\delta_{ab}$, and calculate the matrix element $\langle p_1a_1p_2a_2|T|p\rangle$ for the decay $\sigma(p) \to \pi(p_1a_1) + \pi(p_2a_2)$.

The internal rotations which transform the π_a into each other are called isospin transformations, with generators I_a, $a = 1, 2, 3$. The pion states $|a\rangle$, $a = 1, 2, 3$ (suppressing the momentum label p) transform in the vector (adjoint) representation, in which the isospin operators I_a are represented as $\langle b|I_a|c\rangle = -i\epsilon_{abc}$. The physical pion states with well defined charge are eigenstates $|I, I_3\rangle$ of I^2 and I_3 with $|1, 1\rangle = |\pi^+\rangle$, $|1, 0\rangle = |\pi^0\rangle$ and $|1, -1\rangle = |\pi^-\rangle$. Isospin polarization vectors $e^a_{I_3} = \langle a|1, I_3\rangle$ can be chosen as $e^a_{+1} = (-1, -i, 0)/\sqrt{2}$, $e^a_0 = (0, 0, 1)$, $e^a_{-1} = (1, -i, 0)/\sqrt{2}$.

3. Check that the above polarization vectors are consistent with the standard action of the isospin lowering operator $I_-|1, 1\rangle = \sqrt{2}|1, 0\rangle$, etc. ($I_- = I_1 - iI_2$).

4. Show that

$$\langle\pi^+\pi^-|T|\sigma\rangle = 2\lambda v, \quad \langle\pi^0\pi^0|T|\sigma\rangle = -2\lambda v. \tag{2.292}$$

The differential decay width in the σ rest frame is given by

$$d\Gamma = \frac{k}{32\pi^2 m_\sigma^2}\, |T|^2\, d\Omega, \tag{2.293}$$

where k, Ω are the spherical coordinates of the momentum of one of the pions.

5. Verify that

$$\Gamma(\sigma \to \pi^+\pi^-) = 2\Gamma(\sigma \to \pi^0\pi^0) = \frac{k\lambda^2 v^2}{2\pi m_\sigma^2}, \tag{2.294}$$

keeping in mind that the two π^0 particles are identical.

6. As a check, compute the total decay width also directly from

$$\Gamma = \frac{1}{2m_\sigma}\,\frac{1}{2}\sum_{a_1a_2}\int d\omega_{p_1} d\omega_{p_2}\,(2\pi)^4\delta(p - p_1 - p_2)|\langle p_1a_1p_2a_2|T|p\rangle|^2. \tag{2.295}$$

The explicit factor 1/2 corresponds to $1/n!$ in the formula for the unit operator (for free pions) in the n-particle subspace

$$1 = \sum_n \frac{1}{n!} \sum_{a_1 \cdots a_n} \int d\omega_{p_1} \cdots d\omega_{p_n} \, |p_1 a_1 \cdots p_n a_n\rangle \langle p_1 a_1 \cdots p_n a_n|. \tag{2.296}$$

7. Interpreting the σ enhancement in $\pi\pi$ scattering as an unstable σ-particle, it might have a mass around 900 MeV and a width of rougly 600 MeV. Given that $v = f_\pi = 93$ MeV, derive

$$\Gamma = \frac{3}{32\pi} \frac{(m_\sigma^2 - m_\pi^2)^2 \, (m_\sigma^2 - 4m_\pi^2)^{1/2}}{m_\sigma^2 f_\pi^2} \tag{2.297}$$

and compare with the above physical data. Derive an upper limit for m_σ from requiring $\Gamma/m_\sigma < 1$.

The pi-pi scattering amplitude can be written as

$$\langle p_3 a_3 p_4 a_4 | T | p_1 a_1 p_2 a_2 \rangle = A \delta_{a_1 a_2} \delta_{a_3 a_4} + B \delta_{a_1 a_3} \delta_{a_2 a_4} + C \delta_{a_1 a_4} \delta_{a_2 a_3}. \tag{2.298}$$

The A, B and C can be expressed in the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_3)^2$ and $u = -(p_1 - p_4)^2$. In the c.m. frame, $s = W^2$, $W =$ total energy, $t = -2k^2(1 - \cos\theta)$, $u = -2k^2(1 + \cos\theta)$, $k =$ c.m. momentum, $\theta =$ scattering angle.

8. Derive

$$A = -2\lambda + \frac{(2\lambda v)^2}{m_\sigma^2 - s} = 2\lambda \frac{s - m_\pi^2}{m_\sigma^2 - s}, \tag{2.299}$$

and find the corresponding expressions for B and C.

9. By using the step isospin operator I_- acting on $|\pi^+\pi^+\rangle$, construct total isospin eigenstates $|I, I_3\rangle$ for $I = 0, 1, 2$ in terms of $|\pi^+\pi^+\rangle$, $|\pi^\pm\pi^\mp\rangle$, and $|\pi^0\pi^0\rangle$. Using the isospin polarization vectors $e^a_{I_3}$ defined earlier, derive the following expressions for the scattering amplitudes $T^I = \langle I, I_3 | T | I, I_3 \rangle$ in total isospin channel I:

$$T^0 = B + C, \quad T^1 = B - C, \quad T^2 = 3A + B + C. \tag{2.300}$$

The partial wave expansion for T can be written as

$$T^I = \frac{8\pi W}{k} \sum_l (2l + 1) T_l^I P_l(\cos\theta). \tag{2.301}$$

where the P_l are the Legendre polynomials. Neglecting inelasticity effects, the phase shift in isospin channel I and angular momentum channel l is given by

$$\exp(2i\delta_l^I) = 1 + iT_l^I, \tag{2.302}$$

and the s-wave scattering lengths are defined by

$$a_0^I = \lim_{k\to 0} \delta_0^I / k. \tag{2.303}$$

10. Neglecting terms of order m_π^2/m_σ^2, derive Weinberg's results

$$a_0^0 = \frac{7}{32\pi}\frac{m_\pi}{f_\pi^2}, \quad a_0^2 = \frac{2}{32\pi}\frac{m_\pi}{f_\pi^2}, \tag{2.304}$$

and compare with the experimental values $a_0^0 = 0.26 \pm 0.05$ fm (1 fm $\approx$ $(200\mathrm{MeV})^{-1}$).

Chapter 3

Lorentz invariance

We explore in this chapter some basic aspects of Lorentz transformations and translations.

3.1 Lorentz transformations

The elements of the group of Lorentz transformations can be defined as the matrices Λ which leave the inner product $xy = g_{\mu\nu}\, x^\mu y^\nu$ invariant:

$$x^\mu \to \Lambda^\mu{}_\alpha x^\alpha, \quad y^\nu \to \Lambda^\nu{}_\beta y^\beta, \quad xy = g_{\mu\nu} x^\mu y^\nu \to g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha y^\beta = g_{\alpha\beta} x^\alpha y^\beta. \quad (3.1)$$

Since x and y are arbitrary this invariance expresses that the metric $g_{\mu\nu}$ is an invariant tensor, which is really a condition on Λ,

$$g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\mu{}_\beta = g_{\alpha\beta}. \quad (3.2)$$

In matrix notation we assign the matrix elements of Λ as

$$(\Lambda)_{\mu\nu} = \Lambda^\mu{}_\nu, \quad (3.3)$$

and we order the indices $\mu = 1, 2, 3, 0$ such that

$$(g)_{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \quad (3.4)$$

Then the Λ's satisfy

$$\Lambda^T g \Lambda = g, \quad (3.5)$$

where T denotes transposition. It follows that

$$\det(\Lambda^T g \Lambda) = (\det\Lambda)^2 \det g = \det g \Rightarrow \det\Lambda = \pm 1. \quad (3.6)$$

For $\alpha = 0$, $\beta = 0$, eq. (3.2) gives

$$(\Lambda^0{}_0)^2 = 1 + \sum_{m=1}^{3} (\Lambda^m{}_0)^2 \Rightarrow \Lambda^0{}_0 \geq 1 \text{ or } \Lambda^0{}_0 \leq 1. \tag{3.7}$$

Thus the full Lorentz group consists of four disjoint sets according to wether $\det \Lambda = \pm 1$ and $\Lambda^0{}_0 \geq 1$ or ≤ 1. Important examples are parity,

$$P = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \tag{3.8}$$

with $\det \Lambda = -1$ and $\Lambda^0{}_0 = 1$, and time reversal

$$T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \tag{3.9}$$

with $\det \Lambda = -1$ and $\Lambda^0{}_0 = -1$. The product PT has $\det \Lambda = 1$ and $\Lambda^0{}_0 = -1$.

Transformations with $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$ are elements of the proper or orthochronous Lorentz group $L_+^\uparrow$. From now on we shall omit the adjective 'proper' and call the $L_+^\uparrow$ 'the Lorentz group'. The elements of $L_+^\uparrow$ are continuously connected to the identity and can be written as

$$\Lambda = \exp(F), \quad \operatorname{Tr} F = 0. \tag{3.10}$$

From (3.5) we see that the real matrix F has to satisfy

$$F^T g = -gF, \tag{3.11}$$

which yields the following solution in terms of parameters $\omega^{\kappa\lambda}$ and generators $M_{\kappa\lambda}$,

$$\begin{aligned} F &= -i\frac{1}{2}\omega^{\kappa\lambda} M_{\kappa\lambda}, & (3.12)\\ -i(M_{\kappa\lambda})_{\mu\nu} &= -iM_{\kappa\lambda}{}^{\mu}{}_{\nu} = -(g^\mu_\kappa g_{\lambda\nu} - g^\mu_\lambda g_{\kappa\nu}), & (3.13) \end{aligned}$$

where the somewhat artificial looking factor $(-i)$ is put in for later convenience. The M_{kl} are hermitian and antisymmetric matrices which are explicitly given by

$$M_{kl} \equiv \epsilon_{klm} M_m, \quad -iM_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.14}$$

$$-iM_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad -iM_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

These are the generators of rotations,

$$\Lambda(R) = \exp(-i\omega_m M_m), \quad \omega_m \equiv \frac{1}{2}\epsilon_{klm}\,\omega^{kl}. \tag{3.16}$$

This can be seen more easily from the representation

$$(M_k)_{\lambda\mu} = -i\epsilon_{klm}, \tag{3.17}$$

for $\lambda, \mu = l, m = 1, 2, 3$ and $(M_k)_{\lambda\mu} = 0$ zero otherwise. The first three rows and columns of M_k are just the spin 1 matrices S_k of (1.145). The M_{k0} are symmetric antihermitian matrices which are explicitly given by

$$M_{k0} \equiv N_k, \quad -iN_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.18}$$

$$-iN_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad -iN_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.19}$$

These are the generators of special Lorentz transformations, often called boosts,

$$\Lambda(B) = \exp(-i\chi_k N_k), \quad \chi_k \equiv \omega_{k0}. \tag{3.20}$$

A boost in the 3-direction,

$$= \exp(-i\chi N_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\chi & \sinh\chi \\ 0 & 0 & \sinh\chi & \cosh\chi \end{pmatrix}, \tag{3.21}$$

has the effect

$$x^{1,2} \to x^{1,2}, \quad x^3 \to \gamma x^3 + \gamma\beta x^0, \quad x^0 \to \gamma\beta x^3 + \gamma x^0, \tag{3.22}$$

where $\beta = v/c$ $(c = 1)$ and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\chi, \quad \gamma\beta = \sinh\chi, \tag{3.23}$$

are the usual parameters in special Lorentz transformations. The rotation matrices are orthogonal, $\Lambda(R)^T = \Lambda(R)^{-1}$ and the boosts are symmetric $\Lambda(B)^T = \Lambda(B)$.

The generators satisfy the following commutation relations,

$$
\begin{aligned}
[M_{\kappa\lambda}, M_{\mu\nu}] &= i(g_{\kappa\mu}M_{\lambda\nu} + g_{\lambda\nu}M_{\kappa\mu} - g_{\kappa\nu}M_{\lambda\mu} - g_{\lambda\mu}M_{\kappa\nu}, &(3.24)\\
[M_k, M_l] &= i\epsilon_{klm}M_m, &(3.25)\\
[M_k, N_l] &= i\epsilon_{klm}N_m, &(3.26)\\
[N_k, N_l] &= -i\epsilon_{klm}M_m, &(3.27)
\end{aligned}
$$

The rotations form a group, but the special Lorentz transformations do not form a group, as is clear from the fact that the commutation relations of the N_l are not closed. The boosts generate rotations and only boosts combined with rotations form a group, the Lorentz group $L_+^\uparrow$.

Under parity and time reversal the generators transform as

$$
\begin{aligned}
P\,M_k\,P &= M_k, \quad P\,N_k\,P = -N_k,\\
T\,M_k\,T &= M_k, \quad T\,N_k\,T = -N_k. &(3.28)
\end{aligned}
$$

The effect of T on M_k and may seem strange, since one may expect the angular momentum or spin to change sign under T. This is indeed the case in the quantum theory, where T involves complex conjugation: T is realized by an antiunitary operator in Hilbert space.

3.2 Irreps and SL(2,C)

To find irreducible representations (irreps) of the Lorentz group we consider the linear combinations

$$I_l^\pm = \frac{1}{2}(M_l \mp iN_l), \tag{3.29}$$

which are hermitian matrices satisfying the commutation relations,

$$
\begin{aligned}
[I_k^\pm, I_l^\pm] &= i\epsilon_{klm}\,I_m^\pm, &(3.30)\\
[I_k^\pm, I_l^\mp] &= 0. &(3.31)
\end{aligned}
$$

The Lorentz group $L_+^\uparrow$ is equivalent ‘in the small’ to two independent rotation groups. This enables us to find irreps of the Lorentz group from the knowledge of those of the rotation group. The representations will be labeled by two angular momenta (j^+, j^-), $j^\pm = 0, 1/2, 1, 3/2, \ldots$, with $(I^\pm)^2 = I_k^\pm I_k^\pm = j^\pm(j^\pm + 1)$, and with the eigenvalues of $I_3^\pm$ taking the values $-j^\pm, \ldots, +j^\pm$.

Since the parity operation changes the sign of $\mathbf{N}$ but not of $\mathbf{M}$, the generators $\mathbf{I}^\pm$ are interchanged under P,

$$P\,\mathbf{I}^\pm\,P = \mathbf{I}^\mp. \tag{3.32}$$

The action of P leads outside an irreducible representation (j^+, j^-) of $L_+^\uparrow$, but it can be represented in a reducible representation $(j^+, j^-) + (j^-, j^+)$, where it interchanges the components (j^+, j^-) and (j^-, j^+).

The simplest nontrivial representations are the two dimensional representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, realized by

$$(\frac{1}{2},0) \quad : \quad M_l \to \frac{1}{2}\sigma_l, \quad N_l \to +i\frac{1}{2}\sigma_l, \quad I_l^+ \to \frac{1}{2}\sigma_l, \quad I_l^- \to 0, \tag{3.33}$$

$$(0,\frac{1}{2}) \quad : \quad M_l \to \frac{1}{2}\sigma_l, \quad N_l \to -i\frac{1}{2}\sigma_l, \quad I_l^+ \to 0, \quad I_l^- \to \frac{1}{2}\sigma_l, \tag{3.34}$$

where σ_l are the Pauli matrices acting in a two dimensional representation space. The matrices in the $(\frac{1}{2},0)$ representation have the form

$$\exp[(\chi_k - i\omega_k)\frac{1}{2}\sigma_k] \equiv L, \tag{3.35}$$

while for the $(0,\frac{1}{2})$ representation

$$\exp[(-\chi_k - i\omega_k)\frac{1}{2}\sigma_k] = (L^\dagger)^{-1}. \tag{3.36}$$

These representations are double valued because the rotations are represented double valued in the $j=\frac{1}{2}$ representation. The double valuedness is a nuisance and it is convenient to work with L directly. The matrices L are general complex 2×2 matrices with $\det L = 1$. This is the defining representation of the group SL(2,C), the group of general complex linear unimodular transformations in two dimensions. The representation $L \to L^*$ of SL(2,C) is inequivalent to L, but equivalent to $L \to (L^\dagger)^{-1}$, because

$$\sigma_k^* = -\sigma_2\,\sigma_k\,\sigma_2 \tag{3.37}$$

implies

$$L^* = \sigma_2\,(L^\dagger)^{-1}\,\sigma_2. \tag{3.38}$$

We now interpret a Lorentz transformation Λ to be a representation of SL(2,C), $\Lambda = \Lambda(L)$. It corresponds to the representation $(\frac{1}{2},\frac{1}{2})$, $L \to L \times L^* \simeq \Lambda$, as follows. Let us assemble the components of a four vector x^μ into a matrix X,

$$X = x^\mu\,\sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \quad \sigma_0 \equiv 1\!\!1, \tag{3.39}$$

$$x^\mu = \frac{1}{2}\mathrm{Tr}\,\sigma_\mu\,X, \quad \mathrm{Tr}\,\sigma_\mu\,\sigma_\nu = 2\delta_{\mu\nu}. \tag{3.40}$$

Then

$$\det X = (x^0)^2 - \mathbf{x}^2 = -g_{\mu\nu}x^\mu x^\nu, \tag{3.41}$$

and the transformation

$$X_{\alpha\beta} \to X'_{\alpha\beta} = L_{\alpha\alpha'}\,L^*_{\beta\beta'}\,X_{\alpha'\beta'} = (LXL^\dagger)_{\alpha\beta} \tag{3.42}$$

leaves the determinant invariant, $\det X' = \det X$. It has to correspond to a Lorentz transformation of x^μ,

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \tag{3.43}$$

$$= \frac{1}{2}\mathrm{Tr}\,\sigma_\mu \, X' = \frac{1}{2}\mathrm{Tr}\,[\sigma_\mu L \sigma_\nu L^\dagger]\, x^\nu. \tag{3.44}$$

Hence,

$$\Lambda(L)^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}\,\sigma_\mu L \sigma_\nu L^\dagger \tag{3.45}$$

is an explicit representation of Λ in terms of L. We see that L and $-L$ give the same Λ.

3.3 Representation in Hilbert space

The finite dimensional representations of $SL(2,C)$ are not unitary, but infinite dimensional representations can be unitary. It has been shown that Lorentz symmetry can be represented by a unitary operator $U(L)$ in Hilbert space. This guarantees that transition amplitudes are invariant,

$$|\hat{\psi}_{1,2}\rangle = U(L)|\psi_{1,2}\rangle \to \langle\hat{\psi}_1|\hat{\psi}_2\rangle = \langle\psi_1|\psi_2\rangle, \tag{3.46}$$

in the Heisenberg picture. Here $|\hat{\psi}_{1,2}\rangle$ represent actively transformed states $|\psi_{1,2}\rangle$, e.g. corresponding to rotated or boosted systems. (For example, $|\psi_2\rangle$ can be a state representing a system of particles converging to a scattering region and $|\psi_1\rangle$ can be a state representing particles emerging from the same region). If Lorentz invariance is broken, then $U(L)$ does not exist or is time dependent.

The expectation value of an observable in the state $|\hat{\psi}\rangle$ is related to the expectation value in $|\psi\rangle$ by a Lorentz transformation. For example, for a current operator $j^\mu(x)$,

$$\langle\hat{\psi}|j^\mu(\hat{x})|\hat{\psi}\rangle = \Lambda^\mu{}_\nu \, \langle\psi|j^\mu(x)|\psi\rangle, \tag{3.47}$$

where $\Lambda = \Lambda(L)$ and $\hat{x} = \Lambda x$ (i.e. $\hat{x}^\mu = \Lambda(L)^\mu{}_\nu \, x^\nu$). Instead of transforming the system we can also transform the observables (the passive point of view). From (3.47) we infer that the current operator transforms as

$$U(L)^\dagger \, j^\mu(x) \, U(L) = \Lambda(L)^\mu{}_\nu \, j^\nu(\Lambda(L)^{-1}x). \tag{3.48}$$

A scalar field transforms as

$$U(L)^\dagger \, \varphi(x) \, U(L) = \varphi(\Lambda(L)^{-1}x). \tag{3.49}$$

Fields transforming as L itself or $L^{-1\dagger}$ are called spinor fields; these will be the subject of the next chapter.

The energy-momentum operators P^μ transform as a vector

$$U(L)^\dagger\, P^\mu\, U(L) = \Lambda(L)^\mu{}_\nu\, P^\nu \tag{3.50}$$

A (spacetime) translation by a four vector a^μ is represented by a unitary operator

$$U(a) = e^{-ia^\mu P_\mu} = e^{-i\mathbf{a}\cdot\mathbf{P}+ia^0 P^0}, \tag{3.51}$$

with P^μ the energy-momentum operator. For example, a scalar field transforms as

$$U(a)^\dagger\, \varphi(x)\, U(a) = \varphi(x-a), \tag{3.52}$$

which is consistent with the solution of the Heisenberg equations of motion for $a^0 = -t$, $\mathbf{a} = \mathbf{0}$.

Lorentz transformations combined with translations form the Poincaré group. The generators of Lorentz transformations are represented by hermitian operators $J_{\kappa\lambda}$,

$$M_{\kappa\lambda} \to J_{\kappa\lambda}, \tag{3.53}$$

with $J_l = \frac{1}{2}\epsilon_{lmn}\, J_{mn}$ the angular momentum operators and $K_l = J_{l0}$ the 'kick' operators generating boosts. From (3.50) follow the commutators of $J_{\kappa\lambda}$ with P_μ, and the complete set of commutators of the Poincaré group is given by

$$\begin{aligned}
[J_{\kappa\lambda}, J_{\mu\nu}] &= i(g_{\kappa\mu}J_{\lambda\nu} + g_{\lambda\nu}J_{\kappa\mu} - g_{\kappa\nu}J_{\lambda\mu} - g_{\lambda\mu}J_{\kappa\nu}, && (3.54)\\
[J_{\kappa\lambda}, P_\mu] &= ig_{\kappa\mu}P_\lambda - ig_{\lambda\mu}P_\kappa, && (3.55)\\
[P_\mu, P_\nu] &= 0. && (3.56)
\end{aligned}$$

We end here with the form of the transformation of a one particle state $|p,\lambda\rangle$ (λ is a spin index), which is defined by applying a standard boost to a standard state at a standard momentum $\bar{p}$ (usually at rest, $\bar{p} = (\mathbf{0}, m)$ or some other $\bar{p}$ in case of massless particles):

$$U(L)|p,\lambda\rangle = \sum_{\lambda'} C_{\lambda'\lambda}(L,p)\, |\Lambda(L)p,\lambda'\rangle, \tag{3.57}$$

where $C_{\lambda'\lambda}$ is a unitary matrix depending on L and p. Unfortunately we have no 'time' here to go into details, see e.g. Ryder sect. 2.7 and Weinberg's 1964 Brandeis lectures. We also cannot go into the discrete symmetries P and T here. See for example Bjorken & Drell II ch. 15.

