

QUANTUM INVARIANTS I

Recall that U is the quantized universal enveloping algebra of \mathfrak{sl}_2 over $\mathbb{K} = k(q^{\frac{1}{2}})$, where k is an algebraically closed field of characteristic zero. In this text we show that the braided monoidal category Mod_U^{fd} of finite dimensional type 1 U -modules over \mathbb{K} is semisimple, and we describe the simple modules explicitly. We use the results to represent the skein category in Mod_U^{fd} .

Convention: In this syllabus all U -modules are considered over \mathbb{K} unless specified explicitly otherwise. In other words, an U -module will mean a \mathbb{K} -module M together with an \mathbb{K} -algebra homomorphism $\pi : U \rightarrow \text{End}_{\mathbb{K}}(M)$ (or equivalently, a \mathbb{K} -module M with a \mathbb{K} -bilinear left action $U \times M \rightarrow M$, given by $(u, m) \mapsto um (= \pi(u)(m))$).

1. HIGHEST WEIGHT MODULES OF U

Recall that U is the associative unital algebra over \mathbb{K} with generators $K^{\pm 1}, E, F$ and relations

$$\begin{aligned} KK^{-1} = 1 = K^{-1}K, & \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \\ KEK^{-1} = q^2E, & \quad KFK^{-1} = q^{-2}F. \end{aligned}$$

For a left U -module M and $\lambda \in \mathbb{K}^{\times}$ we write M_{λ} for the K -eigenspace with eigenvalue λ ,

$$M_{\lambda} = \{m \in M \mid Km = \lambda m\}.$$

Definition 1.1. *Let M be a left U -module and $\lambda \in \mathbb{K}^{\times}$.*

- (i) *An element $m \in M$ is called a highest weight vector of weight λ if $0 \neq m \in M_{\lambda}$ and $Em = 0$.*
- (ii) *M is called a highest weight module of highest weight λ if $M = Um$ with $m \in M$ a highest weight vector of weight λ .*

Remark 1.2. If M is a highest weight module of highest weight λ , then the spectrum (=set of eigenvalues) $\text{Spec}_M(K)$ of the action of K on M satisfies $\lambda \in \text{Spec}_M(K) \subseteq \lambda q^{2\mathbb{Z} \leq 0}$. Indeed, if $m \in M$ is the highest weight vector of highest weight λ , then M is spanned by $\{F^i m\}_{i \in \mathbb{Z}_{\geq 0}}$ (use here that $\{F^i E^j K^l\}_{i, j \in \mathbb{Z}_{\geq 0}, l \in \mathbb{Z}}$ is a \mathbb{K} -basis of U and the fact that $M = Um$) and $K(F^i m) = q^{-2i} \lambda F^i m$. In particular, the highest weight of a highest module is unique up to a multiplicative constant.

Let $\lambda \in \mathbb{K}^{\times}$ and let

$$M(\lambda) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{K} m_n(\lambda)$$

1

be an infinite dimensional \mathbb{K} -vector space with \mathbb{K} -linear basis $\{m_n(\lambda)\}_{n \in \mathbb{Z}_{\geq 0}}$. Set $m_{-1}(\lambda) := 0$ in $M(\lambda)$. Recall the q -number

$$(m)_{q^2} := \frac{q^{2m} - 1}{q^2 - 1} \in \mathbb{K}, \quad m \in \mathbb{Z}_{\geq 0}.$$

Proposition 1.3. (i) *The following formulas uniquely define a left U -module structure on $M(\lambda)$,*

$$(1.1) \quad \begin{aligned} Km_n(\lambda) &= \lambda q^{-2n} m_n(\lambda), \\ Em_n(\lambda) &= (n)_{q^2} \left(\frac{q^{2-2n} \lambda - \lambda^{-1}}{q - q^{-1}} \right) m_{n-1}(\lambda), \\ Fm_n(\lambda) &= m_{n+1}(\lambda). \end{aligned}$$

The resulting left U -module $M(\lambda)$ is a highest weight module of weight λ , with highest weight vector $m_0(\lambda)$.

(ii) *Any highest weight left U -module M of weight λ is a quotient of $M(\lambda)$.*

Proof. (i) Consider $I(\lambda) = UE + U(K - \lambda)$, the left ideal of U generated by E and $K - \lambda$. The set

$$\{F^n E^i K^j \mid n, i \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}\}$$

is a \mathbb{K} -linear basis of U by the Poincaré-Birkhoff-Witt theorem, hence the elements

$$(1.2) \quad F^n + I(\lambda), \quad n \geq 0$$

form a \mathbb{K} -linear basis of the quotient space $U/I(\lambda)$. We now identify the \mathbb{K} -vector space $M(\lambda)$ with $U/I(\lambda)$ via the \mathbb{K} -linear isomorphism $M(\lambda) \xrightarrow{\sim} U/I(\lambda)$ mapping the distinguished \mathbb{K} -basis element $m_n(\lambda)$ of $M(\lambda)$ to $F^n + I(\lambda) \in U/I(\lambda)$ for all $n \in \mathbb{Z}_{\geq 0}$. With this particular realization of $M(\lambda)$ as quotient of U by the left ideal $I(\lambda)$, it canonically inherits a left U -module structure. Computing its action explicitly on the basis elements (1.2) we obtain (1.1); the first equation follows from the relation $KF^n = q^{-2n} F^n K$ in U ,

$$Km_n(\lambda) = q^{-2n} F^n K + I(\lambda) = q^{-2n} \lambda (F^n + I(\lambda)) = q^{-2n} \lambda m_n(\lambda).$$