Chapter 4

Spinor fields and fermions

Dirac proposed in 1928 a relativistic generalization of the Schrödinger equation for a quantum mechanical wave function, his famous Dirac equation. It turned out later that this 'wave function' should not be seen as a wave function in the Schrödinger picture, but as a quantum operator field analogous to the scalar and Maxwell fields. Hence the name 'second quantization' (quantizing the wave function a second time to get an operator field) which is sometimes given to quantum field theory. We shall not follow this historical road, as it is tends to be confusing conceptually, but start from the notion that there are spin 1/2 particles which we want to describe by a quantum field transforming in a spinor representation of the Lorentz group. We are then automatically led to the Dirac equation. The basic principles of quantum field theory – in particular locality – lead to the connection between spin and statistics: the spin 1/2 particles have to follow Fermi-Dirac statistics, they are fermions.

Using the principle of gauge invariance we couple the Dirac field (complex spinor field) in the next chapter to the electromagnetic field and derive the Feynman rules the resulting spinor electrodynamics.

4.1 Spinors and Dirac matrices

For a field theory of spin 1/2 particles we need spin 1/2 fields, i.e. fields which transform in the $j = \frac{1}{2}$ representation of the rotation group. This representation is embedded in the spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group. We shall use a notation in which the spinor fields in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ irreps are denoted by ψ_R and ψ_L, respectively. The meaning of L and R will become clear later. The spinor fields transform as

$$\psi_R(x) \to L\,\psi_R(\Lambda(L)^{-1}x), \quad L \in (\frac{1}{2}, 0), \tag{4.1}$$

$$\psi_L(x) \to L^{\dagger-1}\psi_L(\Lambda(L)^{-1}x), \quad L^{\dagger-1} \in (0, \frac{1}{2}), \tag{4.2}$$

where we should not confuse the $L \in$ SL(2,C) with the subscript L of ψ_L. We recall that these irreps can be written as

$$\begin{aligned} L &= e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2+\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2}, &(4.3)\\ L^{\dagger-1} &= e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2}, &(4.4)\end{aligned}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\chi}$ are the angles corresponding to rotations and boosts, respectively.

The representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are complex. Suppose we choose a ψ_R for our description of spin 1/2 particles. We expect that ψ_R^* will occur in our formulas essentially as often as ψ_R. Now ψ_R^* transforms with $L^* \simeq L^{\dagger-1}$, i.e. it transforms like a ψ_L. Let us define ψ_L in terms of ψ_R by

$$\psi_L(x) = \sigma_2\,\psi_R(x)^*. \tag{4.5}$$

Then this ψ_L transforms as in (4.2); recall $L^{\dagger-1} = \sigma_2\,L^*\,\sigma_2$. Since we need to work with both irreps $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, it is advantageous to combine the fields $\psi_{L,R}$ into a four component spinor

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{4.6}$$

which transforms in the reducible representation $(\frac{1}{2},0)+(0,\frac{1}{2})$,

$$\begin{aligned} \psi(x) &\to S(L)\,\psi(\Lambda(L)^{-1}x), &(4.7)\\ S(L) &= \begin{pmatrix} L & 0 \\ 0 & L^{\dagger-1} \end{pmatrix}. &(4.8)\end{aligned}$$

The four components of $\psi(x)$ are not independent because of (4.5); it is called a Majorana field. We shall see in the next section that it can be turned into a real field by a unitary transformation.

We now introduce 4×4 Dirac matrices γ^μ, γ_5, β and α^μ, as follows:

$$\begin{aligned} \beta &= i\gamma^0 = -i\gamma_0 = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix} = \rho_1, &(4.9)\\ \gamma^k = \gamma_k &= \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix} = -\rho_2\,\sigma_k, &(4.10)\\ \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} = \rho_3, &(4.11)\\ \alpha^\mu &= i\beta\,\gamma^\mu, &(4.12)\\ \alpha^k = \alpha_k &= \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} = \rho_3\,\sigma_k, \quad \alpha^0 = -\alpha_0 = 1. &(4.13)\end{aligned}$$

Here the ρ_k are Pauli matrices in block form,

$$\rho_1 = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i1\!\!1 \\ i1\!\!1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}. \tag{4.14}$$

The specification in terms of tensor product matrices $\rho_k \sigma_l$ is very convenient. Note that we often write 1 for the unit matrix (2×2 or 4×4) $\mathbb{1}$. The Dirac matrices have the following hermiticity properties

$$\gamma^{0\dagger} = -\gamma^0, \quad \gamma^{k\dagger} = \gamma^k, \quad \gamma_5^\dagger = \gamma_5, \quad \beta^\dagger = \beta, \quad \alpha_k^\dagger = \alpha_k, \tag{4.15}$$

and

$$\beta\, \gamma^{\mu\dagger}\, \beta = -\gamma^\mu. \tag{4.16}$$

The γ^μ satisfy the algebraic relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \equiv \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}, \quad \{\gamma^\mu, \gamma_5\} = 0, \tag{4.17}$$

i.e. the γ^μ anticommute with each other and with γ_5 and their square is ± 1,

$$\gamma_0^2 = -1, \quad \gamma_k^2 = 1, \quad \gamma_5^2 = 1, \quad \beta^2 = 1, \quad \alpha_k^2 = 1. \tag{4.18}$$

It follows from (4.17), using the identities

$$[ab, c] = a[b, c] + [a, c]b, \tag{4.19}$$
$$[ab, c] = a\,\{b, c\} - \{a, c\}\, b, \tag{4.20}$$

and the combination of these

$$[ab, cd] = a[b, cd] + [a, cd]b = a\{b, c\}d - a\{b, d\}c + \{a, c\}db - c\{a, d\}b, \tag{4.21}$$

that the matrices

$$\Sigma_{\kappa\lambda} = -\Sigma_{\lambda\kappa} = -i\gamma_\kappa\, \gamma_\lambda, \quad \kappa \neq \lambda, \tag{4.22}$$

satisfy the commutation relations of the generators of the Lorentz group, up to a factor 2,

$$[\Sigma_{\kappa\lambda}, \Sigma_{\mu\nu}] = 2i(g_{\kappa\mu}\Sigma_{\lambda\nu} + g_{\lambda\nu}\Sigma_{\kappa\mu} - g_{\kappa\nu}\Sigma_{\lambda\mu} - g_{\lambda\mu}\Sigma_{\kappa\nu}). \tag{4.23}$$

We have a representation of the Lorentz algebra, $M_{\kappa\lambda} \to \frac{1}{2}\Sigma_{\kappa\lambda}$, and in fact

$$S = \exp(-i\frac{1}{4}\omega^{\kappa\lambda}\, \Sigma_{\kappa\lambda}). \tag{4.24}$$

To show this in detail we identify the generators of rotations and boosts,

$$\Sigma_{kl} = \epsilon_{klm}\, \Sigma_m, \quad \Sigma_m = \frac{1}{2}\epsilon_{klm}\Sigma_{kl}, \tag{4.25}$$
$$\Sigma_{k0} = -i\gamma_k\gamma_0 = i\gamma_0\gamma_k = -i\gamma^0\gamma_k = -\beta\gamma_k \tag{4.26}$$
$$= i\alpha_k = i\gamma_5\, \Sigma_k, \tag{4.27}$$

and (4.24) reduces to

$$
\begin{aligned}
S &= \exp(-i\frac{1}{2}\varphi_k\,\Sigma_k - i\frac{1}{2}\chi_k\,i\gamma_5\Sigma_k) && (4.28)\\
&= \exp(-i\frac{1}{2}\varphi_k\,\sigma_k + \frac{1}{2}\chi_k\,\rho_3\sigma_k) && (4.29)\\
\omega^{kl} &= \frac{1}{2}\epsilon_{klm}\varphi_m, \quad \omega^{k0} = -\omega^{0k} = \chi_k, && (4.30)
\end{aligned}
$$

which is identical to (4.8), taking into account (4.3), (4.4).

We note that

$$
\begin{aligned}
\beta\,S^\dagger\,\beta &= \beta\,\left(e^{-\frac{1}{4}\omega^{\mu\nu}\gamma_\mu\gamma_\nu}\right)^\dagger\beta = \beta\,e^{-\frac{1}{4}\omega^{\mu\nu}\gamma_\nu^\dagger\gamma_\mu^\dagger}\beta = e^{-\frac{1}{4}\omega^{\mu\nu}\gamma_\nu\gamma_\mu} = e^{+\frac{1}{4}\omega^{\mu\nu}\gamma_\mu\gamma_\nu}\\
&= S^{-1}, \qquad (4.31)
\end{aligned}
$$

which shows that β plays the role of the metric,

$$S^\dagger\,\beta\,S = \beta. \qquad (4.32)$$

For example, $\psi^\dagger\beta\psi$ is a Lorentz scalar. It is customary and convenient to hide this 'metric' into the 'bar' notation,

$$\bar\psi \equiv \psi^\dagger\,\beta. \qquad (4.33)$$

Under Lorentz transformations we have

$$\psi \to S\,\psi, \quad \bar\psi \to \bar\psi\,S^{-1}, \qquad (4.34)$$

which makes it obvious that $\bar\psi\psi$ is a Lorentz scalar.

Using the identity (4.20) we find the commutation relations between the generators $\Sigma_{\kappa\lambda}$ and γ_μ,

$$[\frac{1}{2}\Sigma_{\kappa\lambda}, \gamma_\mu] = -i\gamma_\kappa\,g_{\lambda\mu} + i\gamma_\lambda\,g_{\kappa\mu}, \qquad (4.35)$$

which imply that γ^μ transforms as a four vector,

$$S^{-1}(L)\,\gamma^\mu\,S(L) = \Lambda(L)^\mu{}_\nu\,\gamma^\nu. \qquad (4.36)$$

It follows that $\gamma^\mu\gamma^\nu$, $\mu\neq\nu$ and $\gamma^\mu\gamma^\nu\gamma^\rho$, $\mu\neq\nu\neq\rho\neq\mu$, transform as antisymmetric tensors, while γ_5, which commutes with S,

$$[\Sigma_{\kappa\lambda}, \gamma_5] = 0, \quad S\gamma_5 = \gamma_5 S, \qquad (4.37)$$

is a Lorentz scalar in this sence.

We can also represent parity P and time reversal T. As mentioned in sect. 3.2, the parity operation can be represented in a reducible representation of the form $(j^+, j^-)+(j^-, j^+)$. This is the case here with $j^+ = \frac{1}{2}$, $j^- = 0$, and inspection shows that P and T can be represented by the matrices

$$P \to S_P = \gamma^0, \quad T \to S_T = i\gamma^0\gamma_5. \tag{4.38}$$

We have,

$$S_P^{-1}\,\gamma^\mu\,S_P = P^\mu_{\ \nu}\,\gamma^\nu, \quad S_T^{-1}\,\gamma^\mu\,S_T = T^\mu_{\ \nu}\,\gamma^\nu, \tag{4.39}$$

and furthermore

$$S_P^{-1}\,\Sigma_{kl}\,S_P = \Sigma_{kl}, \quad S_P^{-1}\,\Sigma_{k0}\,S_P = -\Sigma_{k0}, \tag{4.40}$$
$$S_T^{-1}\,\Sigma_{kl}\,S_T = \Sigma_{kl}, \quad S_T^{-1}\,\Sigma_{k0}\,S_T = -\Sigma_{k0}, \tag{4.41}$$

form a representation of (3.28). Taking P and T into account, γ^μ is a vector and γ_5 is a pseudoscalar,

$$S_P^{-1}\,\gamma_5\,S_P = -\gamma_5, \quad S_T^{-1}\,\gamma_5\,S_T = -\gamma_5, \tag{4.42}$$

which can also be seen from

$$\gamma_5 = i\frac{1}{4!}\,\epsilon_{\mu\nu\rho\sigma}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \quad \epsilon_{0123} = 1, \tag{4.43}$$

and the fact that the Levi-Civita tensor is a pseudotensor under P and T. We now have the following summary:

$$\gamma^\mu \quad \text{is a vector,} \tag{4.44}$$
$$\Sigma_{\mu\nu} \quad \text{is an antisymmetric tensor,} \tag{4.45}$$
$$i\gamma^\mu\gamma_5 \quad \text{is a pseudovector,} \tag{4.46}$$
$$i\gamma_5 \quad \text{is a pseudoscalar,} \tag{4.47}$$

when these matrices are sandwiched between a $\bar{\psi}_1$ and a ψ_2.

The $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ components of the representation $L \to S(L)$ can recovered with the projectors

$$P_R = \frac{1+\gamma_5}{2}, \quad P_L = \frac{1-\gamma_5}{2}, \tag{4.48}$$
$$P_R^2 = P_R, \quad P_L^2 = P_L, \quad P_R P_L = 0. \tag{4.49}$$

4.2 Majorana field and Majorana representation

The algebraic relations among the Dirac matrices and their hermiticity properties are invariant under unitary transformations,

$$\hat{\gamma}^\mu = U\,\gamma^\mu\,U^\dagger, \tag{4.50}$$

and so are all the relations which are written in terms of the γ^μ, β, γ_5, α^μ, $\ldots$, which are made out of products of the γ^μ. This is very useful if we want to transform to another representation of the γ's. The representation (4.9) – (4.13), which is characterized by the fact that γ_5 is diagonal, is called a Weyl representation, or chiral representation.

It is sometimes useful to use a representation in which the γ^μ are real. Such a representation is called a Majorana representation. The transformation

$$U = e^{i\frac{\pi}{4}\rho_2\,\sigma_2} = \frac{1}{\sqrt{2}}\,(1 + i\rho_2\,\sigma_2) \tag{4.51}$$

leads to the real matrices

$$\hat{\gamma}^1 = -\sigma_3, \quad \hat{\gamma}^2 = -\rho_2\sigma_2, \quad \hat{\gamma}^3 = \sigma_1, \quad \hat{\gamma}^0 = i\rho_3\sigma_2 \tag{4.52}$$

(e.g. $U\gamma_1U^\dagger = -U\rho_2\sigma_1U^\dagger = -U^2\rho_2\sigma_1 = -i\rho_2\sigma_2\rho_2\sigma_1 = -\sigma_3$). On the other hand

$$\hat{\gamma}_5 = -\rho_1\sigma_2 \tag{4.53}$$

is imaginary. Our Majorana field, which we introduced in the Weyl representation as a field for which $\psi_L = \sigma_2\,\psi_R^*$ turns into a real field in the above Majorana representation,

$$\hat{\psi} = U\psi = U\begin{pmatrix}\psi_R\\ \psi_L\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi_R + \sigma_2\psi_L\\ \psi_L - \sigma_2\psi_R\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi_R + \psi_R^*\\ \sigma_2(\psi_R^* - \psi_R)\end{pmatrix}, \tag{4.54}$$

which is real. Writing out the real and imaginary parts and the two components $\psi_\pm$ of ψ_R explicitly,

$$\psi_R = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} = \begin{pmatrix}\psi'_+ + i\psi''_+\\ \psi'_- + i\psi''_-\end{pmatrix}, \tag{4.55}$$

we have

$$\hat{\psi} = \sqrt{2}\begin{pmatrix}\psi'_+\\ \psi'_-\\ -\psi''_-\\ \psi''_+\end{pmatrix}, \tag{4.56}$$

which gives $\hat{\psi}$ as a real four component field.

From now on we drop the $\hat{}$, the type of representation will be clear from the context. In a Majorana representation the matrix S representing Lorentz transformations is real,

$$S = e^{-i\frac{1}{4}\omega^{\mu\nu}\,\Sigma_{\mu\nu}} = e^{-\frac{1}{4}\omega^{\mu\nu}\,\gamma_\mu\gamma_\nu} = S^*, \tag{4.57}$$

and the same is true for the matrices representing P and T,

$$S_P = \gamma^0 = S_P^*, \quad S_T = i\gamma^0\gamma_5 = S_T^*. \tag{4.58}$$

Hence, the reality of the Majorana field is preserved under these transformations.

In general hermiticity properties are preserved under a change of representation, in a real representation these become symmetry properties (under transposition). Let us now express the symmetry and reality properties in a representation independent form. For this we need the so-called charge conjugation matrix C. In any representation there is a unitary antisymmetric matrix C,

$$C^\dagger C = 1, \quad C^T = -C, \tag{4.59}$$

relating γ^μ and $(\gamma^\mu)^T$ according to

$$\gamma^{\mu T} = -C^\dagger \, \gamma^\mu \, C. \tag{4.60}$$

In the Majorana representation $\gamma^{\mu T} = \gamma^{\mu\dagger}$ and (cf. (4.16))

$$C = \beta = i\gamma^0 \tag{4.61}$$

$(= -\rho_3\sigma_2)$. In any other representation (indicating the Majorana representation by the ˆ for the moment),

$$\begin{aligned} \gamma^{\mu T} &= (U^\dagger \, \hat{\gamma}^\mu \, U)^T = U^T \, \hat{\gamma}^{\mu T} \, U^* = -U^T \, \hat{\beta}\hat{\gamma}^\mu \hat{\beta} \, U^* && (4.62) \\ &= -U^T U \, \beta\gamma^\mu\beta \, U^\dagger U^*, && (4.63) \end{aligned}$$

and we obtain C in the form

$$C = \beta U^\dagger U^* \equiv \beta\tilde{C}. \tag{4.64}$$

We then also have in any representation

$$\begin{aligned} \gamma^{\mu *} &= (\gamma^\mu)^{\dagger T} = -(\beta\gamma^\mu\beta)^T = C^\dagger\beta\gamma^\mu\beta C && (4.65) \\ &= \tilde{C}^\dagger\gamma^\mu\tilde{C}. && (4.66) \end{aligned}$$

In the Majorana representation $\tilde{C} = 1$. In our Weyl representation

$$\tilde{C} = U^\dagger U^* = e^{-i\frac{\pi}{2}\rho_2\sigma_2} = -i\rho_2\sigma_2, \quad C = \rho_3\sigma_2 = -\gamma^0\gamma^2. \tag{4.67}$$

The charge conjugation matrix derives its usefulness by relating $S(L)^T$ with $S(L)^{-1}$,

$$S(L)^T = C^\dagger \, S(L)^{-1} \, C, \tag{4.68}$$

and S^* with S,

$$S(L)^* = S(L)^{\dagger T} = C^\dagger\beta \, S(L) \, \beta C. \tag{4.69}$$

For example, $C^\dagger_{\alpha\beta}\psi^\alpha\psi^\beta = \psi^T C^\dagger \psi$ is a scalar.

For general complex spinors ψ and $\bar{\psi} = \psi^\dagger\beta$ the so-called charge conjugate spinors $\psi^{(c)}$ and $\bar{\psi}^{(c)}$ are defined as

$$\psi^{(c)} = (\bar{\psi}C)^T = \beta C\psi^*, \quad \bar{\psi}^{(c)} = -(C^\dagger\psi)^T \tag{4.70}$$

(where the formula for $\bar{\psi}^{(c)}$ follows from $\psi^{(c)}$). Under Lorentz transformations $\psi^{(c)}$ transforms like ψ.

Finally in this section, let us express the Majorana property of a spinor field in representation independent form. In the Weyl representation the Majorana property

$$\psi_L = \sigma_2 \psi_R^* \tag{4.71}$$

implies

$$\begin{aligned} \psi^* &= \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \psi & (4.72) \\ &= \tilde{C}^\dagger \psi = C^\dagger \beta \, \psi, & (4.73) \end{aligned}$$

where the last line is a representation independent form. This can also be expressed as

$$\begin{aligned} \bar{\psi} &= \psi^\dagger \beta = \psi^{*T} \beta = \psi^T \beta^T C^{\dagger T} \beta = -\psi^T C^{\dagger T} \\ &= -(C^\dagger \psi)^T. \end{aligned} \tag{4.74}$$

With this definition eq. (4.73) expresses the fact that a Majorana field is self (charge) conjugate

$$\psi^{(c)} = \psi, \quad \bar{\psi}^{(c)} = \bar{\psi}. \tag{4.75}$$

4.3 Polarization spinors

In our description of spin 1/2 particles we will need polarization spinors $u^\alpha(p, \lambda)$, the analogue of the polarization vectors $e^\mu(p, \lambda)$ for the photon field. They are constructed as follows.

A particle at rest transforms under rotations like a two component spinor χ_λ,

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.76}$$

From these two component spinors we make a four component spinor for a particle at rest in the Weyl representation,

$$u(\bar{p}, \lambda) = \sqrt{2m}\, \xi_\lambda, \quad \xi_\lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_\lambda \\ \chi_\lambda \end{pmatrix}, \tag{4.77}$$

where m is the particle mass and

$$\bar{\mathbf{p}} = \mathbf{0}, \;\; \bar{p}^0 = m. \tag{4.78}$$

The curious normalization factor $\sqrt{2m}$ is put in for later convenience. We can characterize $u(\bar{p}, \lambda)$ by the eigenvalues of the two commuting matrices Σ_3 and β,

$$\begin{aligned} \Sigma_3 \, u(\bar{p}, \lambda) &= \lambda u(\bar{p}, \lambda), \quad \Sigma_3 \, \xi_\lambda = \lambda \xi_\lambda & (4.79) \\ \beta u(\bar{p}, \lambda) &= u(\bar{p}, \lambda), \quad \beta \xi_\lambda = \xi_\lambda. & (4.80) \end{aligned}$$

These relations together with $u(\bar{p},-) = \frac{1}{2}(\Sigma_1 - i\Sigma_2)\,u(\bar{p},+)$ serve to characterize $u(\bar{p},\lambda)$ in a general representation.

Polarization spinors $u(p,\lambda)$ for arbitrary momentum p now follow by applying a standard boost B_p which takes $\bar{p}$ into p:

$$\begin{aligned} \Lambda(B_p)\,\bar{p} &= p, && (4.81)\\ B_p &= e^{\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2}, \quad \boldsymbol{\chi} = \chi\,\hat{\mathbf{p}}, \;\; \hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}, \quad \tanh\chi = \frac{|\mathbf{p}|}{p^0}. && (4.82) \end{aligned}$$

Applying this standard boost to $u(\bar{p},\lambda)$ we get

$$\begin{aligned} u(p,\lambda) &= S(B_p)\,u(\bar{p},\lambda) && (4.83)\\ &= (\cosh\frac{\chi}{2} + \sinh\frac{\chi}{2}\,\hat{\mathbf{p}}\cdot\boldsymbol{\Sigma}\gamma_5)u(\bar{p},\lambda) && (4.84)\\ &= (\sqrt{p^0+m} + \sqrt{p^0-m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\Sigma}\gamma_5)\xi_\lambda. && (4.85) \end{aligned}$$

We shall also need conjugate spinors related to $u(p,\lambda)$ by charge conjugation (cf. (4.70)),

$$\begin{aligned} v(p,\lambda) &\equiv u^{(c)}(p,\lambda) && (4.86)\\ &= \beta C\,u(p,\lambda)^* = [\bar{u}(p,\lambda)C]^T, && (4.87)\\ \bar{v}(p,\lambda) &= \bar{u}^{(c)}(p,\lambda) = -[C^\dagger u(p,\lambda)]^T. && (4.88) \end{aligned}$$

In the Majorana representation $C = \beta$, giving simply

$$v(p,\lambda) = u(p,\lambda)^*, \quad \text{Majorana rep.} \qquad (4.89)$$

Since charge conjugate spinors transform under Lorentz transformations like ordinary spinors we have

$$\begin{aligned} v(p,\lambda) &= S(B_p)\,v(\bar{p},\lambda) && (4.90)\\ &= (\sqrt{p^0+m} + \sqrt{p^0-m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\Sigma}\gamma_5)\xi^{(c)}_\lambda, && (4.91)\\ \xi^{(c)}_\lambda &= \beta C\,\xi^*_\lambda. && (4.92) \end{aligned}$$

Furthermore, at rest

$$\begin{aligned} \Sigma_3 v(\bar{p},\lambda) &= -\lambda\,v(\bar{p},\lambda), \quad \Sigma_3\,\xi^{(c)}_\lambda = -\lambda\,\xi^{(c)}_\lambda, && (4.93)\\ \beta\,v(\bar{p},\lambda) &= -v(\bar{p},\lambda), \quad \beta\,\xi^{(c)}_\lambda = -\xi^{(c)}_\lambda, && (4.94) \end{aligned}$$

and

$$\begin{aligned} \bar{u}(\bar{p},\lambda)\,i\gamma^\mu\,u(\bar{p},\lambda') &= 2\bar{p}^\mu\,\delta_{\lambda\lambda'}, && (4.95)\\ \bar{v}(\bar{p},\lambda)\,i\gamma^\mu\,v(\bar{p},\lambda') &= 2\bar{p}^\mu\,\delta_{\lambda\lambda'}, && (4.96)\\ \bar{u}(\bar{p},\lambda)\,u(\bar{p},\lambda') &= 2m\,\delta_{\lambda\lambda'}, && (4.97)\\ \bar{v}(\bar{p},\lambda)\,v(\bar{p},\lambda') &= -2m\,\delta_{\lambda\lambda'}, && (4.98)\\ \bar{u}(\bar{p},\lambda)\,v(\bar{p},\lambda') &= \bar{v}(\bar{p},\lambda)\,u(\bar{p},\lambda') = 0. && (4.99) \end{aligned}$$

The orthogonality of a $u(\bar{p},\lambda)$ and a $v(\bar{p},\lambda')$ follow from the fact that they are eigenvectors of β with different eigenvalues. From the above follow the relations for general p:

$$\begin{aligned}
\bar{u}(p,\lambda)\, i\gamma^\mu\, u(p,\lambda') &= 2p^\mu\, \delta_{\lambda\lambda'}, && (4.100)\\
\bar{v}(p,\lambda)\, i\gamma^\mu\, v(p,\lambda') &= 2p^\mu\, \delta_{\lambda\lambda'}, && (4.101)\\
\bar{u}(p,\lambda)\, u(p,\lambda') &= 2m\, \delta_{\lambda\lambda'}, && (4.102)\\
\bar{v}(p,\lambda)\, v(p,\lambda') &= -2m\, \delta_{\lambda\lambda'}, && (4.103)\\
\bar{u}(p,\lambda)\, v(p,\lambda') &= \bar{v}(p,\lambda)\, u(p,\lambda') = 0. && (4.104)
\end{aligned}$$

For example,

$$\begin{aligned}
\bar{u}(p,\lambda)\, i\gamma^\mu\, u(p,\lambda') &= \bar{u}(\bar{p},\lambda)S(B_p)^{-1}i\gamma^\mu S(B_p)u(\bar{p},\lambda) = \Lambda(B_p)^\mu{}_\nu\, 2\bar{p}^\nu\, \delta_{\lambda\lambda'}\\
&= 2p^\mu\, \delta_{\lambda\lambda'}. && (4.105)
\end{aligned}$$

Since $\bar{u}(p,\lambda)i\gamma^0 = u(p,\lambda)^\dagger$ we can interprete (4.100) and (4.101) for $\mu = 0$ as orthogonality relations. The u's are orthogonal to the v's in the sense

$$\begin{aligned}
u(p,\lambda)^\dagger\, v(\tilde{p},\lambda') &= u(\bar{p},\lambda)^\dagger S(B_p)^\dagger\, S(B_{\tilde{p}})v(\bar{p},\lambda') && (4.106)\\
&= 0, \quad \tilde{p} \equiv (-\mathbf{p},\, p^0), && (4.107)
\end{aligned}$$

where we used $S(B_p)^\dagger = S(B_p) = S(B_{\tilde{p}})^{-1}$.

Similarly, we have completeness type relations at rest,

$$\begin{aligned}
\sum_\lambda u(\bar{p},\lambda)\, \bar{u}(\bar{p},\lambda) &= 2m \sum_\lambda \xi_\lambda\, \xi_\lambda^\dagger = m(1+\beta)\\
&= m - i\gamma^\mu \bar{p}_\mu, && (4.108)
\end{aligned}$$

and for general momentum

$$\begin{aligned}
\sum_\lambda u(p,\lambda)\, \bar{u}(p,\lambda) &= m - i\gamma^\mu p_\mu, && (4.109)\\
\sum_\lambda v(p,\lambda)\, \bar{v}(p,\lambda) &= -m - i\gamma^\mu p_\mu. && (4.110)
\end{aligned}$$

The second relation follows from the first and the definition of $v(p,\lambda)$,

$$\begin{aligned}
\sum_\lambda v^\alpha(p,\lambda)\, \bar{v}^\beta(p,\lambda) &= \sum_\lambda [\bar{u}(p,\lambda)C]^\alpha\, [-C^\dagger u(p,\lambda)]^\beta = -[C^\dagger\, (m - i\gamma^\mu p_\mu)\, C]_{\beta\alpha}\\
&= -[C^\dagger\, (m - i\gamma^\mu p_\mu)\, C]^T_{\alpha\beta} = -[C\, (m - i\gamma^\mu p_\mu)^T\, C^\dagger]_{\alpha\beta}\\
&= -(m + i\gamma^\mu p_\mu)_{\alpha\beta}. && (4.111)
\end{aligned}$$

In the Majorana representation these relations follow more easily from the reality of the γ^μ and $v = u^*$.

Because of the orthogonality relations (4.100), (4.101) and (4.107) the completeness relation in four dimensional spinor space reads

$$\sum_{\lambda}[u(p,\lambda)u(p,\lambda)^{\dagger}+v(\tilde{p},\lambda)v(\tilde{p},\lambda)^{\dagger}]=2p^{0}. \tag{4.112}$$

Eqs. (4.80) and (4.94) generalize to arbitrary p as,

$$i\gamma^{\mu}p_{\mu}\,u(p,\lambda)=-mu(p,\lambda),\quad i\gamma^{\mu}p_{\mu}\,v(p,\lambda)=mv(p,\lambda), \tag{4.113}$$

which turn out to be the free Dirac equation in momentum space.

We conclude this section with the zero mass limit of the polarization spinors, which can also be interpreted as their approximate form for high energies. From eqs. (4.85) and (4.91) we see that for $m\to 0$,

$$\begin{aligned} u(p,\lambda) &\;\to\; \sqrt{|\mathbf{p}|}\,(1+\hat{\mathbf{p}}\cdot\mathbf{\Sigma}\gamma_5)\,\xi_{\lambda}, && (4.114)\\ v(p,\lambda) &\;\to\; \sqrt{|\mathbf{p}|}\,(1+\hat{\mathbf{p}}\cdot\mathbf{\Sigma}\gamma_5)\,\xi_{\lambda}^{(c)}. && (4.115) \end{aligned}$$

The quantity within parenthesis is essentially a projector. Let us change the specification of the ξ_λ such that they become eigenvectors of the helicity matrix

$$\frac{1}{2}\hat{\mathbf{p}}\cdot\mathbf{\Sigma}, \tag{4.116}$$

with eigenvectors $\lambda/2$. This can be done by a standard rotation which brings the three axis along $\hat{\mathbf{p}}$,

$$\begin{aligned} \xi_{\lambda}(\theta,\phi) &= e^{-i\phi\frac{1}{2}\Sigma_3}\,e^{-i\theta\frac{1}{2}\Sigma_2}\,\xi_{\lambda}, && (4.117)\\ \hat{\mathbf{p}} &= (\sin\theta\cos\phi,\,\sin\theta\sin\phi,\,\cos\theta). && (4.118) \end{aligned}$$

Then λ is the sign of the helicity,

$$\hat{\mathbf{p}}\cdot\mathbf{\Sigma}\,\xi_{\lambda}(\theta,\phi)=\lambda\xi_{\lambda}(\theta,\phi),\quad \hat{\mathbf{p}}\cdot\mathbf{\Sigma}\,\xi_{\lambda}^{(c)}(\theta,\phi)=-\lambda\xi_{\lambda}^{(c)}(\theta,\phi), \tag{4.119}$$

and the helicity is tied to γ_5,

$$\begin{aligned} u(|\mathbf{p}|,\theta,\phi,\lambda) &= \sqrt{|\mathbf{p}|}\,(1+\lambda\gamma_5)\,\xi_{\lambda}(\theta,\phi), && (4.120)\\ v(|\mathbf{p}|,\theta,\phi,\lambda) &= \sqrt{|\mathbf{p}|}\,(1-\lambda\gamma_5)\,\xi_{\lambda}^{(c)}(\theta,\phi). && (4.121) \end{aligned}$$

where we recognize the projectors $P_{L,R}=(1\mp\gamma_5)/2$. Since γ_5 commutes with $\mathbf{\Sigma}$ we can choose the helicity ξ's to be eigenvectors of γ_5. The eigenvalue χ of γ_5, which takes values ± 1, is called the chirality ('handedness'). We see that for the u-spinors $\chi=\lambda$, whereas for the v-spinors $\chi=-\lambda$. Then a right handed spinor $u_R=P_R\,u$, which in the Weyl representation has only the upper two components nonzero, has positive helicity, while a left handed spinor $u_L=P_L\,u$, which in the Weyl representation has only the lower two components nonzero, has negative helicity, and vice versa for $v_{R,L}$.

4.4 Spin and statistics

We shall derive here that an operator spinor field has to describe fermions. We assume now a theory of free spin 1/2 particles, in which there is a vacuum state $|0\rangle$ with zero energy-momentum, and one particle states $|p\lambda\rangle$ with energy-momentum p^μ,

$$P^\mu|0\rangle = 0, \quad P^\mu|p\lambda\rangle = p^\mu|p\lambda\rangle, \tag{4.122}$$

where P^μ is the energy-momentum operator; $\lambda = \pm$ is a spin index. The conventions are such that these states are obtained by the action of standard boosts B_p (cf. (4.82)) on a particle state at rest,

$$|p\lambda\rangle = U(B_p)|\bar{p}\lambda\rangle, \quad \bar{\mathbf{p}} = \mathbf{0}, \;\; \bar{p}^0 = m, \tag{4.123}$$

where $U(B_p)$ is the unitary operator representing B_p in Hilbert space. The index $\lambda = \pm$ labels the eigenvalue of the third component of angular momentum J_3 in the rest frame of the particle,

$$J_3|\bar{p}\lambda\rangle = \frac{1}{2}\lambda\,|\bar{p}\lambda\rangle. \tag{4.124}$$

Let $\psi(x)$ now be an operator spinor field of the Majorana type,

$$\psi^\alpha(x)^\dagger = (C^\dagger\beta)_{\alpha\beta}\,\psi^\beta(x), \quad \text{or} \quad \bar{\psi}(x) = -[C^\dagger\psi(x)]^T. \tag{4.125}$$

In the Majorana representation we have a hermitian spinor field

$$\psi^\alpha(x)^\dagger = \psi^\alpha(x), \quad \text{Majorana rep.} \tag{4.126}$$

By analogy to the scalar and Maxwell fields we assume $\psi(x)$ to annihilate spin 1/2 particles to the vacuum according to

$$\langle 0|\psi^\alpha(x)|p,\lambda\rangle = u^\alpha(p,\lambda)\,e^{ipx}. \tag{4.127}$$

The form of this equation is dictated by translation invariance (the factor $\exp(ipx)$) and Lorentz invariance (the factor $u^\alpha(p,\lambda)$, because $|p,\lambda\rangle$ and $u(p,\lambda)$ are constructed in exactly the same way with the boost B_p). The remaining factor ($=1$) is a normalization condition for $\psi(x)$. In general we may have an additional factor $\sqrt{Z_\psi}$ as in sect. 2.11, which we take to be $\sqrt{Z_\psi} = 1$ in case of no interactions. Taking the complex conjugate of (4.127) and multiplying by β gives

$$\langle p,\lambda|\bar{\psi}(x)|0\rangle = \bar{u}(p,\lambda)\,e^{-ipx}. \tag{4.128}$$

On the other hand[1], using the Majorana property of $\psi(x)$, the c.c. of (4.127) can be written as

$$\begin{aligned}\langle p,\lambda|\psi^\alpha(x)^\dagger|0\rangle &= u^\alpha(p,\lambda)^*\,e^{-ipx} && (4.129)\\ &= (C^\dagger\beta)_{\alpha\beta}\,\langle p,\lambda|\psi^\alpha(x)|0\rangle, && (4.130)\end{aligned}$$

[1]The derivations can be shortened by working consistently in the Majorana representation.

and using (4.87),

$$\langle p,\lambda|\psi^\alpha(x)|0\rangle = v^\alpha(p,\lambda)\, e^{-ipx}, \tag{4.131}$$

Taking the complex conjugate of (4.131) again and multiplying by β,

$$\langle 0|\bar\psi(x)|p,\lambda\rangle = \bar v(p,\lambda)\, e^{ipx}. \tag{4.132}$$

In the Majorana representation for the Dirac matrices we have simply,

$$\langle p,\lambda|\psi^\alpha(x)|0\rangle = u^\alpha(p,\lambda)^*\, e^{-ipx}, \quad \text{Majorana rep.}, \tag{4.133}$$

We have seen that free fields create only single particle states out of the vacuum. If we assume this to be the case of our free spinor field as well, we can derive the vacuum expectation value of equal time commutator or anticommutator relations. Using completeness we have

$$\begin{aligned}
\langle 0|\psi(x)\,\bar\psi(y)|0\rangle &= \sum_\lambda \int d\omega_p\, \langle 0|\psi(x)|p,\lambda\rangle\langle p,\lambda|\bar\psi(y)|0\rangle && (4.134)\\
&= \sum_\lambda \int d\omega_p\, e^{ip(x-y)}\, u(p,\lambda)\,\bar u(p,\lambda) && (4.135)\\
&= \int d\omega_p\, e^{ip(x-y)}\,(m - ip^\mu\gamma_\mu), && (4.136)
\end{aligned}$$

where we used (4.109). Similarly, we have

$$\begin{aligned}
\langle 0|\bar\psi^\beta(y)\,\psi^\alpha(x)|0\rangle &= \sum_\lambda \int d\omega_p\, \langle 0|\bar\psi^\beta(y)|p,\lambda\rangle\langle p,\lambda|\psi^\alpha(x)|0\rangle && (4.137)\\
&= \sum_\lambda \int d\omega_p\, e^{ip(y-x)}\, v^\alpha(p,\lambda)\,\bar v^\beta(p,\lambda) && (4.138)\\
&= -\int d\omega_p\, e^{ip(y-x)}\,(m + ip^\mu\gamma_\mu)_{\alpha\beta}, && (4.139)
\end{aligned}$$

using (4.110).

From these relations now follow the vacuum expectation values of equal time commutators or anticommutators:

$$\begin{aligned}
&\langle 0|[\psi^\alpha(x)\,\bar\psi^\beta(y) \pm \bar\psi^\beta(y)\,\psi^\alpha(x)]|0\rangle_{x^0=y^0} && (4.140)\\
&= \int d\omega_p\, [e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\,(m - ip^\mu\gamma_\mu) \mp [e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\,(m + ip^\mu\gamma_\mu)]_{\alpha\beta} && (4.141)\\
&= \int d\omega_p\, e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\,[(m - ip^k\gamma_k)(1\mp 1) + ip^0\gamma^0(1\pm 1)]_{\alpha\beta}. && (4.142)
\end{aligned}$$

It follows that the vacuum expectation value of the commutator $[\psi^\alpha(x),\bar\psi^\beta(y)]$ is given by

$$\langle 0|[\psi^\alpha(x),\bar\psi^\beta(y)]|0\rangle_{x^0=y^0} = 2(m - \gamma^k\partial_k)\int d\omega_p\, e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}. \tag{4.143}$$

This does not vanish for $\mathbf{x} \neq \mathbf{y}$: it is not 'local' for nonzero spacelike $(x-y)^2 = (\mathbf{x}-\mathbf{y})^2$. On the other hand, the vacuum expectation value of the anticommutator is simple and local:

$$\begin{aligned} \langle 0|\{\psi^\alpha(x), \bar{\psi}^\beta(y)\}|0\rangle_{x^0=y^0} &= i\gamma^0_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y}), && (4.144)\\ \langle 0|\{\psi^\alpha(x), \psi^\beta(y)^\dagger\}|0\rangle_{x^0=y^0} &= \delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y}). && (4.145)\end{aligned}$$

The spinor operators at different points in space evidently do not commute at equal times: they anticommute!

The above anticommutator looks similar to the commutator between a field $\varphi(x)$ and its canonical momentum $\pi(y)$ at equal times, apart from spinor indices. By analogy we shall assume now that not only the vacuum expectation value, but the operators satisfy the anticommutation relations,

$$\{\psi^\alpha(\mathbf{x},t), \psi^\beta(\mathbf{y},t)^\dagger\} = \delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y}). \tag{4.146}$$

We have arrived at equal time anticommutation relations for the spinor field.

Next we introduce operators $a(\mathbf{p},\lambda)$ and $a(\mathbf{p},\lambda)^\dagger$ by the expansion

$$\psi(x) = \sum_\lambda \int d\omega_p\,[e^{ipx}\,u(p,\lambda)\,a(\mathbf{p},\lambda) + e^{-ipx}\,v(p,\lambda)\,a(\mathbf{p},\lambda)^\dagger], \tag{4.147}$$

or in the Majorana representation,

$$\psi(x) = \sum_\lambda \int d\omega_p\,[e^{ipx}\,u(p,\lambda)\,a(\mathbf{p},\lambda) + e^{-ipx}\,u(p,\lambda)^*\,a(\mathbf{p},\lambda)^\dagger]. \tag{4.148}$$

Then

$$a(\mathbf{p},\lambda) = \int d^3x\,e^{i\mathbf{px}}\,u(p,\lambda)^\dagger\,\psi(x), \tag{4.149}$$

where we used $u(p,\lambda)^\dagger = \bar{u}(p,\lambda)i\gamma^0$, (4.100) and (4.107). Using (4.101), (4.107) and the Majorana property of ψ we also have

$$\begin{aligned} a(\mathbf{p},\lambda)^\dagger &= \int d^3x\,e^{-i\mathbf{px}}\,v(p,\lambda)^\dagger\psi(x) && (4.150)\\ &= \int d^3x\,e^{-i\mathbf{px}}\,\psi(x)^\dagger u(p,\lambda) = [a(\mathbf{p},\lambda)]^\dagger, && (4.151)\end{aligned}$$

i.e. the 'dagger' on $a(\mathbf{p},\lambda)^\dagger$ indeed means hermitian conjugation, as is obvious from (4.4). These expression give the following anticommutation relations for the $a(\mathbf{p},\lambda)^\dagger$ and $a(\mathbf{p},\lambda)$,

$$\begin{aligned} \{a(\mathbf{p},\lambda), a(\mathbf{p}',\lambda')^\dagger\} &= 2p^0\,(2\pi)^3\,\delta(\mathbf{p}-\mathbf{p}')\,\delta_{\lambda\lambda'}, && (4.152)\\ \{a(\mathbf{p},\lambda), a(\mathbf{p}',\lambda')\} &= \{a(\mathbf{p},\lambda)^\dagger, a(\mathbf{p}',\lambda')^\dagger\} = 0. && (4.153)\end{aligned}$$

For example,

$$\begin{aligned}
\{a(\mathbf{p},\lambda), a(\mathbf{p}',\lambda')\} &= \int d^3x d^3x'\, e^{i\mathbf{px}+i\mathbf{p}'\mathbf{x}'}\, u^\alpha(p,\lambda)^*\, u^{\alpha'}(p',\lambda')^* \\
&\quad \{\psi^\alpha(\mathbf{x}), \psi^{\alpha'}(\mathbf{x}')\} \\
&= (2\pi)^3\, \delta(\mathbf{p}+\mathbf{p}')\, u(p,\lambda)^\dagger \beta C u(p',\lambda')^* \\
&= (2\pi)^3\, \delta(\mathbf{p}+\mathbf{p}')\, u(p,\lambda)^\dagger v(\tilde{p},\lambda')^* = 0,
\end{aligned} \tag{4.154}$$

where we used $\{\psi^\alpha(\mathbf{x}), \psi^{\alpha'}(\mathbf{x}')\} = (\beta C)_{\alpha\alpha'}\,\delta(\mathbf{x}-\mathbf{x}')$ which follows from (4.146) and (4.125), and (4.107).