The second equation of (1.1) follows from

$$\begin{aligned} Em_n(\lambda) &= EF^n + I(\lambda) \\ &= F^n E + (n)_{q^2} F^{n-1} \left(\frac{q^{2-2n} K - K^{-1}}{q - q^{-1}} \right) + I(\lambda) \\ &= (n)_{q^2} \left(\frac{q^{2-2n} \lambda - \lambda^{-1}}{q - q^{-1}} \right) m_{n-1}(\lambda), \end{aligned}$$

with the obvious adjustments for $n = 0$. The third equation of (1.1) is trivial.

(ii) Let M be a highest weight U -module of highest weight λ , with highest weight vector m . Consider U as left U -module by left multiplication in U ($u \cdot u' := uu'$ for $u, u' \in U$). Then the map $U \rightarrow M$, defined by $u \mapsto um$, is a surjective U -module morphism. Since $m \in M$ is a highest weight vector of weight λ , it factors through the ideal $I(\lambda)$, hence it gives rise to a well defined surjective U -module morphism $M(\lambda) = U/I(\lambda) \rightarrow M$. \square

Definition 1.4. $M(\lambda)$ is called the Verma module of highest weight $\lambda \in \mathbb{K}^\times$.

A left U -module M has two trivial U -submodules, namely $\{0\}$ and M .

Definition 1.5. A left U -module M is simple if $M \neq \{0\}$ and if $\{0\}$ and M are the only U -submodules of M .

Under mild conditions we can now describe the finite dimensional simple left U -modules over \mathbb{K} as follows.

Proposition 1.6. Let $\lambda \in \mathbb{K}^\times$, $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$.

(i) $M(\lambda)$ is simple if $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$.

(ii) $M(\epsilon q^n)$ has a unique nontrivial U -submodule $N(\epsilon q^n)$, isomorphic to the Verma module $M(\epsilon q^{-2-n})$.

(iii) The left U -module $L(\epsilon, n) := M(\epsilon q^n)/N(\epsilon q^n)$ is a finite dimensional simple U -module of dimension $n + 1$. Furthermore, $L(\epsilon, n) \simeq L(\epsilon', n')$ if and only if $(\epsilon, n) = (\epsilon', n')$.

(iv) Let M be a finite dimensional, simple left U -module. If M has an eigenvector for the action of K , then $M \simeq L(\epsilon, n)$ for a unique $(\epsilon, n) \in \{\pm 1\} \times \mathbb{Z}_{\geq 0}$.

Proof. We start with some preliminary remarks before embarking on the proof of the four statements. Let $\lambda \in \mathbb{K}^\times$ and let $\{0\} \neq N \subseteq M(\lambda)$ be a U -submodule. Fix $0 \neq m \in N$ and write

$$m = \sum_{j \in J} c_j m_j(\lambda)$$

with $c_j \in \mathbb{K}^\times$ and J a nonempty finite subset of $\mathbb{Z}_{\geq 0}$. For $i \in J$ we then have

$$m_i(\lambda) = c_i^{-1} \prod_{j \in J \setminus i} \left(\frac{K - q^{-2j} \lambda}{q^{-2i} \lambda - q^{-2j} \lambda} \right) m \in N.$$

Furthermore, if $m_i(\lambda) \in N$ for a given $i \geq 1$, then acting by E we conclude from the previous proposition that $m_{i-1}(\lambda) \in N$ if $\lambda \neq \pm q^{i-1}$.

(i) Let $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$ and suppose that $\{0\} \neq N \subseteq M(\lambda)$ is a U -submodule. Let $i \in \mathbb{Z}_{\geq 0}$ such that $m_i(\lambda) \in N$. Since $\lambda \notin \pm q^{\mathbb{Z}_{\geq 0}}$, we conclude from the above remarks that the highest weight vector $m_0(\lambda)$ is also contained in N . Since $M(\lambda) = U m_0(\lambda)$, we conclude that $N = M(\lambda)$, hence $M(\lambda)$ is a simple U -module.

(ii) Suppose that $\lambda = \epsilon q^n$ ($\epsilon \in \{\pm 1\}$, $n \in \mathbb{Z}_{\geq 0}$). Then $E m_{n+1}(\epsilon q^n) = 0$ by the previous proposition, hence $N(\epsilon q^n) := \bigoplus_{j \geq n+1} \mathbb{K} m_j(\epsilon q^n)$ is a U -submodule of $M(\epsilon q^n)$. It is a highest weight module of highest weight ϵq^{-2-n} , with highest weight vector $m_{n+1}(\epsilon q^n)$. It is easily seen to be isomorphic to $M(\epsilon q^{-2-n})$ (hence it is simple by (i)).