It follows that $a(\mathbf{p},\lambda)^\dagger$ and $a(\mathbf{p},\lambda)$ can be interpreted as creation and annihilation operators,

$$a(\mathbf{p},\lambda)|0\rangle = 0, \tag{4.155}$$

$$|p\lambda\rangle = a(\mathbf{p},\lambda)^\dagger|0\rangle, \tag{4.156}$$

$$|p_1\lambda_1, p_2\lambda_2\rangle = a(\mathbf{p}_1,\lambda_1)^\dagger\, a(\mathbf{p}_2,\lambda_2)^\dagger|0\rangle, \tag{4.157}$$

etc. These relations plus (4.147) are consistent with (4.127), (4.131). Furthermore, it is consistent to define the energy momentum operator as

$$P^\mu = \int d\omega_p\, a(\mathbf{p},\lambda)^\dagger\, a(\mathbf{p},\lambda)\, p^\mu, \tag{4.158}$$

since the anticommutation relations between the a's and $a^\dagger$'s imply

$$[P^\mu, a(\mathbf{p},\lambda)^\dagger] = p^\mu\, a(\mathbf{p},\lambda)^\dagger \tag{4.159}$$

$$P^\mu\, a(\mathbf{p},\lambda)^\dagger|0\rangle = p^\mu\, a(\mathbf{p},\lambda)^\dagger|0\rangle, \tag{4.160}$$

etc. Because the $a(\mathbf{p},\lambda)^\dagger$ anticommute among themselves the basis vectors $|p_1\lambda_1,\cdots,p_n\lambda_n\rangle$ are totally antisymmetric: the spin 1/2 particles follow Fermi-Dirac statistics, they are fermions.

Let us list the important ingredients which went into this famous spin-statistics connection:

- Hilbert space (of course with positive metric);
- a vacuum state $|0\rangle$ and one particle states $|p,\lambda\rangle$ with the expected energy momentum eigenvalues (4.122);
- translation invariance and Lorentz invariance, in (4.127)–(4.133);
- locality.

We stress here the relevance of the locality principle, as introduced for the case of the electromagnetic field in sect. 1.16. Imagine constructing local observables

$O(x)$ out of the spinor field. We want these to be local, i.e. they should commute for spacelike separations,

$$[O(x), O(y)] = 0, \quad (x-y)^2 > 0. \tag{4.161}$$

The spinor fields are not local in this sense, because anticommutators are not commutators, and apparently spinor fields are not observables. However, 'bilinears' of the type (Γ is some combination of Dirac matrices)

$$O(x, \Gamma) = \bar{\psi}(x)\Gamma\psi(x), \tag{4.162}$$

and generalizations thereoff, e.g. involving derivatives, do satisfy locality. This follows from application of the identity (4.21) with the help of which we can express commutators of bilinears in terms of anticommutators. The anticommutators satisfy locality, and therefore also the commutators of the bilinears,

$$[O(x, \Gamma_1), O(y, \Gamma_2)] = 0, \quad (x-y)^2 > 0. \tag{4.163}$$

We shall see later that familiar observables like currents and the energy momentum tensor can indeed be expressed as 'bilinears'. Had we insisted on commutation relations for $\psi(x)$, we would have had to assume a nonlocal commutator $[\psi^\alpha(x), \bar{\psi}^\beta(y)]$, as follows from its vacuum expectation value (4.143), and we could not have satisfied the locality principle.

4.5 Vacuum amplitude, propagator and action

At this stage we have and operator field $\psi(x)$ and an energy operator $H = P^0$, but not yet an action or unambiguous field equation which can be used to introduce local interactions. It is obvious from (4.147) that $\psi(x)$ satisfies the Klein-Gordon equation,

$$(m^2 - \partial^2)\, \psi(x) = 0, \tag{4.164}$$

but it also satisfies the Dirac equation:

$$\begin{aligned}(m + \gamma^\mu\partial_\mu)\, \psi(x) &= \sum_\lambda \int d\omega_p\, [e^{ipx}\, (m + i\gamma p)\, u(p,\lambda)\, a(p,\lambda) \\ &\quad + e^{-ipx}\, (m - i\gamma p)\, v(p,\lambda)\, a(p,\lambda)^\dagger] \qquad (4.165) \\ &= 0, \qquad (4.166)\end{aligned}$$

where $\gamma p = \gamma^\mu p_\mu$ and we used (4.113). The Klein-Gordon equation is actually a consequence of the Dirac equation, as follows by applying $m - \gamma^\mu\partial_\mu$ to the above equation and using

$$p_\mu p_\nu\, \gamma^\mu\gamma^\nu = \frac{1}{2} p_\mu p_\nu\, \{\gamma^\mu, \gamma^\nu\} = p_\mu p_\nu\, g^{\mu\nu} = p^2, \tag{4.167}$$

and

$$(m - i\gamma^\mu p_\mu)\,(m + i\gamma^\nu p_\nu) = m^2 + (\gamma p)^2 = m^2 + p^2. \tag{4.168}$$

So the Dirac equation seems favoured. Yet, it is not completely clear at this point that we should invent an action based on the Dirac equation rather than on the Klein-Gordon equation. To resolve this dilemma we shall introduce an external source and discover the action from the vacuum amplitude. It turns out to lead to the introduction of anticommuting numbers.

To streamline the presentation we shall temporarily restrict ourselves to the Majorana representation, in which $\psi^\alpha(x) = \psi^\alpha(x)^\dagger$. We introduce a real external source $\eta_\alpha(x)$ and add a source term

$$H_\eta(x^0) = -\int d^3x\, \overline{\eta}_\alpha(x)\psi^\alpha(x) \tag{4.169}$$

to the hamiltonian such that the total hamiltonian is $H + H_\eta$, where $\overline{\eta}\psi = \eta^T\beta\psi = \eta_{\alpha'}\beta_{\alpha'\alpha}\psi^\alpha$. The vacuum amplitude can then be expressed as usual in the source-interaction picture as

$$\begin{aligned} Z(\eta) &= \langle 0|T\, e^{-i\int dx^0\, H_\eta(x^0)}|0\rangle = \langle 0|T\, e^{i\int d^4x\, \overline{\eta}_\alpha(x)\psi^\alpha(x)}|0\rangle & (4.170)\\ &= 1 + \frac{i^2}{2}\int d^4x d^4y\, \langle 0|T\, \overline{\eta}_\alpha\psi^\alpha(x)\overline{\eta}_\beta\psi^\beta(y)|0\rangle + \cdots. & (4.171) \end{aligned}$$

Consider now the fermion propagator

$$\langle 0|T\, \psi^\alpha(x)\psi^\beta(y)|0\rangle. \tag{4.172}$$

We shall see shortly that we have to modify the definition of the time ordering operator when fermion fields are involved. For Bose fields the time ordered product is symmetric in exchange of labels as if the fields commute, e.g. for a scalar field

$$T\,\varphi(x)\varphi(y) = T\,\varphi(y)\varphi(x). \tag{4.173}$$

Thus it is natural to define the T product for fermion fields such that it is antisymmetric, as if they anticommute,

$$T\,\psi^\alpha(x)\psi^\beta(y) = -T\,\psi^\beta(y)\psi^\alpha(x), \tag{4.174}$$

that is

$$T\,\psi^\alpha(x)\psi^\beta(y) = \theta(x^0 - y^0)\,\psi^\alpha(x)\psi^\beta(y) - \theta(y^0 - x^0)\,\psi^\beta(y)\psi^\alpha(x). \tag{4.175}$$

With this definition the vacuum expectation value of the time ordered product takes the form, for $x^0 > y^0$, using (4.127), (4.128),

$$\langle 0|T\,\psi^\alpha(x)\psi^\beta(y)|0\rangle \;=\; \langle 0|\psi^\alpha(x)\psi^\beta(y)|0\rangle, \quad x^0 > y^0 \tag{4.176}$$

$$
\begin{aligned}
&= \sum_\lambda \int d\omega_p \, \langle 0|\psi^\alpha(x)|p,\lambda\rangle\langle p,\lambda|\psi^\beta(x)|0\rangle && (4.177)\\
&= \int d\omega_p \, e^{ip(x-y)} \sum_\lambda u^\alpha(p,\lambda)u^\beta(p,\lambda)^*, && (4.178)\\
&= \int d\omega_p \, e^{ip(x-y)} \, [(m - i\gamma^\mu p_\mu)\beta]_{\alpha\beta} && (4.179)\\
&= (m\beta - \gamma^\mu\beta\partial_\mu)_{\alpha\beta} \int d\omega_p \, e^{ip(x-y)}, && (4.180)
\end{aligned}
$$

where ∂_μ acts on x, and we must not confuse the index β with the matrix $\beta = i\gamma^0$. Similarly, we have for $x^0 < y^0$,

$$
\begin{aligned}
\langle 0|T\,\psi^\alpha(x)\psi^\beta(y)|0\rangle &= -\langle 0|\psi^\beta(y)\psi^\alpha(x)|0\rangle, \quad x^0 < y^0 && (4.181)\\
&= -\int d\omega_p \, e^{ip(y-x)} \, [(m - i\gamma^\mu p_\mu)\beta]_{\beta\alpha} && (4.182)\\
&= \int d\omega_p \, e^{-ip(x-y)} \, [(m + i\gamma^\mu p_\mu)\beta]_{\alpha\beta} && (4.183)\\
&= (m\beta - \gamma^\mu\beta\partial_\mu)_{\alpha\beta} \int d\omega_p \, e^{-ip(x-y)}, && (4.184)
\end{aligned}
$$

where we used the fact that in the Majorana representation the hermitian $\beta = i\gamma^0$ is purely imaginary, hence antisymmetric, and the antihermitian $\beta\gamma^\mu$ and $\gamma^\mu\beta$ are also purely imaginary, hence symmetric,

$$
\beta^T = -\beta, \quad (\beta\gamma^\mu)^T = \beta\gamma^\mu \quad (\gamma^\mu\beta)^T = \gamma^\mu\beta. \tag{4.185}
$$

Summarizing, we have

$$
\begin{aligned}
\langle 0|T\,\psi(x)\psi^T(y)|0\rangle &= \theta(x^0 - y^0)\,(m\beta - \gamma^\mu\beta\partial_\mu) \int d\omega_p \, e^{ip(x-y)} \\
&\quad + \theta(y^0 - x^0)\,(m\beta - \gamma^\mu\beta\partial_\mu) \int d\omega_p \, e^{-ip(x-y)} \\
&= (m\beta - \gamma^\mu\beta\partial_\mu)\,[\theta(x^0 - y^0) \int d\omega_p \, e^{ip(x-y)} \\
&\quad + \theta(y^0 - x^0) \int d\omega_p \, e^{-ip(x-y)}], && (4.186)
\end{aligned}
$$

where in the last line we pulled the time differentiation through the θ functions, which is allowed because the difference vanishes:

$$
-\gamma^0\beta\,[\delta(x^0 - y^0) \int d\omega_p \, e^{ip(x-y)} - \delta(x^0 - y^0) \int d\omega_p \, e^{-ip(x-y)}] = 0. \tag{4.187}
$$

We now use the relation

$$
\theta(x^0 - y^0) \int d\omega_p \, e^{ip(x-y)} + \theta(y^0 - x^0) \int d\omega_p \, e^{-ip(x-y)} \tag{4.188}
$$

$$= -i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}}{m^2+p^2-i\epsilon} \tag{4.189}$$
$$= -iG_{\rm scal}(x-y), \tag{4.190}$$

familiar from the scalar field and find the fermion propagator

$$\langle 0|T\,\psi^\alpha(x)\psi^\beta(y)|0\rangle = [(m-\gamma^\mu\partial_\mu)\beta]_{\alpha\beta}(-i)G_{\rm scal}(x-y) \tag{4.191}$$
$$= -i\int \frac{d^4p}{(2\pi)^4}\,e^{ip(x-y)}\,\frac{[(m-i\gamma^\mu p_\mu)\beta]_{\alpha\beta}}{m^2+p^2-i\epsilon} \tag{4.192}$$
$$\equiv -iG^{\alpha\beta'}(x-y)\,\beta_{\beta'\beta}, \tag{4.193}$$
$$\langle 0|T\,\psi^\alpha(x)\bar{\psi}^\beta(y)|0\rangle = -iG^{\alpha\beta}(x-y). \tag{4.194}$$

Had we used the Bose field definition of the T product for the fermion fields, the expression in (4.188) would have appeared with a minus sign in front of the second θ function and the resulting expression would not be a propagator.

The vacuum amplitude (4.171) is now expected to contain the expression

$$\int d^4x d^4y\,\overline{\eta}_\alpha(x)\,[G(x-y)\beta]^{\alpha\beta}\,\overline{\eta}_\beta(y). \tag{4.195}$$

However, we now have a problem: this expression is identically zero (when the sources are ordinary numbers)! This is because $G(x-y)\beta$ is antisymmetric when viewed as a continuous matrix:

$$[G(y-x)\beta]_{\beta\alpha} = \int \frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\,\frac{[(m\beta-i\gamma^\mu\beta p_\mu)]_{\beta\alpha}}{m^2+p^2-i\epsilon} \tag{4.196}$$
$$= -[G(x-y)\beta]_{\alpha\beta}, \tag{4.197}$$

where we used (4.185) and changed variables $p\to -p$. It follows that (4.195) vanishes identically when the $\overline{\eta}$ are ordinary numbers. To resolve the problem we have to introduce sources $\eta_\alpha(x)$ which are anticommuting:

$$\eta_\kappa(x)\eta_\lambda(y) = -\eta_\lambda(y)\eta_\kappa(x). \tag{4.198}$$

These are called anticommuting numbers or Grassmann 'variables'. They are generators of a Grassmann algebra. We will explain how to use them as we go along. For more information see for example Brown sect. 2.4. With anticommuting sources the expression (4.195) is algebraically nontrivial. However, it is not an ordinary complex number but an element of a Grassmann algebra, The same holds for the vacuum amplitude, and $|Z(\eta)|^2$ can no longer be interpreted as a probability. Yet we shall see that anticommuting numbers are very convenient and allow for a treatment of fermion fields similar to boson fields. They widely used in field theory.

The anticommuting character of all anticommuting numbers includes the fermion operator fields,

$$\eta_\alpha(x)\psi^\beta(y) = -\psi^\beta(y)\eta_\alpha(x). \tag{4.199}$$

Then there is no ambiguity in the introduction of the source term in the hamiltonian,

$$\overline{\eta}\psi = \eta^T \beta\psi = -\psi^T \beta^T \eta = \psi^T \beta\eta = \bar{\psi}\eta, \tag{4.200}$$

where the first minus sign appears because of the anticomuting η and ψ. We can now show how the definition of the time ordering operator for fermion fields appears naturally from (4.171), where T has its usual 'bosonic meaning' since it came from the evolution operator in the interaction picture. The product of two fermionic objects is commuting, apart, of course, from the noncommutativity of the operator fields. The T product of pairs of fermionic objects is a commuting T product in exchanges of the pairs. We have for $x^0 > y^0$:

$$\begin{aligned} T\,\overline{\eta}_\alpha(x)\psi^\alpha(x)\overline{\eta}_\beta(y)\psi^\beta(y) &= \overline{\eta}_\alpha(x)\psi^\alpha(x)\overline{\eta}_\beta(y)\psi^\beta(y) && (4.201)\\ &= -\overline{\eta}_\alpha(x)\psi^\alpha(x)\psi^\beta(y)\overline{\eta}_\beta(y), && (4.202)\end{aligned}$$

while for $x^0 < y^0$,

$$\begin{aligned} T\,\overline{\eta}_\alpha(x)\psi^\alpha(x)\overline{\eta}_\beta(y)\psi^\beta(y) &= \overline{\eta}_\beta(y)\psi^\beta(y)\overline{\eta}_\alpha(x)\psi^\alpha(x) = \overline{\eta}_\alpha(x)\overline{\eta}_\beta(y)\psi^\beta(y)\psi^\alpha(x)\\ &= \overline{\eta}_\alpha(x)\psi^\beta(x)\psi^\alpha(x)\overline{\eta}_\beta(x). && (4.203)\end{aligned}$$

Hence,

$$T\,\overline{\eta}_\alpha(x)\psi^\alpha(x)\overline{\eta}_\beta(y)\psi^\beta(y) = -\overline{\eta}_\alpha(x)\,T\left[\psi^\alpha(x)\psi^\beta(y)\right]\overline{\eta}_\beta(y), \tag{4.204}$$

and we find for the vacuum amplitude

$$\begin{aligned} Z(\eta) &= 1 - i\frac{1}{2}\int d^4x d^4y\,\overline{\eta}_\alpha(x)\left[G(x-y)\beta\right]^{\alpha\beta}\overline{\eta}_\beta + \ldots && (4.205)\\ &= 1 + i\frac{1}{2}\int d^4x d^4y\,\overline{\eta}(x)\,G(x-y)\,\eta(y) + \ldots, && (4.206)\end{aligned}$$

where we used $\overline{\eta}_\beta(y) = \eta_{\beta'}(y)\,\beta_{\beta'\beta} = -\beta_{\beta\beta'}\,\eta_{\beta'}(y)$.

We now look for the inverse (in the matrix sense) of the propagator. Using (4.168) we find that the inverse of the propagator is a differential operator,

$$\beta(m + \gamma^\mu\partial_\mu)\,G(x-y)\beta = \int \frac{d^4p}{(2\pi)^4}\,e^{ip(x-y)}\,\frac{m^2+p^2}{m^2+p^2-i\epsilon} = \delta^4(x-y), \tag{4.207}$$

and the propagator is the Green function of this differential operator with Feynman boundary conditions. By analogy with the Bose case, minus inverse of the Green function, contracted with classical anticommuting fields ψ_c, is now the candidate for the action of the free Fermi field,

$$\begin{aligned} S &= -\int d^4x\,\frac{1}{2}\psi_c^\alpha(x)\left[\beta(m+\gamma^\mu\partial_\mu)\right]_{\alpha\beta}\psi_c^\beta(x) && (4.208)\\ &= -\int d^4x\,\frac{1}{2}\bar{\psi}_c(x)\,(m+\gamma^\mu\partial_\mu)\,\psi_c(x). && (4.209)\end{aligned}$$

If ψ_c were commuting rather than anticommuting, S would vanish identically, as can be shown by interchanging the order of the ψ_c (transposition) and partial integration.

Making $\hbar$ explicit we can interpret $\psi_c^\alpha(x)$ as a formal classical limit of $\psi^\alpha(x)$:

$$\{\psi^\alpha(x), \psi^\beta(y)\} = O(\hbar) \to \{\psi_c^\alpha(x), \psi_c^\beta(y)\} = 0, \tag{4.210}$$

as $\hbar \to 0$.

4.6 Anticommuting variables

Because fermion variables anticommute, the variation of the action can be written in two equivalent ways but different ways[2],

$$\delta S = \int d^4x\, \delta\psi^\alpha(x)\, \frac{\delta S}{\delta\psi^\alpha(x)} = \int d^4x\, S\, \frac{\overleftarrow{\delta}}{\delta\psi^\alpha(x)}\, \delta\psi^\alpha(x), \tag{4.211}$$

and correspondingly we have to distinguish between left and right derivatives. To see this in more detail let us write the action in the condensed notation used earlier for the Bose fields, using capital letters for indices attached to Fermi fields,

$$S = -\int d^4x\, \frac{1}{2}\, \psi^T \beta(m + \gamma\partial)\psi \equiv \frac{1}{2}\, S_{KL}\, \psi^K \psi^L, \tag{4.212}$$

where $S_{KL} = -S_{LK}$. Then

$$\begin{aligned}
\delta S &= S(\psi + \delta\psi) - S(\psi) \\
&= \frac{1}{2}\, S_{KL}\, [(\psi^K + \delta\psi^K)(\psi^L + \delta\psi^L) - \psi^K \psi^L] \\
&= \frac{1}{2}\, S_{KL}\, (\delta\psi^K\, \psi^L + \psi^K\, \delta\psi^L) \\
&= S_{KL}\, \delta\psi^K\, \psi^L = -S_{KL} \psi^L\, \delta\psi^K.
\end{aligned} \tag{4.213}$$

Hence

$$\frac{\delta}{\delta\psi^K}\, S = S_{KL}\, \psi^L, \quad S\, \frac{\overleftarrow{\delta}}{\delta\psi^K} = -S_{KL}\, \psi^L. \tag{4.214}$$

The differentiations also behave like anticommuting variables, e.g.

$$\frac{\delta}{\delta\psi^K}\, \psi^L = \delta_K^L, \tag{4.215}$$

$$\frac{\delta}{\delta\psi^K}\, (\psi^L \psi^M) = \delta_K^L\, \psi^M - \psi^L\, \delta_K^M. \tag{4.216}$$

---

[2]For simplicity we drop the c on the classical ψ_c.

We shall always use left derivatives. Notice that $S,_{KL}$ (i.e. first $\delta/\delta\psi^K$ then $\delta/\delta\psi^L$) equals S_{KL}.

Using

$$\frac{\delta}{\delta\psi^\alpha(x)}\int d^4y\,\overline{\eta}\psi = -\overline{\eta}_\alpha(x) = -\eta_{\alpha'}\,\beta_{\alpha'\alpha} = \beta_{\alpha\alpha'}\,\eta_{\alpha'}, \tag{4.217}$$

the field equation with external source can now be derived as

$$\begin{aligned} 0 &= \frac{\delta}{\delta\psi^\alpha(x)}\,(S + \int d^4x\,\overline{\eta}\psi) \\ &= \beta_{\alpha\alpha'}\,[-(m+\gamma\partial)]\,\psi(x) + \eta(x)]_{\alpha'}, \end{aligned} \tag{4.218}$$

in which we recognize the Dirac equation found earlier in (4.166).

We end this section by giving the rule for complex conjugation (or hermitian conjugation when operator fields are involved),

$$(\psi^{K_1}\cdots\psi^{K_n})^* = \psi^{K_n}\cdots\psi^{K_1}. \tag{4.219}$$

Although the individual ψ^K are real (with our present use of Majorana fields in the Majorana representation), the order of fermion variables gets reversed as for hermitian conjugation. With this rule the action is real,

$$S^* = (\frac{1}{2}S_{KL}\,\psi^K\psi^L)^* = \frac{1}{2}S^*_{KL}\,\psi^L\psi^K = S, \tag{4.220}$$

since $S^*_{KL} = -S_{KL}$.

4.7 Dirac field

From two Majorana fields ψ_a, $a = 1, 2$, which are real in the Majorana representation for of the gamma matrices, we can form a complex fermion field. Such a field is called a Dirac field. We introduce it here by analogy to the complex scalar field. The action for two classical Majorana fields with idential mass m is given by

$$S = -\int d^4x\,\frac{1}{2}\,\psi_a^T\,\beta(\gamma^\mu\partial_\mu + m)\,\psi_a, \tag{4.221}$$

where there is a summation over a and with $\beta = -\beta^T$ in the Majorana representation. This action has a global SO(2) symmetry,

$$\psi_a \;\to\; R_{ab}\,\psi_b, \tag{4.222}$$

$$R_{ab} \;=\; (e^{-i\omega q})_{ab} = R^{-1}_{ba}, \quad q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{4.223}$$

The corresponding Noether current follows in the same way as for the scalar field,

$$\delta\psi_a = -iq_{ab}\,\psi_b\,\delta\omega, \tag{4.224}$$

$$\delta S = \int d^4x\, j^\mu\, \partial_\mu\delta\omega, \tag{4.225}$$

$$j^\mu = \frac{1}{2}\bar{\psi}_a\, i\gamma^\mu q_{ab}\psi_b, \tag{4.226}$$

and j^μ is conserved as a consequence of the field equations,

$$\partial_\mu j^\mu = 0. \tag{4.227}$$

The eigenstates of the charge matrix q define the Dirac fields

$$\psi = \frac{1}{\sqrt{2}}\,(\psi_1 - i\psi_2), \quad \psi^* = \frac{1}{\sqrt{2}}\,(\psi_1 + i\psi_2), \tag{4.228}$$

$$\psi_1 = \frac{1}{\sqrt{2}}\,(\psi + \psi^*), \quad \psi_2 = \frac{i}{\sqrt{2}}\,(\psi - \psi^*), \tag{4.229}$$

and keeping in mind that the ψ_a are anticommuting we find the action for the Dirac fields,

$$S = -\int d^4x\,\bar{\psi}\,(\gamma^\mu\partial_\mu + m)\,\psi, \tag{4.230}$$

where $\bar{\psi} = \psi^\dagger\beta$. From now on we shall work almost exclusively with Dirac fields treating ψ and $\bar{\psi}$ as independent variables, and assume no longer the Majorana representation for the gamma matrices. (Real fermion fields are used for example in Schwinger I, II, but complex fields are more common.).

In the quantum theory the following free field expressions are now similar to those for the scalar field,

$$\psi(x) = \sum_\lambda \int d\omega_p\,[e^{ipx}\,u(p,\lambda)\,a(\mathbf{p},\lambda,+) + e^{-ipx}\,v(p,\lambda)\,a(\mathbf{p},\lambda,-)^\dagger], \tag{4.231}$$

$$\bar{\psi}(x) = \sum_\lambda \int d\omega_p\,[e^{-ipx}\,\bar{u}(p,\lambda)\,a(\mathbf{p},\lambda,+)^\dagger + e^{ipx}\,\bar{v}(p,\lambda)\,a(\mathbf{p},\lambda,-)], \tag{4.232}$$

$$a(\mathbf{p},\lambda,\pm) = \frac{1}{\sqrt{2}}\,[a_1(\mathbf{p},\lambda) \mp ia_2(\mathbf{p},\lambda)], \tag{4.233}$$

$$a(\mathbf{p},\lambda,\pm)^\dagger = \frac{1}{\sqrt{2}}\,[a_1(\mathbf{p},\lambda)^\dagger \pm ia_2(\mathbf{p},\lambda)^\dagger]. \tag{4.234}$$

From the anticommutation relations of the Majorana fields ψ_a we find those of the Dirac fields,

$$\{\psi^\alpha(x), \psi^\beta(y)^\dagger\}_{x^0=y^0} = \delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y}),) \tag{4.235}$$

$$\{\psi^\alpha(x), \psi^\beta(y)\}_{x^0=y^0} = \{\psi^\alpha(x)^\dagger, \psi^\beta(y)^\dagger\}_{x^0=y^0} = 0. \tag{4.236}$$

The creation and annihilation operators satisfy the anticommutator relations

$$\{a(\mathbf{p},\lambda,c), a(\mathbf{p}',\lambda',c')^\dagger\} = 2p^0\,(2\pi)^3\,\delta(\mathbf{p}-\mathbf{p}')\,\delta_{\lambda\lambda'}\,\delta_{cc'}, \tag{4.237}$$

and zero otherwise; $c = +$ denotes a particle and $c = -$ an antiparticle.

In the quantum theory we have to pay attention to the ordering of operators in the current j^μ, as for the scalar field,

$$\begin{aligned} j^\mu &= \frac{1}{2}\,\bar\psi_a\, i\gamma^\mu\, q_{ab}\,\psi_b \\ &= \frac{1}{2}\,[\bar\psi\, i\gamma^\mu\,\psi - \psi^T\, i\gamma^{\mu T}\,\bar\psi^T]. \end{aligned} \tag{4.238}$$

Then the charge operator is given by

$$\begin{aligned} Q &= \int d^3x\, j^0 = \int d^3x\,\frac{1}{2}\,[\psi^{\alpha\dagger},\psi^\alpha] \\ &= \sum_\lambda \int d\omega_p\,[a(\mathbf{p},\lambda,+)^\dagger a(\mathbf{p},\lambda,+) - a(\mathbf{p},\lambda,-)^\dagger a(\mathbf{p},\lambda,-)], \end{aligned} \tag{4.239}$$

which is just the number operator for particles minus the number operator for antiparticles. It has the following commutation relations with the creation and annihilation operators,

$$[Q, a(\mathbf{p},\lambda,\pm)] = \mp a(\mathbf{p},\lambda,\pm), \quad [Q, a(\mathbf{p},\lambda,\pm)^\dagger] = \pm a(\mathbf{p},\lambda,\pm)^\dagger, \tag{4.240}$$

and with the Dirac fields

$$[Q,\psi] = -\psi, \quad [Q,\bar\psi] = \bar\psi. \tag{4.241}$$

As for the scalar field the theory is invariant under the charge conjugation transformation (4.70), $S(\psi^{(c)},\bar\psi^{(c)}) = S(\psi,\bar\psi)$. In the quantum theory charge conjugation is represented by a unitary operator U_C,

$$U_C^\dagger\,\psi\,U_C = \psi^{(c)} = (\bar\psi C)^T, \quad U_C^\dagger\,\bar\psi\,U_C = \bar\psi^{(c)} = -(C^\dagger\psi)^T. \tag{4.242}$$

The current changes sign under C,

$$U_C^\dagger\, j^\mu\, U_C = -j^\mu \tag{4.243}$$

(this would not be the case had we ignored the operator ordening subtlety in (4.238)).

4.8 Energy-momentum tensor and vacuum energy

In (4.158) we have constructed the energy-momentum operator P^μ of the free fermion field. It is still of interest to know the form of the energy-momentum tensor $T^{\mu\nu}$. Now we know the action we can use translation invariance to find a suitable $T^{\mu\nu}$ via the 'Noether procedure'. (The coupling of a spinor field to the gravitational field is much too involved to use here for the definition of $T^{\mu\nu}$.) In the case of global U(1)$\simeq$SO(2) invariance we found the conserved current j^μ by making a local SO(2) rotation. We follow the same strategy here for finding $T^{\mu\nu}$. We make an infinitesimal local translation $x \to x + \delta\xi(x)$ on the (classical) fields,

$$\psi'(x) = \psi(x + \delta\xi(x)) = \psi(x) + \delta\xi^\kappa(x)\partial_\kappa\psi(x), \tag{4.244}$$
$$\bar\psi'(x) = \bar\psi(x) + \delta\xi^\kappa(x)\partial_\kappa\bar\psi(x), \tag{4.245}$$

and identify $T^{\mu\nu}$ from the change in the (classical) action,

$$\delta S = \int d^4x\, \delta\xi^\kappa\, \partial_\mu T^\mu_{\ \kappa}. \tag{4.246}$$

In the calculation of δS appears a derivative of ξ^κ which is converted to the fields by partial integration,

$$\begin{aligned}
\delta S &= S(\psi', \bar\psi') - S(\psi, \bar\psi) \\
&= -\int d^4x\, [(\bar\psi + \delta\xi^\kappa\partial_\kappa\bar\psi)(m + \gamma^\mu\partial_\mu)(\psi + \delta\xi^\kappa\partial_\kappa\psi) - \bar\psi(m + \gamma^\mu\partial_\mu)\psi] \\
&= -\int d^4x\, \{\delta\xi^\kappa[\partial_\kappa\bar\psi(m + \gamma^\mu\partial_\mu)\psi + \bar\psi(m + \gamma^\mu\partial_\mu)\partial_\kappa\psi] + \partial_\mu\delta\xi^\kappa\, \bar\psi\gamma^\mu\partial_\kappa\psi\} \\
&= \int d^4x\, \delta\xi^\kappa[\partial_\kappa\mathcal{L} + \partial_\mu(\bar\psi\gamma^\mu\partial_\kappa\psi)],
\end{aligned} \tag{4.247}$$

where $\mathcal{L} = -\bar\psi(m + \gamma\partial)\psi$ is the lagrangian. Since δS is stationary when ψ and $\bar\psi$ are solutions of the equations of motion, we have the local conservation relation

$$\partial_\mu T^\mu_{\ \kappa} = 0, \quad T^\mu_{\ \kappa} = \bar\psi\gamma^\mu\partial_\kappa\psi + \delta^\mu_\kappa\mathcal{L}. \tag{4.248}$$

However, this expression for the energy-momentum tensor is not real, in fact, the lagrangian itself is not real. This can be repaired by symmetrizing the derivative in $\mathcal{L}$, writing

$$S = \int d^4x\, \mathcal{L}, \quad \mathcal{L} = -\bar\psi(m + \gamma^\mu \overleftrightarrow{\partial}_\mu)\psi, \tag{4.249}$$
$$\overleftrightarrow{\partial}_\mu \equiv \frac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu), \tag{4.250}$$

which leads to the real energy-momentum tensor

$$T^{\mu\nu} = \bar{\psi}\,\gamma^{\mu}\;\overset{\leftrightarrow}{\partial^{\nu}}\;\psi + g^{\mu\nu}\,\mathcal{L}. \tag{4.251}$$

In the case of the electromagnetic field the Noether form of the energy-momentum tensor would not be the gauge invariant expression (1.45). For a discussion see e.g. De Wit & Smith sect. 1.5.

The energy density is given by

$$\begin{aligned} T^{00} &= -\bar{\psi}\gamma^0 \overset{\leftrightarrow}{\partial_0}\psi + \bar{\psi}(m + \gamma^k \overset{\leftrightarrow}{\partial_k} + \gamma^0 \overset{\leftrightarrow}{\partial_0})\psi \\ &= \bar{\psi}(m + \gamma^k \overset{\leftrightarrow}{\partial_k})\psi. \end{aligned} \tag{4.252}$$

Of course, we still have to check the normalization and sign – these will turn out to be correct. The total energy is given by

$$H = \int d^3x\, T^{00} = \int d^3x\, \psi^{\dagger}\, \mathcal{H}\, \psi, \tag{4.253}$$

$$\mathcal{H} = m\beta - i\alpha_k \partial_k, \tag{4.254}$$

where $\alpha_k = i\beta\gamma_k$ and the hermitian differential operator $\mathcal{H}$ is called the Dirac hamiltonian.

Consider next the quantum theory. There is no operator ordering ambiguity in H, since β and α_k are traceless matrices. The operator H generates time translations in the way we expect for a hamiltonian,

$$i[H, \psi(x)] = i\int d^3y, [\psi^{\dagger}(y)\beta(m + \gamma^k\partial_k)\psi(y), \psi(x)] \tag{4.255}$$

$$= -i\beta(m + \gamma^k\partial_k)\psi(x) \tag{4.256}$$

$$= \partial_0\psi(x), \tag{4.257}$$

where we chose $y^0 = x^0$ (which we are free to do since H is time independent) and in the last line used the Dirac equation. Conversely, if we assume that H generates the time development according to the Heisenberg equation $\partial_0\psi = i[H, \psi]$, then the Dirac equation follows. We see here a glimpse of a canonical formalism for anticommuting variables (see Schwinger III, for example). We do not need this here since we have now enough at our disposal to turn to the covariant action (and path integral) formalism.

To express H in the creation and annihilation operators we use the fact that in the expansion of the fields, e.g. at time zero,

$$\psi(\mathbf{x}) = \int d\omega_p\, [e^{i\mathbf{px}}\, u(p, \lambda)\, a(\mathbf{p}, \lambda, +) + e^{-i\mathbf{px}}\, v(p, \lambda)\, a(\mathbf{p}, \lambda, -)^{\dagger}], \tag{4.258}$$

appear orthogonal eigenfunctions of $\mathcal{H}$:

$$\mathcal{H}(\mathbf{p})\, u(p, \lambda) \equiv (m\beta + \boldsymbol{\alpha}\cdot\mathbf{p})u(p, \lambda) = p^0\, u(p, \lambda), \tag{4.259}$$

$$\mathcal{H}(-\mathbf{p})\, v(p, \lambda) = -p^0\, v(p, \lambda), \tag{4.260}$$

where we used the Dirac equation in momentum space (4.113). We then find

$$\begin{aligned} H &= \int d\omega_p \,[a(\mathbf{p},\lambda,+)^\dagger a(\mathbf{p},\lambda,+) - a(\mathbf{p},\lambda,-)a(\mathbf{p},\lambda,-)^\dagger]\, p^0 \quad (4.261) \\ &= \int d\omega_p \,[a(\mathbf{p},\lambda,+)^\dagger a(\mathbf{p},\lambda,+) + a(\mathbf{p},\lambda,-)^\dagger a(\mathbf{p},\lambda,-)]\, p^0 \\ &\quad + E_0, \quad (4.262) \end{aligned}$$

with ground state energy density

$$\frac{E_0}{V} = -\frac{1}{2}\sum_{\lambda=\pm,\, c=\pm}\int d\omega_p\, p^0 p^0 = -2\int d\omega_p\, p^0 p^0. \quad (4.263)$$

In obtaining this expression we replaced $(2\pi)^3\,\delta(\mathbf{0}) \to V$, with $V\to\infty$ the total volume of the system (a more careful treatment giving the same result was given in the photon case, using a finite periodic box).

We see that this expression for for the ground state energy of free fermions is similar to that of an infinite set of bosonic harmonic oscillators, except that it has opposite sign, it is negative. As before we have to cancel the energy density with a suitable bare cosmological constant. The intruiging posibility of canceling, in an interacting theory, the positive bosonic contribution against a negative fermionic contribution is one of the aims of introducing supersymmetry.

4.9 Vacuum amplitude to all orders in η, $\overline{\eta}$

We shall now determine the exact vacuum amplitude for the free Dirac field with external sources. The coupling of the complex field to the complex sources η and $\overline{\eta}$ is described by the total action

$$S(\psi_c,\bar\psi_c) + \int d^4x\,(\overline{\eta}\psi_c + \bar\psi_c\eta), \quad (4.264)$$

where ψ_c and $\bar\psi_c$ are classical fermion fields and

$$S(\psi_c,\bar\psi_c) = -\int d^4x\,\bar\psi_c(m+\gamma^\mu\partial_\mu)\psi_c. \quad (4.265)$$

The field equations are given by

$$\begin{aligned} 0 &= \frac{\delta S}{\delta\bar\psi_c(x)} = -(m+\gamma^\mu\partial_\mu)\,\psi_c(x) + \eta(x), \quad (4.266) \\ 0 &= \frac{-\delta S}{\delta\psi_c(x)} = -\bar\psi_c(x)\,(m-\gamma^\mu\overleftarrow{\partial}_\mu) + \overline{\eta}(x). \quad (4.267) \end{aligned}$$

In the quantum theory the source terms lead to the additional term in the hamiltonian

$$H_\eta(x^0) = \int d^3x\,[\overline{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)], \tag{4.268}$$

which enters in the evolution operator in the source-interaction picture

$$U_\eta(t_1, t_2) = T\, e^{-i\int_{t_2}^{t_1} dx^0\, H_\eta(x^0)}. \tag{4.269}$$

Then the vacuum amplitude is given by

$$\begin{aligned} Z(\eta,\overline{\eta}) &= \langle 0|U_\eta(\infty,-\infty)|0\rangle = \langle 0|T\, e^{-i\int dx^0\, H_\eta(x^0)}|0\rangle & (4.270)\\ &= \langle 0|T\, e^{i\int d^4x\,(\overline{\eta}\psi+\bar{\psi}\eta)}|0\rangle. & (4.271)\end{aligned}$$

Since the combinations of pairs of fermionic objects $\overline{\eta}\psi$ and $\bar{\psi}\eta$ are commuting within the T product, taking a functional derivative with respect to $\overline{\eta}$ or η goes initially as in the bosonic case,

$$\delta Z = i\int d^4y\,\langle 0|T\,[\delta\overline{\eta}(x)\psi(x) + \bar{\psi}(x)\delta\eta(x)]\, e^{i\int d^4y\,(\overline{\eta}\psi+\bar{\psi}\eta)}|0\rangle. \tag{4.272}$$

For the derivatives we then get

$$\begin{aligned} \frac{\delta Z}{i\delta\overline{\eta}(x)} &= \langle 0|T\,\psi(x)\, e^{i\int d^4y\,(\overline{\eta}\psi+\bar{\psi}\eta)}|0\rangle, & (4.273)\\ &= \langle 0|U_\eta(\infty,x^0)\,\psi(x)\,U_\eta(x^0,-\infty)|0\rangle & (4.274)\\ \frac{\delta Z}{i\delta\eta(x)} &= -\langle 0|T\,\bar{\psi}(x)\, e^{i\int d^4y\,(\overline{\eta}\psi+\bar{\psi}\eta)}|0\rangle, & (4.275)\end{aligned}$$

where we recall that we use left derivates. Taking ∂_0 of (4.274) and using the anticommutators (4.235), (4.236) gives

$$\begin{aligned} \partial_0\frac{\delta Z}{i\delta\overline{\eta}(x)} &= \langle 0|U_\eta(\infty,x^0)\, i[H_\eta(x^0),\psi(x)]\,U_\eta(x^0,-\infty)|0\rangle \\ &\quad + \langle 0|U_\eta(\infty,x^0)\,\partial_0\psi(x)\,U_\eta(x^0,-\infty)|0\rangle & (4.276)\\ &= -\gamma^0\eta(x)\langle 0|U_\eta(\infty,-\infty)|0\rangle & (4.277)\\ &\quad + \gamma^0(m+\gamma^k\partial_k)\,\langle 0|U_\eta(\infty,x^0)\,\psi(x)\,U_\eta(x^0,-\infty)|0\rangle. \end{aligned}$$

This can be rewritten in the form

$$\left[-(m+\gamma^\mu\partial_\mu)\,\frac{\delta}{i\delta\overline{\eta}(x)} + \eta(x)\right]\, Z(\eta,\overline{\eta}) = 0, \tag{4.278}$$

which is just the field equation for ψ_c with $\psi_c \to \delta/i\delta\overline{\eta}$, as might be expected from our experience with Bose fields. Similarly, we have the conjugate equation corresponding to the field equation for $\bar{\psi}_c$,

$$\left[-m\frac{-\delta}{i\delta\eta(x)} + \partial_\mu\frac{-\delta}{i\delta\eta(x)}\,\gamma^\mu + \overline{\eta}(x)\right]\, Z(\eta,\overline{\eta}) = 0. \tag{4.279}$$

The solution to these equations with Feynman boundary conditions in time is easily written down by analogy with the Bose case,

$$Z(\eta,\overline{\eta}) = e^{i\int d^4x d^4y\, \overline{\eta}(x) G(x-y)\eta(y)}, \tag{4.280}$$

with $G(x-y)$ the fermion propagator (4.194). Let us check this for eq. (4.278):

$$\begin{aligned}\delta Z &= i\int d^4x d^4y\, [\delta\overline{\eta}(x)G(x-y)\eta(y) + \overline{\eta}(x)G(x-y)\delta\eta(y)] \\ &\quad e^{i\int d^4u d^4v\, \overline{\eta}(u)G(u-v)\eta(v)}, \end{aligned}\tag{4.281}$$

$$\frac{\delta Z}{i\delta\overline{\eta}(x)} = \int d^4y\, G(x-y)\eta(y)\, Z, \tag{4.282}$$

and using the fact that $G(x-y)$ is the inverse of $m+\gamma\partial$ gives

$$(m+\gamma^\mu\partial_\mu)\,\frac{\delta Z}{i\delta\overline{\eta}(x)} = \eta(x)\, Z, \tag{4.283}$$

which was to be shown.