Let $0 \neq N \subseteq M(\epsilon q^n)$ be an U -submodule. Let i be the smallest nonnegative integer such that $m_i(\epsilon q^n) \in N$. We conclude from the remarks so far that there are two possibilities: $i = 0$ (in which case $N = M(\epsilon q^n)$) or $i = n+1$ (in which case $N = N(\epsilon q^n)$). This completes the proof of (ii).

(iii) Let $\pi : M(\epsilon q^n) \rightarrow L(\epsilon, n)$ be the canonical map. It is a surjective U -module morphism. If $M \subseteq L(\epsilon, n)$ is an U -submodule, then $N(\epsilon q^n) \subseteq \pi^{-1}(M) \subseteq M(\epsilon q^n)$ is an

U -submodule of $M(\epsilon q^n)$, hence $\pi^{-1}(M)$ equals $N(\epsilon q^n)$ or $M(\epsilon q^n)$ by the observation in the previous paragraph. Thus $M = \{0\}$ or $M = L(\epsilon, n)$, hence $L(\epsilon, n)$ is a simple U -module.

Since $N(\epsilon q^n) = \bigoplus_{j \geq n+1} \mathbb{K}m_j(\epsilon q^n)$ it is clear that $\{m_i(\epsilon q^n) + N(\epsilon q^n)\}_{i=0}^n$ is a \mathbb{K} -linear basis of $L(\epsilon, n)$, hence $\dim_{\mathbb{K}}(L(\epsilon, n)) = n + 1$.

If $L(\epsilon, n) \simeq L(\epsilon', n')$ as U -modules then the corresponding sets of eigenvalues for the actions of K on $L(\epsilon, n)$ and $L(\epsilon', n')$ should be the same. This forces $(\epsilon, n) = (\epsilon', n')$.

(iv) Let M be a finite dimensional, simple left U -module with $M_\lambda \neq \{0\}$ for some $\lambda \in \mathbb{K}^\times$. The algebraic sum $\sum_{s=0}^{\infty} M_{q^{2s}\lambda}$ in M is direct, since the summands are eigenspaces of the action of K on M for different eigenvalues. But M is finite dimensional over \mathbb{K} by assumption, hence $M_{q^{2s}\lambda} \neq \{0\}$ for only finitely many $s \in \mathbb{Z}_{\geq 0}$. Since $E^r M_\lambda \subseteq M_{q^{2r}\lambda}$ there exists a $m' \in M_\lambda$ and $r \in \mathbb{Z}_{\geq 0}$ such that $m := E^r m' \in M_{q^{2r}\lambda}$ is nonzero while $Em = E^{r+1} m' = 0$. Set $\mu = q^{2r}\lambda$. Since M is simple, we have $M = Um$, hence M is a highest weight U -module of highest weight μ , with highest weight vector m . Let $\phi : M(\mu) \rightarrow M$ be the corresponding surjective U -module morphism, characterized by $\phi(m_0(\mu)) = m$. Then $\text{Ker}(\phi) \subset M(\mu)$ is a nontrivial left U -submodule of $M(\mu)$ and $M(\mu)/\text{Ker}(\phi) \simeq M$. By the previous observations, $\mu = \epsilon q^n$ for some $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$, and $\text{Ker}(\phi) = N(\epsilon q^n)$. Hence $M \simeq L(\epsilon, n)$. \square

Corollary 1.7. For $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$ choose

$$\bar{m}_j(\epsilon q^n) := m_j(\epsilon q^n) + N(\epsilon q^n), \quad (j = 0, \dots, n)$$

as \mathbb{K} -linear basis of $L(\epsilon, n) = M(\epsilon q^n)/N(\epsilon q^n)$. Furthermore set $\bar{m}_{-1}(\epsilon q^n) = 0 = \bar{m}_{n+1}(\epsilon q^n)$. The U -action on $L(\epsilon, n)$ is explicitly given by

$$\begin{aligned} K\bar{m}_j(\epsilon q^n) &= \epsilon q^{n-2j} \bar{m}_j(\epsilon q^n), \\ E\bar{m}_j(\epsilon q^n) &= \epsilon q^{1-n} (j)_{q^2} (n+1-j)_{q^2} \bar{m}_{j-1}(\epsilon q^n), \\ F\bar{m}_j(\epsilon q^n) &= \bar{m}_{j+1}(\epsilon q^n) \end{aligned}$$

for $j = 0, \dots, n$.

Proof. Direct computation. \square

We now investigate the morphisms between U -modules.

Lemma 1.8. (i) Suppose M and N are non-isomorphic simple left U -modules. Then

$$\text{Hom}_U(M, N) = \{0\}.$$

(ii) If M is a highest weight module then $\text{End}_U(M) = \text{span}_{\mathbb{K}}\{\text{Id}_M\}$.

Proof. (i) Let $\phi \in \text{Hom}_U(M, N)$. The kernel of ϕ is an U -submodule of M , hence it is $\{0\}$ or M . If ϕ is injective, then the image of ϕ is a nonzero U -submodule of N , hence it is equal to N . In this case ϕ is an isomorphism, contradicting the fact that $M \not\cong N$. Thus the kernel of ϕ is M , which means that $\phi = 0$.