We can now use this result to express arbitrary vacuum expectation values of time ordered products as a sum of products of propagators. The only difference with the bosonic case are the signs corresponding to permutations of the fermion operators ψ or $\bar{\psi}$ in the fermionic T-product:

$$\langle 0|T\,\psi(x)\bar{\psi}(y)|0\rangle = \frac{\delta}{\delta\overline{\eta}(x)}\frac{-\delta}{\delta\eta(y)}\, Z_{|\eta=\overline{\eta}=0} \tag{4.284}$$

$$= -iG(x-y), \tag{4.285}$$

$$\begin{aligned}\langle 0|T\,\psi(x_1)\bar{\psi}(y_1)\psi(x_2)\bar{\psi}(y_2)|0\rangle &= \frac{\delta}{\delta\overline{\eta}(x_1)}\frac{-\delta}{\delta\eta(y_1)}\frac{\delta}{\delta\overline{\eta}(x_2)}\frac{-\delta}{\delta\eta(y_2)}\, Z_{|\eta=\overline{\eta}=0} \\ &= (-i)^2\,[G(x_1-y_1)\,G(x_2-y_2) \\ &\quad - G(x_1-y_2)\,G(x_2-y_1)], \end{aligned}\tag{4.286}$$

and so on. The reader is urged to verify the second relation by successive differentiation of Z with respect to the sources. Notice that for a nonzero result there have to be an equal number of ψ's and $\bar{\psi}$'s, in accordance with charge conservation. Eq. (4.286) is illustrated in fig. 4.1. The generalization to an arbitrary number of pairs is evidently

$$\langle 0|T\,\psi(x_1)\bar{\psi}(y_1)\cdots\psi(x_n)\bar{\psi}(y_n)|0\rangle = (-i)^n\sum_P \epsilon_P\, G(x_1-y_{P1})\cdots G(x_n-y_{Pn}), \tag{4.287}$$

where P denotes permutations of 1, ..., n with signature ϵ_P.

The vacuum amplitude can of course also be expressed in effective action form

$$Z(\eta,\overline{\eta}) = e^{iW(\eta,\overline{\eta})}, \tag{4.288}$$

$$W(\eta,\overline{\eta}) = S(\psi_c,\bar{\psi}_c) + \int d^4x\,(\overline{\eta}\psi_c + \bar{\psi}_c\eta), \tag{4.289}$$

Figure 4.1: Diagrams for eq. (4.286).

with ψ_c and $\bar{\psi}_c$ solutions of the classical field equations with Feynman boundary conditions,

$$\psi_c(x) = \int d^4y\, G(x-y)\, \eta(y), \tag{4.290}$$

$$\bar{\psi}_c(x) = \int d^4y\, \overline{\eta}(y)\, G(x-y). \tag{4.291}$$

4.10 Problems

1. For a free Dirac fermion field let

$$|p_1\lambda_1c_1, \cdots, p_n\lambda_nc_n\rangle = a(\mathbf{p}_1, \lambda_1, c_1)^\dagger \cdots a(\mathbf{p}_n, \lambda_n, c_n)^\dagger |0\rangle, \tag{4.292}$$

and

$$\begin{aligned} \langle p_1\lambda_1c_1, \cdots, p_n\lambda_nc_n| &\equiv (|p_1\lambda_1c_1, \cdots, p_n\lambda_nc_n\rangle)^\dagger, \\ &= \langle 0|a(\mathbf{p}_n, \lambda_n, c_n) \cdots a(\mathbf{p}_1, \lambda_1, c_1), \end{aligned} \tag{4.293}$$

where $c = +$ denotes a particle and $c = -$ an antiparticle. Verify

$$\langle p'\lambda'c'|p\lambda c\rangle = 2p^0\,(2\pi)^3\,\delta(\mathbf{p}' - \mathbf{p})\,\delta_{\lambda'\lambda}\,\delta_{c'c}, \tag{4.294}$$

and in the two particle subspace ($n = 2$) at least, verify the orthogonality and completeness relations

$$\langle p_1'\lambda_1'c_1', \cdots, p_m'\lambda_m'c_m'|p_1\lambda_1c_1, \cdots, p_n, \lambda_nc_n\rangle = \tag{4.295}$$
$$\delta_{mn} \sum_P \epsilon_P\, \langle p_1'\lambda_1'c_1'|p_{P1}\lambda_{P1}c_{P1}\rangle \cdots \langle p_n'\lambda_n'c_n'|p_{Pn}\lambda_{Pn}c_{Pn}\rangle,$$

$$\sum_n \frac{1}{n!} \sum_{\lambda_1\cdots\lambda_n} \sum_{c_1\cdots c_n} \int d\omega_{p_1} \cdots d\omega_{p_n}$$
$$|p_1\lambda_1, \cdots, p_n, \lambda_n\rangle\langle p_1\lambda_1, \cdots, p_n, \lambda_n| = 1, \tag{4.296}$$

where $P1 \cdots Pn$ is a permutation of $1 \cdots n$ with signature ϵ_P.

2. For a free Dirac field, verify the following matrix elements of the current $j^\mu = \bar\psi i\gamma^\mu\psi$:

$$\begin{aligned}
\langle 0|j^\mu(x)|0\rangle &= 0, && (4.297)\\
\langle p'\lambda'|j^\mu(x)|p\lambda\rangle &= \bar u' i\gamma^\mu u\, e^{i(p-p')x}, && (4.298)\\
\langle\overline{p'\lambda'}|j^\mu(x)|\overline{p\lambda}\rangle &= -\bar v i\gamma^\mu v'\, e^{i(p-p')x}, && (4.299)\\
\langle p'\lambda', \overline{p\lambda}|j^\mu(x)|0\rangle &= \bar u' i\gamma^\mu v\, e^{-i(p+p')x}, && (4.300)\\
\langle 0|j^\mu(x)|p\lambda, \overline{p'\lambda'}\rangle &= \bar v i\gamma^\mu u'\, e^{i(p+p')x}, && (4.301)
\end{aligned}$$

where $u = u(p,\lambda)$, $\bar u' = \bar u(p',\lambda')$, etc., and the 'bar' in $\overline{p\lambda}$ denotes an antiparticle. It may be convenient to use the (conventional) notation $b(\mathbf{p},\lambda) = a(\mathbf{p},\lambda,+)$, $d(\mathbf{p},\lambda) = a(\mathbf{p},\lambda,-)$ for the particle and antiparticle annihilation operators in (4.231), (4.232).

3. Using the charge conjugation matrix C verify

$$\langle p'\lambda'|j^\mu(x)|p\lambda\rangle = -\langle\overline{p'\lambda'}|j^\mu(x)|\overline{p\lambda}\rangle \qquad (4.302)$$

from the explicit answers obtained above.

4. Verify $\partial_\mu j^\mu = 0$ in the above matrix elements of the current j^μ.

5. For the explicit expressions obtained above for the matrix elements of j^μ verify that

$$\langle p'\lambda'|Q|p\lambda\rangle = \langle p'\lambda'|p\lambda\rangle, \qquad (4.303)$$

etc., where $Q = \int d^3x\, j^0(x)$.

6. For general μ and ν we have

$$\Sigma_{\mu\nu} = \frac{-i}{2}[\gamma_\mu, \gamma_\nu]. \qquad (4.304)$$

Let $u = u(p,\lambda)$, $\bar u' = \bar u(p',\lambda')$. Verify that

$$(p-p')^\nu\, \bar u'\, \Sigma_{\mu\nu}\, u = 2m\, \bar u'\gamma_\mu u + i(p+p')_\mu\, \bar u' u, \qquad (4.305)$$

and the Gordon decomposition

$$\bar u' i\gamma^\mu u = \frac{1}{2m}\left[(p+p')^\mu \bar u'\, u + i(p-p')_\nu\, \bar u'\, \Sigma^{\mu\nu}\, u\right]. \qquad (4.306)$$

Compare with the expression of the matrix element of the current for a scalar particle.

Chapter 5

Spinor electrodynamics

Spinor electrodynamics is the theory of interacting spinor and electromagnetic fields. We discuss the Feynman rules and present some applications.

5.1 Defining the theory

The coupling of the Dirac field to the electromagnetic field is completely analagous to that of the complex scalar field. We start with

$$S = S_A + S_{A\psi}, \tag{5.1}$$

with

$$S_A = -\int d^4x\, \frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} \tag{5.2}$$

the action for the electromagnetic field and $S_{A\psi}$ the action of the fermion fields in which the derivative is replaced by the covariant derivative D_μ,

$$S_{A\psi} = -\int d^4x\, \bar{\psi}\,(m + \gamma^\mu D_\mu)\,\psi. \tag{5.3}$$

In view of the applications to particle physics we have to decide what the charge is of the particles to be described by ψ. For example, in an effective description of a proton by a spinor field ψ_p we have

$$\begin{aligned} D_\mu\,\psi_p &= (\partial_\mu - ieA_\mu)\,\psi_p, & (5.4)\\ D_\mu\,\bar{\psi}_p &= (\partial_\mu + ieA_\mu)\,\bar{\psi}, & (5.5) \end{aligned}$$

with the convention

$$e = |e|, \tag{5.6}$$

since the proton has positive charge. On the other hand, the electron which has negative charge is described by an electron field ψ_e,

$$\begin{aligned} D_\mu\,\psi_e &= (\partial_\mu + ieA_\mu)\,\psi_e, & (5.7)\\ D_\mu\,\bar{\psi}_e &= (\partial_\mu - ieA_\mu)\,\bar{\psi}_e, & (5.8) \end{aligned}$$

So the charge of the particles (p^+ and e^-) determines the sign in the covariant derivative. The fact that antiparticles have opposite charge is taken care of automatically by the formalism. In the following we shall take the electron case as an example.

In the quantum theory, in the Coulomb gauge, the bosonic operators have the usual equal time commutation relations, the fermionic operators the anticommutation relations, while boson operators commute with fermion operators. The Coulomb gauge is awkward to work with and as in the case of the scalar field we replace the Maxwell action by the modified action

$$S_A = -\int d^4x \, [\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A^\mu)^2], \tag{5.9}$$

to be used in the equation for the vacuum amplitude Z, with external sources J^μ, η and $\overline{\eta}$. This equation for Z follows from the clasical field equations for the classical fields A_c^μ, ψ_c and $\bar{\psi}_c$,

$$\begin{aligned} 0 &= [-\partial^2 g^{\mu\nu} + (1-\frac{1}{\xi})\partial_\mu\partial_\nu]\, A_c^\nu - e\bar{\psi}_c i\gamma^\mu\psi_c + J^\mu, && (5.10)\\ 0 &= -[m + \gamma^\mu(\partial_\mu + ieA_{c\mu})]\, \psi_c + \eta, && (5.11)\\ 0 &= -[\bar{\psi}_c(m + ie\gamma^\mu A_{c\mu}) - \partial_\mu\bar{\psi}_c\gamma^\mu] + \overline{\eta}, && (5.12)\end{aligned}$$

by replacing

$$A_c^\mu \to \frac{\delta}{i\delta J_\mu}, \quad \psi_c \to \frac{\delta}{i\delta\overline{\eta}}, \quad \bar{\psi}_c \to \frac{-\delta}{i\delta\eta}, \tag{5.13}$$

and letting the thus obtained functional differential operator act on $Z(J,\eta,\overline{\eta})$. In addition we should replace the coupling constants and masses by bare parameters, $e \to e_0$, $m \to m_0$, $\xi \to \xi_0$, and calculate the renormalized values e, m, ξ in terms of the bare parameters in a given regularization. This becomes relevant beyond the semiclassical approximation.

As mentioned in sect. 2.8, the solution of the equations for the vacuum amplitude can be represented by a path integral,

$$Z(J,\eta,\overline{\eta}) = \frac{\int DA\, D\bar{\psi}D\psi\, e^{iS(A,\psi,\bar{\psi}) + i\int d^4x\,(J_\mu A^\mu + \overline{\eta}\psi + \bar{\psi}\eta)}}{\int DA\, D\bar{\psi}D\psi\, e^{iS(A,\psi,\bar{\psi})}}, \tag{5.14}$$

where the integration variables are classical (anticomuting for ψ and $\bar{\psi}$), and formally

$$\begin{aligned} \int DA &= \prod_{x\mu} \int_{-\infty}^{\infty} dA_\mu(x), && (5.15)\\ \int D\bar{\psi}D\psi &= \prod_{x\alpha} \int d\bar{\psi}^\alpha(x) d\psi^\alpha(x). && (5.16)\end{aligned}$$

The demonstration that (5.14) is the solution of the equations for Z uses only translation invariance of the integration, as in (2.165), and the Feynman boundary conditions in time. As for ordinary integrals, fermionic integration can be defined rigourously for a finite number of variables, see e.g. Brown sect. 2.4. The path integral can be defined with the lattice regularization, using a finite number of modes or 'along the way' in a perturbative expansion in the coupling constant e_0. Perturbation theory leads to expressions involving Z in the free theory which we know how to evaluate, as in (2.170). The path integral then becomes a convenient tool in obtaining this expansion.

For us, the stage has been set already by the example of scalar electrodynamics involving only boson fields. The vacuum amplitude $Z(\eta,\overline{\eta},J)$ can be written in terms of an effective action $\Gamma(\psi_c,\bar{\psi}_c,A)$ by making a Legendre transformation from $W(\eta,\overline{\eta},J) = -i\ln Z(\eta,\overline{\eta},J)$ to $\Gamma(\psi_c,\bar{\psi}_c,A_c)$,

$$Z(\eta,\overline{\eta},J) = e^{iW(\eta,\overline{\eta},J)}, \tag{5.17}$$

$$W(\eta,\overline{\eta},J) = \Gamma(\psi_c,\bar{\psi}_c,A_c) + \int d^4x\,(\overline{\eta}\psi_c + \bar{\psi}_c\eta + J_\mu A_c^\mu), \tag{5.18}$$

and functional derivatives of W with respect to the sources give the correlation functions (connected Green functions). The equation for Z can be converted into an equation for Γ, which generates the Dyson-Schwinger equations. Keeping track of Planck's constant leads again to the conclusion

$$\Gamma(\psi_c,\bar{\psi}_c,A_c) = S(\psi_c,\bar{\psi}_c,A_c) + O(\hbar). \tag{5.19}$$

Since the only terms in S of higher order in the fields than bilinear are the $\bar{\psi}^\alpha\psi^\beta A^\mu$ couplings, there is only one bare vertex function,

$$S_{\bar{\psi}^\alpha\psi^\beta A^\mu}(x,y,z) = -ie_0\,(\gamma_\mu)_{\alpha\beta}\,\delta^4(x-y)\,\delta^4(x-z), \tag{5.20}$$

$$S_{\bar{\psi}^\alpha\psi^\beta A^\mu}(p,q,k) = -ie_0\,(\gamma_\mu)_{\alpha\beta}. \tag{5.21}$$

This is represented by the vertex in fig. 5.1, which also shows the propagators

$$G_\psi^{\alpha\beta}(p) \equiv G^{\psi^\alpha\bar{\psi}^\beta}(p,-p) = \frac{(m-i\gamma p)_{\alpha\beta}}{m^2+p^2-i\epsilon}, \tag{5.22}$$

$$G_A^{\mu\nu}(k) \equiv G^{A^\mu A^\nu}(k,-k) = \frac{g^{\mu\nu}-(1-\xi)k^\mu k^\nu/(k^2-i\epsilon)}{k^2-i\epsilon}. \tag{5.23}$$

In principle this is all straightforward. However, in practise the details are cumbersome because we have to keep track of minus signs due to the anticommuting character of fermion variables. In the condensed notation it can be useful to use capital letters to indicate fermionic variables,

$$J_K \leftrightarrow \eta_\alpha(x),\ \overline{\eta}_\alpha(x), \quad \phi^K \leftrightarrow \psi_c^\alpha(x),\ \bar{\psi}_c^\alpha(x), \tag{5.24}$$

$$J_k \leftrightarrow J_\mu(x), \quad \phi^k \leftrightarrow A^\mu(x). \tag{5.25}$$

Figure 5.1: Propagators and vertexfunctions in spinor electrodynamics.

Then all objects are antisymmetric in permutations of $K, L, \ldots$. We shall not go into details, but observe that when we ignore the anticommuting character of fermion variables the final result will be as we have seen before with boson variables only, up to possible minus signs. The determination of these signs will now be illustrated in the examples in the next section.

5.2 Scattering amplitudes

As a warm up we recapitulate the determination of the scattering amplitudes for $\gamma + \pi^\pm \to \gamma + \pi^\pm$, involving only scalar particles and photons:

1. Determine the two point correlation functions $G^{\phi\phi^*}$ and $G^{A^\mu A^\nu}$ for large time separations to find polarization vectors $e^\mu e^{\nu *}$ and wave function renormalization constants Z_ϕ, Z_A:

$$G^{A_1 A_2} \overset{x_1^0 \gg x_2^0}{\rightarrow} i \int d\omega_k \, e^{ik(x_1 - x_2)} \, Z_A \sum_\lambda e^{\mu_1}(k,\lambda) e^{\mu_2}(k,\lambda)^*, \tag{5.26}$$

$$G^{\phi_3 \phi_4^*} \overset{x_3^0 \gg x_4^0}{\rightarrow} i \int d\omega_p \, e^{ip(x_3 - x_4)}. \tag{5.27}$$

2. Determine the four point correlation function $G^{A_1 A_2 \phi_3 \phi_4^*}$ and identify the external propagators and $H_{A_1 A_2 \phi_3^* \phi_4}$; schematically

$$G^{A_1 A_2 \phi_3 \phi_4^*} = G^{A_1 A_{1'}} G^{A_{2'} A_2} G^{\phi_3 \phi_{3'}^*} G^{\phi_{4'} \phi_4^*} H_{A_{1'} A_{2'} \phi_{3'}^* \phi_{4'}}. \tag{5.28}$$

Factors $\sqrt{Z_A}\, e_1$ and $\sqrt{Z_A}\, e_2^*$ (i.e. $\sqrt{Z_A}\, e^{\mu_1}(k_1, \lambda_1)$ etc.) 'belong' to the absorption and emission amplitudes, while factors $\sqrt{Z_A} e_{1'}^*$ and $\sqrt{Z_A} e_{2'}$ belong to the scattering amplitude.

Figure 5.2: $\gamma_2 + \pi_4^+ \to \gamma_1 + \pi_3^+$ or $\gamma_2 + e_4^- \to \gamma_1 + e_3^-$ scattering (a), and $\gamma_2 + \pi_3^- \to \gamma_1 + \pi_4^-$ or $\gamma_2 + e_3^+ \to \gamma_1 + e_4^+$ scattering (b).

3. The scattering amplitude for $\gamma_2 + \pi_4^+ \to \gamma_1 + \pi_3^+$ is schematically given by

$$T = Z_A Z_\phi \, e_1^* \, H_{A_1 A_2 \phi_3^* \phi_4} \, e_2, \tag{5.29}$$

with appropriate ingoing and outgoing momenta, while the amplitude for the process $\gamma_2 + \pi_3^- \to \gamma_1 + \pi_4^-$ is derived from the same H-function, again with appropriate momenta. Denoting the antiparticles by a 'bar' in ket and bra, we have in detail

$$\begin{aligned} \langle k_1 \lambda_1, p_3 | T | k_2 \lambda_2, p_4 \rangle &= Z_A Z_\phi \, e^{\mu_1 *}(k_1, \lambda_1) \, H_{A^{\mu_1} A^{\mu_2} \phi^* \phi}(k_1, -k_2, p_3, -p_4) \\ & \quad e^{\mu_2}(k_2, \lambda_2), & (5.30) \\ \langle k_1 \lambda_1, \overline{p_4} | T | k_2 \lambda_2, \overline{p_3} \rangle &= Z_A Z_\phi \, e^{\mu_1 *}(k_1, \lambda_1) \, H_{A^{\mu_1} A^{\mu_2} \phi^* \phi}(k_1, -k_2, -p_3, p_4) \\ & \quad e^{\mu_2}(k_2, \lambda_2), & (5.31) \end{aligned}$$

as illustrated in fig. 5.2. Notice that e.g. G and H are completely symmetric under exchange of indices, which is a reflection of the fact that boson operators behave as commuting in time ordered products ('conn' = connected)

$$\langle 0 | T \, A^{\mu_1}(x_1) A^{\mu_2}(x_2) \varphi(x_3) \varphi(x_4)^\dagger | 0 \rangle_{\rm conn} = (-i)^3 G^{A_1 A_2 \phi_3 \phi_4^*}. \tag{5.32}$$

Therefore, $H_{A^{\mu_1} A^{\mu_2} \phi^* \phi}(k_1, -k_2, -p_3, p_4)$ appearing in (5.31) equals $H_{A^{\mu_1} \phi A^{\mu_2} \phi^*}(k_1, p_3, -k_2, -p_3)$, which might look more natural for process (b) in fig. 5.2.

Consider next the processes $\gamma + e^\mp \to \gamma + e^\mp$. We go again through the steps 1 – 3 above, in more detail for the fermion aspects:

1. The fermion propagator has poles with residue modified by a factor Z_ψ because of the interactions[1]. Then

$$G^{\psi_3 \bar{\psi}_4} \quad \overset{x_3^0 \gg x_4^0}{\longrightarrow} \quad i \int d\omega_p \, e^{ip(x_3 - x_4)} \, Z_\psi \sum_\lambda u^{\alpha_3}(p, \lambda) \bar{u}^{\alpha_4}(p, \lambda), \tag{5.33}$$

[1] This is true in the covariant gauges we are using. In the Coulomb gauge this is not the whole story and the situation is more complicated than suggested in Bjorken & Drell II sect. 17.9.

$$\stackrel{x_3^0 \ll x_4^0}{\rightarrow} \quad -i \int d\omega_p \, e^{ip(x_4-x_3)} \, Z_\psi \sum_\lambda v^{\alpha_3}(p,\lambda)\bar{v}^{\alpha_4}(p,\lambda), \tag{5.34}$$

which reflects the one particle contributions for the two time orderings

$$\langle 0|T\,\psi_3\bar{\psi}_4|0\rangle \stackrel{x_3^0 > x_4^0}{=} \sum_\lambda \int d\omega_p \, \langle 0|\psi_3|p,\lambda\rangle\langle p,\lambda|\bar{\psi}_4|0\rangle + \text{mpc} \tag{5.35}$$

$$\stackrel{x_3^0 < x_4^0}{=} -\sum_\lambda \int d\omega_p \, \langle 0|\bar{\psi}_4|\overline{p\lambda}\rangle\langle\overline{p\lambda}|\psi_3|0\rangle + \text{mpc}, \tag{5.36}$$

where mpc denotes the multiparticle contributions. The formulas reflect the free particle expressions (4.135), (4.138), and (4.231), (4.232) which for clarity we repeat here in a conventional notation for the creation and annihilation operators:

$$\psi(x) = \sum_\lambda \int d\omega_p \, [e^{ipx}\, u(p,\lambda)\, b(\mathbf{p},\lambda) + e^{-ipx}\, v(p,\lambda)\, d(\mathbf{p},\lambda)^\dagger], \tag{5.37}$$

$$\bar{\psi}(x) = \sum_\lambda \int d\omega_p \, [e^{-ipx}\, \bar{u}(p,\lambda)\, b(\mathbf{p},\lambda)^\dagger + e^{ipx}\, \bar{v}(p,\lambda)\, d(\mathbf{p},\lambda)]. \tag{5.38}$$

i.e. the particles are annihiliated by $b \equiv a(+)$ and the antiparticles to $d \equiv a(-)$.

2. Consider

$$\begin{aligned} \langle 0|T\, A^{\mu_1}(x_1)A^{\mu_2}(x_2)\psi^{\alpha_3}(x_3)\bar{\psi}^{\alpha_4}(x_4)|0\rangle &= (-i)^2 G^{A_1A_2}G^{\psi_3\bar{\psi}_4} + \\ &\quad (-i)^3 G^{A_1A_2\psi_3\bar{\psi}_4}. \end{aligned} \tag{5.39}$$

For the time ordering corresponding to fig. 5.2a this is equal to

$$\langle 0|A^{\mu_1}(x_1)A^{\mu_2}(x_2)\psi^{\alpha_3}(x_3)\bar{\psi}^{\alpha_4}(x_4)|0\rangle \leftrightarrow \langle 13|S|24\rangle, \tag{5.40}$$

where we indicated the resulting scattering matrix element on the right hand side (recall $S_{\text{conn}} = iT$). The reasoning behind this is that particles are created at x_2 and x_4, which evolve in time and may scatter, and get annihilated at x_1 and x_3. On the other hand, for the time ordering corresponding to fig. 5.2b the expression (5.39) is equal to

$$-\langle 0|A^{\mu_1}(x_1)A^{\mu_2}(x_2)\bar{\psi}^{\alpha_4}(x_4)\psi^{\alpha_3}(x_3)|0\rangle \leftrightarrow -\langle 1\overline{4}|S|2\overline{3}\rangle, \tag{5.41}$$

with the reasoning that in this case an antiparticle is created at x_3 and annihilated at x_4. Note again that one time ordered product (5.39) leads to several scattering matrix elements (we have mentioned only two of these) and note the minus sign in (5.41).

Figure 5.3: Diagrams for $\gamma_2 + e_4^- \to \gamma_1 + e_3^-$ scattering (a), and $\gamma_2 + e_3^+ \to \gamma_1 + e_4^+$ scattering (b).

The H function is defined in terms of the connected Green function without attention to time ordering. However, we like to keep the natural $\psi\bar{\psi}$-type charge ordering, since ordering matters for fermion Green functions,

$$G^{A_1 A_2 \psi_3 \bar{\psi}_4} = G^{A_1 A_{1'}} G^{A_{2'} A_2} G^{\psi_3 \bar{\psi}_{3'}} H_{A_{1'} A_{2'} \bar{\psi}_{3'} \psi_{4'}} G^{\psi_{4'} \bar{\psi}_4}. \tag{5.42}$$

3. The scattering amplitudes for $\gamma_2 + e_4^\mp \to \gamma_1 + e_3^\mp$ are now given by

$$\begin{aligned} \langle k_1\lambda_1, p_3\lambda_3|T|k_2\lambda_2, p_4\lambda_4\rangle &= Z_A Z_\psi \, e^{\mu_1 *}(k_1,\lambda_1)\bar{u}^{\alpha_3}(p_3,\lambda_3) \\ &\quad H_{A^{\mu_1} A^{\mu_2} \bar{\psi}^{\alpha_3} \psi^{\alpha_4}}(k_1, -k_2, p_3, -p_4) \\ &\quad u^{\alpha_4}(p_4,\lambda_4) e^{\mu_2}(k_2,\lambda_2), \qquad (5.43) \\ \langle k_1\lambda_1, \overline{p_4\lambda_4}|T|k_2\lambda_2, \overline{p_3\lambda_3}\rangle &= -Z_A Z_\psi \, e^{\mu_1 *}(k_1,\lambda_1)\bar{v}^{\alpha_3}(p_3,\lambda_3) \\ &\quad H_{A^{\mu_1} A^{\mu_2} \bar{\psi}^{\alpha_3} \psi^{\alpha_4}}(k_1, -k_2, -p_3, p_4) \\ &\quad v^{\alpha_4}(p_4,\lambda_4) e^{\mu_2}(k_2,\lambda_2). \qquad (5.44) \end{aligned}$$

Notice the $\bar{v}$-v structure: $\bar{v}$ corresponds to the initial state and v to the final state (compare also with (5.34)). The minus sign in (5.44) comes from the minus sign in (5.41).

It is straightforward to write down the explicit expressions for in the semiclassical approximation, see fig. 5.3. Fig. 5.3a represents

$$\begin{aligned} iT(13;24) &= e^{\mu *}(k_1,\lambda_1)\bar{u}(p_3,\lambda_3)\, [e\gamma_\mu(-i)\frac{m - i\gamma q}{m^2+q^2}\, e\gamma_\nu \\ &\quad + e\gamma_\nu(-i)\frac{m - i\gamma r}{m^2+r^2}\, e\gamma_\mu]\, u(p_4,\lambda_4) e^\nu(k_2,\lambda_2), \end{aligned} \tag{5.45}$$

where $q = p_1 + p_3 = p_2 + p_4$ and $r = p_4 - p_1 = p_3 - p_2$. Fig. 5.3b represents

$$\begin{aligned} -iT(1\bar{4};2\bar{3}) &= e^{\mu *}(k_1,\lambda_1)\bar{v}(p_3,\lambda_3)\, [e\gamma_\nu(-i)\frac{m - i\gamma q}{m^2+q^2}\, e\gamma_\mu \\ &\quad + e\gamma_\mu(-i)\frac{m - i\gamma r}{m^2+r^2}\, e\gamma_\nu]\, v(p_4,\lambda_4) e^\nu(k_2,\lambda_2), \end{aligned} \tag{5.46}$$

Figure 5.4: Disconnected and connected contributions to (5.47) in the semiclasical approximation.

Figure 5.5: $e_2^- + e_4^- \to e_1^- + e_3^-$ scattering (a) and $e_2^- + e_3^+ \to e_1^- + e_4^+$ scattering (b).

where $q = p_1 + p_4 = p_2 + p_3$ and $r = p_4 - p_2 = p_3 - p_1$.

A second class of examples is given by $e^- + e^\pm \to e^- + e^\pm$. These are derived from

$$\begin{aligned}\langle 0|T\,\psi_1\bar\psi_3\psi_3\bar\psi_4|0\rangle &= (-i)^2[G^{\psi_1\bar\psi_2}G^{\psi_3\bar\psi_4} - G^{\psi_1\bar\psi_4}G^{\psi_2\bar\psi_3}] \\ &\quad + (-i)^3 G^{\psi_1\bar\psi_2\psi_3\bar\psi_4}. \end{aligned} \tag{5.47}$$

The minus sign in the disconnected part shows already the signs to be given to the individual diagrams. Fig. 5.4 shows the diagrams for (5.47), with their signs, in the semiclasical approximation. No choice of time ordering is assumed. Fig. 5.5 shows the diagrams for scattering. For figure (a) we have taken the time ordering $x_3^0 > x_1^0 \gg x_2^0 > x_4^0$, for which (5.47) takes the form

$$\langle 0|T\,\psi_1\bar\psi_3\psi_3\bar\psi_4|0\rangle = +\langle 0|\psi_3\psi_1\bar\psi_2\bar\psi_4|0\rangle \leftrightarrow \langle 13|S|24\rangle, \tag{5.48}$$

while for (b) we have taken the time ordering $x_4^0 > x_1^0 \gg x_2^0 > x_3^0$, for which

$$\langle 0|T\,\psi_1\bar\psi_3\psi_3\bar\psi_4|0\rangle = -\langle 0|\bar\psi_4\psi_1\bar\psi_2\psi_3|0\rangle \leftrightarrow -\langle 1\bar4|S|2\bar3\rangle, \tag{5.49}$$

Figure 5.6: A closed fermion loop.

and the diagrams in fig. 5.5b represent $-iT(1\overline{4};2\overline{3})$. Fig. 5.5a now gives in self evident notation

$$\begin{aligned} iT(13;24) &= \bar{u}_1 e\gamma_\mu u_2\, \bar{u}_3 e\gamma_\nu u_4\,(-i)G^{\mu\nu}(k) \\ &\quad - \bar{u}_1 e\gamma_\nu u_4\, \bar{u}_3 e\gamma_\mu u_2\,(-i)G^{\mu\nu}(l), \end{aligned} \tag{5.50}$$

where $G^{\mu\nu}$ is the photon propagator and $k = p_1 - p_2 = p_4 - p_3$, $l = p_3 - p_2 = p_4 - p_1$. For the process involving the antiparticles e^+ we get from fig. 5.5b and (5.49),

$$\begin{aligned} -iT(1\overline{4};2\overline{3}) &= \bar{u}_1 e\gamma_\mu u_2\, \bar{v}_3 e\gamma_\mu v_4\,(-i)G^{\mu\nu}(k) \\ &\quad - \bar{u}_1 e\gamma_\mu v_4\, \bar{v}_3 e\gamma_\nu u_2(-i)G^{\mu\nu}(l), \end{aligned} \tag{5.51}$$

with $k = p_1 - p_2 = p_3 - p_4$ and $l = p_1 + p_4 = p_2 + p_3$.

The above examples show how the polarization spinors enter in scattering amplitudes. The various minus signs reflect the antisymmetry of multipoint Green functions in exchange of labels of external fermion lines. We end this section with the rule:

with each closed fermion loop goes a minus sign,

which follows from the derivation using Dyson-Schwinger equations, and which is also evident in the perturbation expansion of the path integral. The rule applies to the diagram in fig. 5.6, which represents the expression (excluding the minus sign)

$$\begin{aligned} \frac{1}{2}\, iS_{mKL}(-i)G^{LM} iS_{nMN}(-i)G^{NK} &= \frac{1}{2}\, S_{A^\mu \bar{\psi}_1 \psi_2}\, G^{\psi_2 \bar{\psi}_3}\, S_{A^\nu \bar{\psi}_3 \psi_4}\, G^{\psi_4 \bar{\psi}_1}, \\ &\quad + \frac{1}{2}\, S_{A^\mu \psi_1 \bar{\psi}_2}\, G^{\bar{\psi}_2 \psi_3}\, S_{A^\nu \psi_3 \bar{\psi}_4}\, G^{\bar{\psi}_4 \psi_1} \\ &= S_{A^\mu \bar{\psi}_1 \psi_2}\, G^{\psi_2 \bar{\psi}_3}\, S_{A^\nu \bar{\psi}_3 \psi_4}\, G^{\psi_4 \bar{\psi}_1} \end{aligned} \tag{5.52}$$

(there is an even number of sign changes when converting the ψ and $\bar{\psi}$ in the second term to 'natural order', and the two contributions are identical).

Figure 5.7: $e_2^- + e_3^+ \to \mu_1^- + \mu_4^+$ scattering.

5.3 Example, $e^- + e^+ \to \mu^- + \mu^+$ scattering

A simple example in fermion-fermion scattering is the process $e^- + e^+ \to \mu^- + \mu^+$, for which we shall evaluate the unpolarized differential cross section. This serves to illustrate a trace technique for the evaluation polarization sums.

We introduce fermion fields for the muon as well as for the electron, and since the two are independent there is only one relevant diagram, shown in fig. 5.7. To make the comparison with the diagrams in fig. 5.4b and the expression (5.51) easy, we use the same labeling. Then

$$\begin{aligned}
\langle \mu^-(1), \mu^+(4)|T|e^-(2), e^+(3)\rangle &= T(1\overline{4}; 2\overline{3}) \\
&= -\, \bar{u}_1 e\gamma_\mu v_4\, \bar{v}_3 e\gamma_\nu u_2\, G^{\mu\nu}(l), \qquad (5.53)\\
&= -\, e^2 \bar{u}(p_1, \lambda_1)\gamma_\mu v(p_4, \lambda_4) \\
&\quad \bar{v}(p_3, \lambda_3)\gamma_\nu u(p_2, \lambda_2) \\
&\quad \frac{g^{\mu\nu} - (1-\xi) l^\mu l^\nu / l^2}{l^2}, \qquad (5.54)
\end{aligned}$$

with $l = p_1 + p_4 = p_2 + p_3$. If we denote the electron and muon masses by m and M, respectively, then

$$p_2^2 = p_3^2 = -m^2, \quad p_3^2 = p_4^2 = -M^2. \tag{5.55}$$

The gauge terms $\propto l^\mu l^\nu$ in the photon propagator do not contribute because of current conservation. For example,

$$\bar{u}(p_1, \lambda_1) i\gamma_\mu v(p_4, \lambda_4)\, (p_1 + p_4)^\mu = 0, \tag{5.56}$$

where we used the fact that the polarization spinors satisfy the (momentum space version of the) Dirac equation (cf. 4.113)),

$$\bar{u}(p_1, \lambda)\, i\gamma p_1 = -M \bar{u}(p_1, \lambda_1), \quad i\gamma p_4\, v(p_4, \lambda_4) = M v(p_4, \lambda_4). \tag{5.57}$$

To calculate the cross section we need T^*, which leads to

$$(\bar{u}_1 \gamma_\rho v_4)^* = v_4^\dagger \gamma_\rho^\dagger \beta u_1 = -\bar{v}_4 \gamma_\rho u_1, \tag{5.58}$$

$$(\bar{v}_3 \gamma_\sigma u_2)^* = u_2^\dagger \gamma_\sigma^\dagger \beta v_3 = -\bar{u}_2 \gamma_\sigma v_3. \tag{5.59}$$

Averaging over initial spins and summing over final spins gives

$$\overline{|T|^2} = e^4\,\frac{1}{4}\sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \bar{u}_1\gamma_\mu v_4\,\bar{v}_3\gamma_\nu u_2\,\bar{v}_4\gamma_\rho u_1\,\bar{u}_2\gamma_\sigma v_3\,\frac{g^{\mu\nu}g^{\rho\sigma}}{s^2}, \tag{5.60}$$

where $s = -(p_1+p_4)^2 = -(p_2+p_3)^2$ is one of the Mandelstam variables (equal to the total cm energy). To evaluate the polarization sums we order the spinor factors in a suggestive way, interpreting $u^\alpha\bar{u}^\beta$ and $v^\alpha\bar{v}^\beta$ as matrices and using for example

$$\bar{u}_1\gamma_\mu v_4\bar{v}_4\gamma_\rho u_1 = \mathrm{Tr}\,[\gamma_\mu v_4\bar{v}_4\gamma_\rho u_1\bar{u}_1]. \tag{5.61}$$

Then

$$\begin{aligned}\overline{|T|^2} \;&=\; e^4\,\frac{g^{\mu\nu}g^{\rho\sigma}}{s^2}\,\frac{1}{4}\,\mathrm{Tr}\,[\gamma_\mu(\sum_{\lambda_4} v_4\bar{v}_4)\gamma_\rho(\sum_{\lambda_1} u_1\bar{u}_1)] \\ &\qquad \mathrm{Tr}\,[\gamma_\nu(\sum_{\lambda_2} u_2\bar{u}_2)\gamma_\sigma(\sum_{\lambda_3} v_3\bar{v}_3)].\end{aligned} \tag{5.62}$$

We now use the properties (4.109), (4.110),

$$\sum_{\lambda_4} v(p_4,\lambda_4)\bar{v}(p_4,\lambda_4) \;=\; -(M + i\gamma p_4), \tag{5.63}$$

$$\sum_{\lambda_2} u(p_2,\lambda_2)\bar{u}(p_2,\lambda_2) \;=\; m - i\gamma p_2, \tag{5.64}$$

etc. and obtain the form

$$\begin{aligned}\overline{|T|^2} \;&=\; e^4\,\frac{g^{\mu\nu}g^{\rho\sigma}}{s^2}\,\frac{1}{4}\,\mathrm{Tr}\,[\gamma_\mu(M + i\gamma p_4)\gamma_\rho(M - i\gamma p_1)] \\ &\qquad \mathrm{Tr}\,[\gamma_\nu(m - i\gamma p_2)\gamma_\sigma(m + i\gamma p_3)].\end{aligned} \tag{5.65}$$

To evaluate this we use the trace formulas

$$\mathrm{Tr}\,\gamma_\kappa\gamma_\lambda \;=\; 4g_{\kappa\lambda}, \tag{5.66}$$

$$\mathrm{Tr}\,\gamma_\kappa\gamma_\lambda\gamma_\mu \;=\; 0, \tag{5.67}$$

$$\mathrm{Tr}\,\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu \;=\; 4(g_{\kappa\lambda}g_{\mu\nu} - g_{\kappa\mu}g_{\lambda\nu} + g_{\kappa\nu}g_{\lambda\mu}). \tag{5.68}$$

These follow from the fact that (1) the trace of a product of gamma matrices vanishes unless each γ_0, …, γ_3 appears an even number of times, (2) $\gamma_0^2 = -1$, $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1$, (3) the gamma's anticommute and (4) $\mathrm{Tr}\,1 = 4$. For more information and derivations of trace theorems see Bjorken & Drell I, sect. 7.2 and De Wit & Smith sect. E.4. The two traces in (5.65) are given by

$$4(M^2\gamma_{\mu\rho} + p_{4\mu}p_{1\rho} - \gamma_{\mu\rho}p_1p_4 + p_{1\mu}p_{4\rho}) \tag{5.69}$$

and

$$4(m^2\gamma_{\nu\sigma} + p_{2\nu}p_{3\sigma} - \gamma_{\nu\sigma}p_2p_3 + p_{3\nu}p_2\sigma). \tag{5.70}$$

The evaluation of $\overline{|T|^2}$ is now straightforward and results in a large number of scalar products of the momenta. Using the Mandelstams variables

$$s = -(p_1 + p_4)^2 = 2M^2 - 2p_1p_4 = -(p_2 + p_3)^2 = 2m^2 - 2p_2p_3, \tag{5.71}$$

$$t = -(p_1 - p_2)^2 = -m^2 - M^2 + 2p_1p_2 = -(p_3 - p_4)^2 = -m^2 - M^2 + 2p_3p_4, \tag{5.72}$$

$$u = -(p_1 - p_3)^2 = -m^2 - M^2 + 2p_1p_3 = -(p_2 - p_4)^2 = -m^2 - M^2 + 2p_2p_4, \tag{5.73}$$

The result simplifies to

$$\begin{aligned} \overline{|T|^2} &= \frac{4e^4}{s^2}\,[4m^2M^2 + M^2(s - 2m^2) + m^2(s - 2M^2) \\ &\quad + \frac{1}{2}\,(t + m^2 + M^2)^2 + \frac{1}{2}\,(u + m^2 + M^2)^2], \end{aligned} \tag{5.74}$$

where we recall that u can be eliminated in favor of s and t by the relation $s + t + u = 2m^2 + 2M^2$. At high energies where we can neglect the electron and muon masses ($m \approx 0.511$ MeV, $M \approx 106$ MeV). Then

$$\overline{|T|^2} \approx \frac{2e^4}{s^2}\,(t^2 + u^2). \tag{5.75}$$

Under these cricumstances t and u are related to the scattering angle in the centre of mass frame by

$$t \approx -\frac{1}{2}\,s\,(1 - \cos\theta), \quad u \approx -\frac{1}{2}\,s\,(1 + \cos\theta), \tag{5.76}$$

and we get for the differential crossection at high energies

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\frac{k_f}{k_i}\,\overline{|T|^2} \tag{5.77}$$

$$\approx \frac{\alpha^2}{4s}\,(1 + \cos^2\theta). \tag{5.78}$$

The total cross section is given by

$$\sigma = 2\pi\,\frac{\alpha^2}{4s}\int_{-1}^{1} d\cos\theta\,(1 + \cos^2\theta) = \frac{4\pi\alpha^2}{3s}. \tag{5.79}$$

It is instructive to rederive these formulas by evaluating first the high energy form of T for given polarization combinations using helicity spinors, and from this $\overline{|T|^2}$. For further discussion see e.g. De Wit & Smith ch. 6.