(ii) Let $\lambda \in \mathbb{K}^\times$ be the highest weight of M . Choose a corresponding highest weight vector $m \in M$ of highest weight λ . Then $M_\lambda = \text{span}_{\mathbb{K}}\{m\}$ since M is a quotient of $M(\lambda)$, and $M(\lambda)_\lambda$ is one-dimensional.

Fix now $\phi \in \text{End}_U(M)$. By the previous paragraph we have $\phi(m) = \mu m$ for some $\mu \in \mathbb{K}$. Let now $m' \in M$. Since M is a highest weight module with highest weight vector m , there exists an $u \in U$ so that $m' = um$. Then $\phi(m') = \phi(um) = u\phi(m) = \mu m'$. We conclude that $\phi = \mu \text{Id}_M$. \square

Exercise 1.9. Recall that the linear dual $V^* = \text{Hom}_{\mathbb{K}}(V, \mathbb{K})$ of a left U -module V is a left U -module by

$$(X\phi)(v) := \phi(S(X)v), \quad \phi \in V^*, v \in V, X \in U.$$

Show that $L(\epsilon, n)^* \simeq L(\epsilon, n)$ as U -modules for all $\epsilon \in \{\pm 1\}$ and $n \in \mathbb{Z}_{\geq 0}$.

2. THE CATEGORY Mod_U^{fd} AGAIN

Recall that a left U -module M is of type 1 if

$$M = \bigoplus_{n \in \mathbb{Z}} M_{q^n}.$$

We defined before the braided monoidal category Mod_U^{fd} consisting of the finite dimensional type 1 left U -modules. From the last section we immediately get the following explicit description of the simple modules in Mod_U^{fd} .

Corollary 2.1. The simple U -modules in Mod_U^{fd} are the $L(n) := L(+1, n)$ ($n \in \mathbb{Z}_{\geq 0}$), up to isomorphism.

Exercise 2.2. Let M be a finite dimensional type 1 U -module and consider the subspace

$$M^e = \{m \in M \mid Em = 0\}.$$

Show that

$$M^e = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M_{q^n}^e$$

with $M_{q^n}^e = M^e \cap M_{q^n}$ the space of highest weight vectors in M of highest weight q^n .

We now proceed to show that a finite dimensional type 1 U -module M is semisimple, i.e. that M is isomorphic to a direct sum of the simple modules $L(n)$ ($n \geq 0$).

Theorem 2.3. Let $M \in \text{Ob}(\text{Mod}_U^{fd})$. Then

$$M \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} L(n)^{\oplus k_n}$$

as U -modules, with $k_n = \dim_{\mathbb{K}}(M_{q^n}^e)$.

We give some preparatory results before embarking on the proof of the theorem.

Lemma 2.4. Let $M \in \text{Ob}(\text{Mod}_U^{fd})$ and let $N \subseteq M$ be a U -submodule. Then $N, M/N \in \text{Ob}(\text{Mod}_U^{fd})$.

Proof. Only the statement that N is of type 1 requires proof. Clearly

$$\bigoplus_{n \in \mathbb{Z}} N \cap M_{q^n} \subseteq N \subset M.$$

We have to show that the first inclusion is an equality. Let $m \in N$ and write $m = \sum_{j \in J} m_j$ with $J \subset \mathbb{Z}$ finite and $m_j \in M_{q^j}$. Then

$$m_n = \left(\prod_{j \in J \setminus \{n\}} \frac{K - q^j}{q^n - q^j} \right) m \in N$$

for all $n \in J$, hence $m \in \bigoplus_{n \in \mathbb{Z}} N \cap M_{q^n}$. \square

The following basic observation will be of use.

Lemma 2.5. *Let $M \in \text{Ob}(\text{Mod}_U^{fd})$. There exists a finite sequence of finite dimensional type 1 U -submodules*

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{r-1} \subset M_r = M$$

such that, for $1 \leq j \leq r$, the quotient U -module M_j/M_{j-1} is simple (hence isomorphic to $L(n)$ for some $n \in \mathbb{Z}_{\geq 0}$). Such a sequence is called a composition series for M of length r .

Proof. Fix a nonzero module $M \in \text{Ob}(\text{Mod}_U^{fd})$. Using induction to the \mathbb{K} -dimension of M , it suffices to show that there exists a U -submodule $N \subsetneq M$ such that M/N is simple.

If M is simple, then we are done ($N = \{0\}$ does the job). If M is not simple, then there exists an U -submodule $\{0\} \neq L \subsetneq M$. The quotient U -module $\overline{M} := M/L$ is of type 1 and of strictly smaller dimension than M . By the induction hypothesis there exists an U -submodule $\overline{N} \subset \overline{M}$ such that $\overline{M}/\overline{N}$ is simple.