5.4 Magnetic moment of the electron

In the nonrelativistic quantum mechanics, an electron in an external electromagnetic potential is described by the hamiltonian

$$H = \frac{\mathbf{p}^2 + e[\mathbf{p}\cdot\mathbf{A}(\mathbf{x}) + \mathbf{A}(\mathbf{x})\cdot\mathbf{p}]}{2m} - eA^0 + \frac{eg}{2m}\mathbf{S}\cdot\mathbf{B}, \tag{5.80}$$

where $\mathbf{S}$ is the spin operator, g is the gyromagnetic ratio and $B_k = \epsilon_{klm}\,\partial_l A_m$ is the magnetic field. The terms $\mathbf{p}\cdot\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{x})\cdot\mathbf{p}$ come from 'minimal substitution' $(\mathbf{p}+e\mathbf{A})^2$ (the charge of the electron is negative and $e>0$), and we have subtracted a term $e^2\mathbf{A}^2$ as it plays no dynamical role for an external potential. It will be shown in this section that in the approximation where (5.65) is valid, spinor electrodynamics predicts $g=2$.

We first derive the form of H in the momentum representation and then identify the same form in spinor electrodynamics. Using momentum states with relativistic normalization,

$$\langle p'\lambda'|p\lambda\rangle = 2p^0\,(2\pi)^3\,\delta(\mathbf{p}'-\mathbf{p})\,\delta_{\lambda'\lambda} \tag{5.81}$$

(just for convenience later), we have

$$\langle \mathbf{x}\lambda'|\mathbf{p}\lambda\rangle = \sqrt{2p^0}\,\delta_{\lambda'\lambda}\,e^{i\mathbf{p}\mathbf{x}} \tag{5.82}$$

and the momentum representation of H takes the form

$$\begin{aligned}\langle p'\lambda'|H|p\lambda\rangle &= \sqrt{4p^0p'^0}\,[\frac{\mathbf{p}^2\,(2\pi)^3\,\delta(\mathbf{p}'-\mathbf{p})\ + e(\mathbf{p}'+\mathbf{p})\cdot\tilde{\mathbf{A}}(\mathbf{p}'-\mathbf{p})}{2m}\,\delta_{\lambda'\lambda} \\ &\quad - e\tilde{A}^0(\mathbf{p}'-\mathbf{p})\,\delta_{\lambda'\lambda} + \frac{eg}{4m}\,\boldsymbol{\sigma}_{\lambda'\lambda}\cdot\tilde{\mathbf{B}}(\mathbf{p}'-\mathbf{p})],\end{aligned} \tag{5.83}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and we used

$$\begin{aligned}\langle p'\lambda'|A^\mu(\mathbf{x})|p\lambda\rangle &= \sum_{\lambda''}\int d^3x\,\langle p'\lambda'|\mathbf{x}\lambda''\rangle\,\langle \mathbf{x}\lambda''|p\lambda\rangle\,A^\mu(\mathbf{x}) && (5.84)\\ &= \sqrt{4p^0p'^0}\,\delta_{\lambda'\lambda}\,\tilde{A}^\mu(\mathbf{p}'-\mathbf{p}), && (5.85)\\ \tilde{A}^\mu(\mathbf{k}) &= \int d^3x\,e^{-i\mathbf{k}\mathbf{x}}\,A^\mu(\mathbf{x}). && (5.86)\end{aligned}$$

Notice that

$$\tilde{B}_k(\mathbf{k}) = i\epsilon_{lmn}\,k_m\tilde{A}_n(\mathbf{k}). \tag{5.87}$$

In spinor electrodynamics, the approximation where (5.83) is valid, is the nonrelativistic approximation in which radiation effects due to the quantized photon field are neglected. So we consider the spinor field in an external static electromagnetic potential $A^\mu(\mathbf{x})$. The hamiltonian of this system can be derived by the Noether argument, since the system is translation invariant if transform the external potential as well as the dynamical variables ψ and $\bar{\psi}$. It is given by

$$H = P^0 - \int d^3x\, (-e) j^\mu(x)\, A_\mu(\mathbf{x}). \tag{5.88}$$

Its matrix element in the one particle subspace is given by

$$\langle p'\lambda'|H|p\lambda\rangle \;=\; p^0\, \langle p'\lambda'|p\lambda\rangle + e \int d^3x\, \langle p'\lambda'|j^\mu(x)|p\lambda\rangle\, A_\mu(\mathbf{x}). \tag{5.89}$$

Using the result derived in the problems in the previous chapter we have

$$\langle p'\lambda'|j^\mu(x)|p\lambda\rangle = \bar{u}(p',\lambda')\, i\gamma^\mu\, u(p,\lambda)\, e^{i(p-p')x}, \tag{5.90}$$

$$\bar{u}(p',\lambda')\, i\gamma^\mu\, u(p,\lambda) = \frac{1}{2m}\, \bar{u}(p',\lambda')\, [(p+p')^\mu + i(p-p')_\nu\, \Sigma^{\mu\nu}]\, u(p,\lambda). \tag{5.91}$$

Using the explicit form (4.85) for the spinors we get in the nonrelativistic approximation

$$\begin{aligned}
\bar{u}(p',\lambda')\, u(p,\lambda) &= 2m\, [\delta_{\lambda'\lambda} + O(\mathbf{p}^2/m^2)], \\
\bar{u}(p',\lambda')\, \Sigma^{0n}\, u(p,\lambda) &= 2m\, [O(|\mathbf{p}|/m)], && (5.92) \\
\bar{u}(p',\lambda')\, \Sigma^{mn}\, u(p,\lambda) &= 2m\, [(\sigma_l)_{\lambda'\lambda}\, \epsilon_{lmn} + O(|\mathbf{p}|/m)] && (5.93)
\end{aligned}$$

(since $\bar{u}'u$ is a scalar its corrections are $O(\mathbf{p}^2/m^2)$). Substitution in (5.89) now gives (5.83) with $g = 2$, plus a rest energy m which is omitted in the usual nonrelativistic expressions.

5.5 Problems

1. *Trace and other identities*

   In sect. 5.3 we encountered traces over products of gamma matrices. The following identities can be derived (see for example Bjorken & Drell sect. 7.2):

$$\begin{aligned}
\mathrm{Tr}\, \gamma^{\mu_1} \cdots \gamma^{\mu_n} &= 0, \quad n = \text{odd}, && (5.94) \\
\mathrm{Tr}\, 1 = 4, & && (5.95) \\
\mathrm{Tr}\, \gamma^\mu \gamma^\nu &= 4g^{\mu\nu}, && (5.96) \\
\mathrm{Tr}\, \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu &= 4(g^{\kappa\lambda} g^{\mu\nu} - g^{\kappa\mu} g^{\lambda\nu} + g^{\kappa\nu} g^{\lambda\nu}), && (5.97) \\
\mathrm{Tr}\, \gamma_5 \gamma^{\mu_1} \cdots \gamma^{\mu_n} &= 0, \qquad n = 0,1,2,3, && (5.98) \\
&= 4i\epsilon^{\mu_1 \cdots \mu_4}, \quad n = 4, && (5.99) \\
\gamma_\mu \gamma^\mu &= 4, && (5.100) \\
\gamma_\mu \gamma^\kappa \gamma^\mu &= -2\gamma^\kappa, && (5.101) \\
\gamma_\mu \gamma^\kappa \gamma^\lambda \gamma^\mu &= 4g^{\kappa\lambda}, && (5.102) \\
\gamma_\mu \gamma^\alpha \gamma^\kappa \gamma^\beta \gamma^\mu &= -2\gamma^\beta \gamma^\kappa \gamma^\alpha. && (5.103)
\end{aligned}$$

2. *Elastic electron scattering*

In sect. 5.2 we derived the amplitude for the process $e^- + e^- \to e^- + e^-$. In this problem we shall work out the unpolarized cross section. Consider the amplitude for $e_1^- + e_2^- \to e_3^- + e_4^-$,

$$T(34;12) = -e^2\,[\bar{u}_3\gamma_\mu u_1\,\bar{u}_4\gamma_\nu u_2\,G^{\mu\nu}(k) - \bar{u}_3\gamma_\nu u_2\,\bar{u}_4\gamma_\mu u_1\,G^{\mu\nu}(l), \tag{5.104}$$

which differes from (5.50) only by a change in numbering the particles.

a. Show using the Dirac equation in momentum space that $\bar{u}_3\gamma_\mu u_1\,k^\mu = 0$, $\bar{u}_3\gamma_\nu u_2\,l^\nu = 0$, and verify that this corresponds to current conservation (cf. Problem 4.4). Consequently the amplitude can be simplied to

$$T = -e^2\,[\bar{u}_3\gamma_\mu u_1\,\bar{u}_4\gamma^\mu u_2\,\frac{1}{k^2} - \bar{u}_3\gamma_\nu u_2\,\bar{u}_4\gamma^\nu u_1\,\frac{1}{l^2}. \tag{5.105}$$

b. Derive along similar lines as in sect. 5.3 that

$$\overline{|T|^2} = \frac{e^4}{4}\,[\frac{T_1}{((p_1-p_3)^2)^2} - \frac{T_1}{(p_1-p_3)^2(p_1-p_4)^2} + (p_3 \leftrightarrow p_4)], \tag{5.106}$$

where, using the convenient 'slash' notation $\not{p} = p_\mu\gamma^\mu$,

$$T_1 = \mathrm{Tr}\,[\gamma_\mu(m - i\not{p}_1)\gamma_\nu(m - i\not{p}_3)]\,\mathrm{Tr}\,[\gamma^\mu(m - i\not{p}_2)\gamma^\nu(m - i\not{p}_4)], \tag{5.107}$$

$$T_2 = \mathrm{Tr}\,[\gamma_\mu(m - i\not{p}_1)\gamma_\nu(m - i\not{p}_4)\gamma^\mu(m - i\not{p}_2)\gamma^\nu(m - i\not{p}_3)]. \tag{5.108}$$

c. Using the identities in Problem 1 and of course momentum conservation $p_1 + p_2 = p_3 + p_4$ and $p_i^2 = -m^2$, show that

$$\begin{aligned} T_1 &= 32[2m^4 + 2m^2 p_1p_3 + (p_1p_2)^2 + (p_1p_4)^2], && (5.109)\\ T_1 &= -32[2m^2 p_1p_2 + (p_1p_2)^2]. && (5.110)\end{aligned}$$

d. These expressions are to be evaluated in the center of mass frame. Let θ be the scattering angle between particles 1 and 3, $p_1p_3 = -m^2 - |\mathbf{p}|^2(1-\cos\theta)$. From now on we use the notation $k \equiv |\mathbf{p}|$. Verify that

$$\begin{aligned} T_1 &= 64[m^4 + 4m^2k^2\cos^2\frac{\theta}{2} + 2k^4(1+\cos^4\frac{\theta}{2})], && (5.111)\\ T_2 &= -32(-m^4 + 4k^4), && (5.112)\\ \overline{|T|^2} &= \frac{e^4}{64k^4}\,[\frac{T_1}{\sin^4\frac{\theta}{2}} - \frac{T_2}{\cos^2\frac{\theta}{2}\,\sin^2\frac{\theta}{2}} + (\theta \to \pi - \theta)]. && (5.113)\end{aligned}$$

e. Under ultrarelativistic conditions we may neglect the electron mass m. Verify that in the center of mass frame

$$\frac{d\sigma}{d\Omega}_{|\mathrm{ur}} = \frac{\alpha^2}{8k^2}\,[\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} + \frac{2}{\cos^2\frac{\theta}{2}\,\sin^2\frac{\theta}{2}} + \frac{1+\sin^4\frac{\theta}{2}}{\cos^4\frac{\theta}{2}}]. \tag{5.114}$$

f. Under nonrelativistic conditions we may neglect p compared to m. Verify that

$$\frac{d\sigma}{d\Omega}_{|\mathrm{nr}} = \frac{\alpha^2 m^2}{16k^4}\Big[\frac{1}{\sin^4\frac{\theta}{2}} - \frac{1}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}} + \frac{1}{\cos^4\frac{\theta}{2}}\Big]. \tag{5.115}$$

The middle term is due to the interference of the two diagrams contributing to the amplitude. The first term goes over in the Rutherford formula for Coulomb scattering off a heavy target, upon expressing it in terms of the reduced mass $m_{\mathrm{red}} = mm/(m+m) = m/2$.

The total cross section is infinite because the integration over angles diverges at $\theta = 0$. This can be attributed to the infinite range of the Coulomb potential.

3. *The decays $\pi^- \to \mu^- + \bar{\nu}_\mu$ and $\pi^- \to e^- + \bar{\nu}_e$.* The charged pions $\pi^\mp$ are unstable and decay mainly into muons $\mu^\mp$ and muon neutrinos $(\bar{\nu}_\mu)\nu_\mu$, with a life time of 2.60×10^{-8} s, or $\Gamma^{-1}c = 780$ cm. There is a corresponding decay into electrons $e^\mp$ and electron neutrinos $(\bar{\nu}_e)\nu_e$, with a much smaller rate. These processes can be described by an effective action of the form $S = S_0 + S_1$, where S_1 is the interaction

$$\begin{aligned} S_1 &= c\int d^4x\,[\partial_\kappa\varphi^*\,\bar{\psi}_\mu i\gamma^\kappa(1-\gamma_5)\psi_{\nu_\mu} + \partial_\kappa\varphi\,\bar{\psi}_{\nu_\mu} i\gamma^\kappa(1-\gamma_5)\psi_\mu \\ &+ (\mu \to e), \end{aligned} \tag{5.116}$$

and S_0 is the sum of the actions for the free pions, muons, electrons, muon neutrinos and electron neutrinos,

$$S_0 = S_\pi + S_\mu + S_e + S_{\nu_\mu} + S_{\nu_e}, \tag{5.117}$$

$$S_\pi = -\int d^4x\,(\partial_\mu\varphi^*\partial^\mu\varphi + m_\pi^2\varphi^*\varphi), \tag{5.118}$$

$$S_\mu = -\int d^4x\,\bar{\psi}_\mu(\gamma^\kappa\partial_\kappa + m_\mu)\psi_\mu, \tag{5.119}$$

and similar for e, ν_μ and ν_e with $m_{\nu_\mu} = m_{\nu_e} = 0$. The constant c is given by

$$c = f_\pi\, G_F\, \cos\theta_C, \tag{5.120}$$

with f_π the pion decay constant, G_F the Fermi weak interaction constant and θ_C the Cabibbo angle.

Notice that the interaction S_1 does not conserve parity P, as it is the sum of terms odd and even under parity.

a. Verify the position space vertex function

$$S_{\varphi^*\bar{\psi}_e^\alpha\psi_{\nu_e}^\beta}(u,v,w) = c\int d^4x\,\partial_\kappa\delta(x-u)\,[i\gamma^\kappa(1-\gamma_5)]_{\alpha\beta}\,\delta(x-v)\delta(x-w), \tag{5.121}$$

and derive similarly the other vertex functions.

b. Verify the momentum space vertex function

$$S_{\varphi^* \bar{\psi}_e^\alpha \psi_{\nu_e}^\beta}(p,k,l) = c[\gamma p(1-\gamma_5)]_{\alpha\beta}, \tag{5.122}$$

and derive similarly the other vertex functions. Draw the diagrams for these vertex functions.

c. Draw the diagram for the decay $\pi^-(p) \to \mu^-(k,\lambda) + \bar{\nu}_\mu(k',\lambda')$ and verify that the decay amplitude is given by

$$\langle k\lambda, \overline{k'\lambda'}|T|p\rangle = -c\bar{u}(k,\lambda)\gamma p(1-\gamma_5)v(k',\lambda'). \tag{5.123}$$

d. Verify the polarization sum

$$\overline{|T|^2} = c^2 \mathrm{Tr}\,[\gamma p(1-\gamma_5)(i\gamma k')\gamma p(1-\gamma_5)(m_\mu - i\gamma k)]. \tag{5.124}$$

e. Using the anticommutation relations of the gamma matrices, the properties of the right and lefthanded projectors $P_{R,L} = (1 \pm \gamma_5)/2$ (cf. (4.49)) and the identities in Problem 1 above, show that

$$\overline{|T|^2} = 8c^2[2(pk)(pk') - p^2 kk']. \tag{5.125}$$

f. In the rest frame of the pion, verify

$$|\mathbf{k}| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}, \quad k^0 - |\mathbf{k}| = \frac{m_\mu^2}{m_\pi}, \tag{5.126}$$

and

$$\overline{|T|^2} = 4c^2(m_\pi^2 - m_\mu^2)m_\mu^2, \tag{5.127}$$

and

$$\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu) = \frac{c^2}{4\pi}\frac{(m_\pi^2 - m_\mu^2)^2 m_\mu^2}{m_\pi^3}. \tag{5.128}$$

g. The masses of the particles are given by $m_{\pi^\pm} = 139.6$ MeV, $m_\mu = 105.7$ MeV, $m_e = 0.5110$ Mev (the neutrino masses are assumed to be zero). Using $G_F \approx 1.17 \times 10^{-5}$ GeV^{-2}, $\theta_C \approx 13°$, and the fact that π^- decays for 99.988% into $\mu^- + \bar{\nu}_\mu$, verify that $f_\pi \approx 93$ MeV from the rate $\Gamma = 780$ cm^{-1}.

h. Calculate the branching ratio

$$\frac{\Gamma(\pi^- \to e^- + \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} \tag{5.129}$$

and compare this with the experimental value 1.22×10^{-4}.

The striking smallness of the above branching ratio is a consequence of the combination $\gamma^\kappa(1-\gamma_5)$ in the interaction S_1. The interaction conserves chirality: $1-\gamma_5$ projects on to chirality –1, in the neutrino fields as well as in the electron or muon fields (recall that $\bar\psi$ contains $\beta = i\gamma^0$ and γ_5 commutes with $i\gamma^0\gamma^\kappa$). For the massless antineutrinos, chirality –1 means helicity +1/2 (cf. (4.121)). For the electron and muon, chirality –1 would mean helicity –1/2 if these particles were massless (cf. (4.120)). However, angular momentum conservation requires that the muon or electron have the same helicity (+1/2) as the antineutrino, since the pion at rest has angular momentum zero. Hence, if m_μ and m_e would be zero, the decay amplitude would vanish (since $1-\gamma_5$ acting on a massless helicity +1/2 *particle* spinor gives zero). So we may expect that the decay amplitude is proportional to $m_{\mu,e}$ as $m_{\mu,e}$ goes to zero. In fact, it can be shown using helicity spinors that the decay amplitude is given by

$$T = 2icm_\pi\sqrt{|\mathbf{k}|}\,(\sqrt{k^0+m}-\sqrt{k^0-m})\,\delta_{\lambda,\lambda'}\,\delta_{\lambda',+}, \tag{5.130}$$

with $k^0 = \sqrt{\mathbf{k}^2+m^2}$ and $m = m_\mu$ or m_e. In this way we can understand why the above branching ratio $\propto m_e^2/m_\mu^2$ is so small.

It is instructive to go through the derivation of (5.130) in the Weyl representation, using helicity spinors (cf. (4.85) and (4.91)),

$$\begin{aligned} u(|\mathbf{k}|,\theta,\phi,\lambda) &= (\sqrt{k^0+m}+\lambda\sqrt{k^0-m}\gamma_5)\,\xi_\lambda(\theta,\phi), &(5.131)\\ v(|\mathbf{k}'|,\theta',\phi',\lambda') &= \sqrt{|\mathbf{k}'|}\,(1-\lambda'\gamma_5)\,\xi^{(c)}_{\lambda'}(\theta',\phi'), &\\ &= \sqrt{|\mathbf{k}|}\,(1-\lambda'\gamma_5)\,\xi^{(c)}_{\lambda'}(\pi-\theta,\phi+\pi) &(5.132)\end{aligned}$$

where $\mathbf{k}' = -\mathbf{k} = (|\mathbf{k}|,\pi-\theta,\phi+\pi)$ in spherical coordinates. Because of the factor $(1-\gamma_5)$ in T we may replace $\gamma_5 \to -1$ in $u^\dagger$ and v', and the amplitude (5.123) reduces to

$$\begin{aligned} T &= -2ic\,m_\pi\sqrt{|\mathbf{k}|}\,(\sqrt{k^0+m}-\lambda\sqrt{k^0-m}) &(5.133)\\ &\quad\ \xi_\lambda(\theta,\phi)^\dagger(1-\gamma_5)\xi^{(c)}_{\lambda'}(\pi-\theta,\phi+\pi)\,\delta_{\lambda',+}. &(5.134)\end{aligned}$$

We now use (4.118) and (4.92), with $\beta = \rho_1$, $\gamma_5 = \rho_3$, $\Sigma_k = \sigma_k$, $\beta C = -i\rho_2\sigma_2$ and $\xi^*_\lambda = \xi_\lambda$, in the Weyl representation. Then

$$\begin{aligned} \xi^{(c)}_{\lambda'}(\pi-\theta,\phi+\pi) &= e^{-i\phi\sigma_3/2}\,e^{-i\pi\sigma_3/2}\,e^{-i\pi\sigma_2/2}\,e^{i\theta\sigma_2/2}\,\beta C\xi_{\lambda'} &\\ &= e^{-i\phi\sigma_3/2}\,e^{-i\theta\sigma_2/2}\,(i\sigma_1)(-i\rho_2\sigma_2)\,\xi_{\lambda'}, &(5.135)\end{aligned}$$

and

$$\xi_\lambda(\theta,\phi)^\dagger(1-\gamma_5)\xi^{(c)}_{\lambda'}(\pi-\theta,\phi+\pi) = \xi^\dagger_\lambda(i\rho_2-\rho_1)\sigma_3\xi_{\lambda'} = -\lambda'\,\delta_{\lambda,\lambda'}, \tag{5.136}$$

which leads to (5.130).