Consider the canonical surjective U -module morphism $\pi : M \rightarrow \overline{M}$. Then $N := \pi^{-1}(\overline{N}) \subset M$ is a U -submodule, hence N is of type 1, and $M/N \simeq \overline{M}/\overline{N}$ is simple. \square

A composition series does not have to be unique, but the subsequent simple quotients are unique up to reshuffling (this is the Jordan-Hölder theorem):

Lemma 2.6. *Let $M \in \text{Ob}(\text{Mod}_U^{fd})$ and suppose that*

$$(2.1) \quad \begin{aligned} \{0\} &= M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M, \\ \{0\} &= M'_0 \subset M'_1 \subset \cdots \subset M'_{s-1} \subset M'_s = M \end{aligned}$$

are two composition series of M . Then $r = s$ and for some permutation σ of $\{1, \dots, r\}$ we have

$$(2.2) \quad M_j/M_{j-1} \simeq M'_{\sigma(j)}/M'_{\sigma(j)-1}, \quad j = 1, \dots, r.$$

Proof. We call two composition series (2.1) of M equivalent if $r = s$ and if (2.2) holds true for some permutation σ of $\{1, \dots, r\}$ (this is indeed an equivalence relation). We are thus required to show that all composition series of M are equivalent.

We prove it by induction to the minimum $\min(r, s)$ of the lengths of the two composition series. If it is zero (respectively one) then $M = \{0\}$ (respectively M is simple), so there is nothing to prove. Let (2.1) be two composition series of M with $r, s \geq 2$ and suppose that the lemma is correct for composition series with one of the two composition series of length $< \min(r, s)$. Without loss of generality we may assume that $r \leq s$.

Case (i): Suppose that $M_{r-1} = M'_{s-1}$. We will denote this U -module by N . Then

$$\begin{aligned} \{0\} &= M_0 \subset M_1 \subset \cdots \subset M_{r-2} \subset M_{r-1} = N, \\ \{0\} &= M'_0 \subset M'_1 \subset \cdots \subset M'_{s-2} \subset M'_{s-1} = N \end{aligned}$$

are composition series for the U -submodule N of M . By the induction hypothesis, they are equivalent, in particular $r = s$. Since $M/M_{r-1} = M/N = M/M'_{r-1}$, the composition series (2.1) for M are equivalent.

Case (ii): Suppose $M_{r-1} \neq M'_{s-1}$. Set $N = M_{r-1} + M'_{s-1}$. Then $M_{r-1} \subsetneq N \subset M$ is a U -submodule. The nonzero U -submodule N/M_{r-1} of M/M_{r-1} equals M/M_{r-1} since M/M_{r-1} is simple, hence $N = M$. We thus obtain isomorphisms

$$\begin{aligned} M/M_{r-1} &= (M_{r-1} + M'_{s-1})/M_{r-1} \simeq M'_{s-1}/(M_{r-1} \cap M'_{s-1}), \\ M/M'_{s-1} &= (M_{r-1} + M'_{s-1})/M'_{s-1} \simeq M_{r-1}/(M_{r-1} \cap M'_{s-1}) \end{aligned}$$

of U -modules. We choose now a composition series

$$\{0\} = K_0 \subset \cdots \subset K_{t-1} \subset K_t = M_{r-1} \cap M'_{s-1}$$

for $M_{r-1} \cap M'_{s-1}$, which can be complemented to form a composition series for M in two ways,

$$(2.3) \quad \begin{aligned} \{0\} &= K_0 \subset \cdots \subset K_{t-1} \subset K_t = M_{r-1} \cap M'_{s-1} \subset M_{r-1} \subset M, \\ \{0\} &= K_0 \subset \cdots \subset K_{t-1} \subset K_t = M_{r-1} \cap M'_{s-1} \subset M'_{s-1} \subset M. \end{aligned}$$

Clearly the two composition series (2.3) for M are equivalent.

The first composition series of (2.1), respectively the first composition series of (2.3), contains a composition series for M_{r-1} of length $r - 1$, respectively of length $t + 1$. By the induction hypothesis, these two composition series for M_{r-1} are equivalent. In particular, $t = r - 2$. It follows that the first composition series for M of (2.1) is equivalent to the second composition series for M of (2.3).

The second composition series of (2.3), respectively the second composition series of (2.1), contains a composition series for M'_{s-1} of length $t + 1 = r - 1$, respectively of length $s - 1$. Again by the induction hypothesis, these two composition series for M'_{s-1} are equivalent, in particular $r = s$. Consequently the second composition series for M of (2.3) is equivalent to the second composition series for M of (2.1). By the transitivity of the equivalence relation, we conclude now that the two composition series of (2.1) for M are equivalent. \square

We can now prove the theorem in a special case.

Lemma 2.7. *Let $n \in \mathbb{Z}_{\geq 0}$. Suppose that $M \in \text{Ob}(\text{Mod}_U^{fd})$ has a composition series*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

with $M_j/M_{j-1} \simeq L(n)$ for all $j = 1 \dots, r$. Then M is semisimple, i.e. $M \simeq L(n)^{\oplus r}$ as U -modules.

Proof. Using that $L(n)_{q^n}$ is one-dimensional and that $L(n)_{q^{n+2}} = \{0\}$, we claim that $M_{j,q^{n+2}} = \{0\}$, $M_{j,q^n} = M_{j,q^n}^e$ (with the notations from Exercise 2.2) and $\dim_{\mathbb{K}}(M_{j,q^n}^e) = j$. We proceed by induction to j . It is clearly true for $j = 1$. Suppose it is true for $j - 1$. Let

$$\pi_j : M_j \rightarrow M_j/M_{j-1} \simeq L(n)$$

be the canonical surjective morphism of U -modules. If $m \in M_{j,q^{n+2}}$ then $m \in \text{Ker}(\pi_j)$, hence $m \in M_{j-1,q^{n+2}} = \{0\}$. Thus $M_{j,q^{n+2}} = \{0\}$.

Since M_j is of type 1, there exists a nonzero $m \in M_{j,q^n}$ mapping under π_j to a highest weight vector of $L(n)$. In particular $m \notin M_{j-1}$, hence $\mathbb{K}m + M_{j-1,q^n} \subseteq M_{j,q^n}$ is a direct sum. Let $v \in M_{j,q^n}$, then for some $c \in \mathbb{K}$ we have $v - cm \in M_{j,q^n} \cap \text{Ker}(\pi_j) = M_{j-1,q^n}$. We conclude that

$$\mathbb{K}m \oplus M_{j-1,q^n} = M_{j,q^n},$$

hence $\dim_{\mathbb{K}}(M_{j,q^n}) = j$. Since $M_{j-1,q^n} = M_{j-1,q^n}^e$ by the induction hypothesis, it remains to show that $Em = 0$, but this is clear since $Em \in M_{j,q^{n+2}} = \{0\}$.

Fix now a \mathbb{K} -basis $\{v_1, \dots, v_r\}$ of $M_{q^n}^e$. For $j = 1, \dots, r$ the U -submodule $V_j := Uv_j \subseteq M$ generated by $v_j \in M_{q^n}^e$ is a finite dimensional highest weight U -module of weight q^n , with highest weight vector v_j . Thus $V_j \simeq L(n)$ as U -modules. Write V for the U -submodule $\sum_{j=1}^r V_j$ of M . By construction, $V_{q^n} = \bigoplus_{i=1}^r \mathbb{K}v_i = M_{q^n}$. Let $\pi : M \rightarrow M/V$ be the canonical surjective U -module morphism. Since M is of type 1, $\pi(M_{q^n}) = (M/V)_{q^n}$. Then $\text{Ker}(\pi) \cap M_{q^n} = V \cap M_{q^n} = V_{q^n}$, hence $(M/V)_{q^n} \simeq M_{q^n}/V_{q^n} = \{0\}$. We claim that $M/V = \{0\}$.

Suppose that $M/V \neq \{0\}$. Then we can construct composition series

$$\begin{aligned} \{0\} &= N_0 \subset N_1 \subset \dots \subset N_{s-1} \subset N_s = V, \\ \{0\} &= \overline{M}_0 \subset \dots \subset \overline{M}_{t-1} \subset \overline{M}_t = M/V \end{aligned}$$

with $t \geq 1$. Then the above two composition series can be combined to give a composition series for M ,

$$\{0\} = N_0 \subset \dots \subset N_s = V = \pi^{-1}(\overline{M}_0) \subset \pi^{-1}(\overline{M}_1) \subset \dots \subset \pi^{-1}(\overline{M}_t) = M.$$

Using the previous lemma, we conclude that $\pi^{-1}(\overline{M}_1)/V$ is a simple U -submodule of M/V , isomorphism to $L(n)$. But this is impossible since $(M/V)_{q^n} = \{0\}$. Thus we conclude that $M = V = \sum_{j=1}^r V_j$. Observe that $\dim_{\mathbb{K}}(M) = (n+1)^r$ equals $\sum_{j=1}^r \dim_{\mathbb{K}}(V_j)$ since $V_j \simeq L(n)$, hence the sum $M = \sum_{j=1}^r V_j$ is direct. We conclude that

$$M \simeq \bigoplus_{j=1}^r V_j \simeq L(n)^{\otimes r}$$

as U -modules, as desired. \square

Denote $Z(U) = \{X \in U \mid XY = YX \quad \forall Y \in U\}$ for the center of U . Let M be a left U -module. The map $m \mapsto Zm$ defines an endomorphism $Z_M \in \text{End}_U(M)$ (since Z

is central). This gives a natural transformation $Z : \text{Id}_{\mathcal{C}} \rightarrow \text{Id}_{\mathcal{C}}$ for the category \mathcal{C} of left U -modules since

$$f \circ Z_M = Z_N \circ f, \quad f \in \text{Hom}_U(M, N).$$

Note that Z restricts to a natural transformation for the identity functors of the full subcategory Mod_U^{fd} of \mathcal{C} .

Write

$$(2.4) \quad C = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} \in U.$$

Proposition 2.8. (i) $C \in Z(U)$.

(ii) If M is highest weight U -module of highest weight $\lambda \in \mathbb{K}^\times$, then

$$C_M = \frac{q\lambda + q^{-1}\lambda^{-1}}{(q - q^{-1})^2} \text{Id}_M.$$

In particular,

$$C_{L(n)} = \frac{q^{1+n} + q^{-1-n}}{(q - q^{-1})^2} \text{Id}_{L(n)}$$

for $n \in \mathbb{Z}_{\geq 0}$.

Proof. (i) We have to show that C commutes with the algebraic generators $K^{\pm 1}$, E and F of U . This is verified by direct computations. We show here the computation for E ,

$$\begin{aligned} EC &= EFE + \frac{qEK + q^{-1}EK^{-1}}{(q - q^{-1})^2} \\ &= \left(FE + \frac{K - K^{-1}}{q - q^{-1}} \right) E + \left(\frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} \right) E \\ &= \left(FE + \frac{(q - q^{-1})(K - K^{-1}) + q^{-1}K + qK^{-1}}{(q - q^{-1})^2} \right) E \\ &= \left(FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} \right) E = CE. \end{aligned}$$

(ii) Let M be a highest weight U -module of highest weight $\lambda \in \mathbb{K}^\times$. Then $C_M = c_M \text{Id}_M$ for some $c_M \in \mathbb{K}$ by Lemma 1.8. To compute the constant c_M , let $m \in M$ be a highest weight vector of M , so that $Km = \lambda m$ and $Em = 0$. Then

$$C_M(m) = C m = \frac{q\lambda + q^{-1}\lambda^{-1}}{(q - q^{-1})^2} m$$

by the explicit expression for the central element C . Hence $c_M = \frac{q\lambda + q^{-1}\lambda^{-1}}{(q - q^{-1})^2}$. \square

Exercise 2.9. Let $\lambda, \mu \in \mathbb{K}^\times$. Show that

$$\frac{q\lambda + q^{-1}\lambda^{-1}}{(q - q^{-1})^2} = \frac{q\mu + q^{-1}\mu^{-1}}{(q - q^{-1})^2}$$

if and only if $\lambda = \mu$ or $\lambda = q^{-2}\mu^{-1}$.

The exercise in particular shows that C acts by different scalars on the simple U -modules $L(n)$ ($n \in \mathbb{Z}_{\geq 0}$) in Mod_U^{fd} .

To prove the theorem, we investigate C_M for an arbitrary finite dimensional type 1 U -module M . We begin with the following basic observation.

Lemma 2.10. *Let M be a finite dimensional type 1 U -module.*

We have

$$M = \bigoplus_{\mu \in \mathbb{K}} M_{(\mu)}, \quad M_{(\mu)} = \{m \in M \mid (C_M - \mu \text{Id}_M)^n m = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

Proof. We write

$$c_n = \frac{q^{1+n} + q^{-1-n}}{(q - q^{-1})^2} \in \mathbb{K}, \quad n \in \mathbb{Z}_{\geq 0}$$

for the scalar such that $C_{L(n)} = c_n \text{Id}_{L(n)}$. Choose a composition series $\{0\} = M_0 \subset \cdots \subset M_{r-1} \subset M_r = M$ of M . Then $M_j/M_{j-1} \simeq L(n_j)$ for some $n_j \in \mathbb{Z}_{\geq 0}$ ($1 \leq j \leq r$). Choosing a \mathbb{K} -basis of M compatible with the composition series (so first fixing a basis of M_0 , extending it to a basis of M_1 and continuing this procedure until we have a \mathbb{K} -basis of M), we see that the characteristic polynomial of $C_M \in \text{End}_{\mathbb{K}}(M)$ is of the form

$$\det(C_M - \lambda \text{Id}_M) = \prod_{j=1}^r (c_{n_j} - \lambda)^{s_j} \in \mathbb{K}[\lambda]$$

with $s_j = \dim_{\mathbb{K}}(L(n_j))$. The characteristic function of $C_M \in \text{End}_{\mathbb{K}}(M)$ thus decomposes as a product of linear factors over \mathbb{K} . A well known result from linear algebra now says that there exists a \mathbb{K} -basis of M such that the associated matrix of C_M is in Jordan normal form (see, e.g., [6, XIV, §2]). For a fixed Jordan block J of size $t \times t$ with eigenvalue $\mu \in \mathbb{K}$ on its diagonal we have $(J - \mu \text{I})^t = 0$, with I (resp. 0) the $t \times t$ unit matrix (resp zero matrix). This proves the lemma. \square

We can now easily complete the proof of Theorem 2.3. Let M be a finite dimensional type 1 U -module, and write

$$M = \bigoplus_{\mu \in \mathbb{K}} M_{(\mu)}$$

for its generalized eigenspace decomposition under the action of C . Since $C \in Z(U)$ (thus $C_M \in \text{End}_U(M)$), the generalized eigenspaces $M_{(\mu)}$ are U -submodules of M . It thus remains to show that the finite dimensional type 1 U -submodules $M_{(\mu)}$ is semisimple.

Choose a composition series

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_r = M_{(\mu)}$$

for $M_{(\mu)}$. For $1 \leq t \leq r$ we thus have $N_t/N_{t-1} \simeq L(n_t)$ for some $n_t \in \mathbb{Z}_{\geq 0}$. The only eigenvalue of $C_{M_{(\mu)}}$ is μ , hence

$$\mu = \frac{q^{1+n_t} + q^{-1-n_t}}{(q - q^{-1})^2}$$

for $t = 1, \dots, r$. This implies that $n_t = n$ is independent of t . Lemma 2.7 now shows that $M_{(\mu)}$ is semisimple, which completes the proof of Theorem 2.3.

Exercise 2.11. *Let M be a finite dimensional type 1 U -module. Show that*

$$\dim_{\mathbb{K}}(\text{End}_U(M)) = \sum_{n=0}^{\infty} \dim_{\mathbb{K}}(M_{q^n}^e)^2.$$

3. A REPRESENTATION OF THE SKEIN CATEGORY

In this section we give a representation of the skein category $\mathcal{S}(q^{-\frac{1}{2}})$ over \mathbb{K} . Recall that $\mathcal{S}(q^{-\frac{1}{2}})$ is the strict tensor category with objects the nonnegative integers $\mathbb{Z}_{\geq 0}$, and with morphisms $\text{Hom}_{\mathcal{S}}(k, l)$ the linear skein $E_{kl}(q^{-\frac{1}{2}})$ over \mathbb{K} .

We call a strict tensor category \mathcal{M} linear over \mathbb{K} if the set $\text{Hom}_{\mathcal{M}}(U, V)$ of morphisms from U to V are vector spaces over \mathbb{K} for all objects U and V of \mathcal{M} , and if the tensor product and composition maps

$$\begin{aligned} \circ : \text{Hom}_{\mathcal{M}}(V, W) \times \text{Hom}_{\mathcal{M}}(U, V) &\rightarrow \text{Hom}_{\mathcal{M}}(U, W), \\ \otimes : \text{Hom}_{\mathcal{M}}(U, V) \times \text{Hom}_{\mathcal{M}}(U', V') &\rightarrow \text{Hom}_{\mathcal{M}}(U \otimes U', V \otimes V') \end{aligned}$$

are \mathbb{K} -bilinear for all objects U, V, U', V', W of \mathcal{M} . The skein category $\mathcal{S}(q^{-\frac{1}{2}})$ is an example of a \mathbb{K} -linear strict tensor category.

For \mathbb{K} -linear strict tensor categories the constructions of last week on generators and relations can be adjusted to incorporate the \mathbb{K} -linearity of the category in a natural way. As a consequence we get the following result.

Proposition 3.1. *Let \mathcal{M} be a \mathbb{K} -linear strict tensor category with unit object \mathbb{I} . Let V be an object of \mathcal{M} and suppose that $\phi \in \text{Hom}_{\mathcal{M}}(V \otimes V, \mathbb{I})$ and $\psi \in \text{Hom}_{\mathcal{M}}(\mathbb{I}, V \otimes V)$ are morphisms satisfying*

$$\begin{aligned} \phi \circ \psi &= -(q + q^{-1})\text{id}_{\mathbb{I}}, \\ (\phi \otimes \text{id}_V) \circ (\text{id}_V \otimes \psi) &= \text{id}_V = (\text{id}_V \otimes \phi) \circ (\psi \otimes \text{id}_V). \end{aligned}$$

Then there exists a unique strict tensor functor $F : \mathcal{S}(q^{-\frac{1}{2}}) \rightarrow \mathcal{M}$ satisfying

- (1) $F(1) = V$,
- (2) $F : E_{kl}(q^{-\frac{1}{2}}) \rightarrow \text{Hom}_{\mathcal{M}}(V^{\otimes k}, V^{\otimes l})$ is \mathbb{K} -linear,
- (3) $F(\cup) = \psi$ and $F(\cap) = \phi$.

Proof. See appendix 4 for a detailed proof. □

Composing the \mathbb{K} -linear strict tensor functor $F : \mathcal{S}(q^{-\frac{1}{2}}) \rightarrow \mathcal{M}$ from Proposition 3.1 with the strict tensor functor

$$G : \mathcal{T} \rightarrow \mathcal{S}(q^{-\frac{1}{2}})$$

from Section 4.4 in the syllabus of week 15, we obtain a strict tensor functor

$$P = F \circ G : \mathcal{T} \rightarrow \mathcal{M}.$$

Given an isotopy class $[T]$ of a (k, l) -ribbon tangle T , we thus obtain an *isotopy invariant* $P([T]) \in \text{Hom}_{\mathcal{M}}(V^{\otimes k}, V^{\otimes l})$ of T . The invariant $P([T])$ for a (k, l) -ribbon tangle T can be computed as follows. Represent T by a generic (k, l) -tangle diagram D_T (with the proper number of curls in the diagram to account for the framing of T). Now view D_T as element in $E_{kl}(q^{-\frac{1}{2}})$ and smoothen the crossings by the skein relation in $E_{kl}(q^{-\frac{1}{2}})$. In the resulting \mathbb{K} -linear sum of (k, l) -tangle diagrams without crossings we get rid of a closed loop on the cost of a factor $-(q + q^{-1})$. We replace minima (respectively maxima) by the morphism ψ (respectively ϕ) in \mathcal{M} and we replace id_r by $\text{id}_{V^{\otimes r}}$, while respecting the tensor product and composition rules in both the categories $\mathcal{S}(q^{-\frac{1}{2}})$ and \mathcal{M} . The resulting morphism in \mathcal{M} is our desired invariant of T .

For ribbon links we immediately re-obtain the Kauffman bracket,

Corollary 3.2. *In the above set-up we have*

$$P([T]) = -(q + q^{-1}) \langle T \rangle_{q^{-\frac{1}{2}}} \text{id}_{\mathbb{1}}$$

for a ribbon link T .

We now are going to construct an explicit \mathbb{K} -linear strict tensor functor $F_{sk} : \mathcal{S}(q^{-\frac{1}{2}}) \rightarrow \widetilde{\text{Mod}}_U^{fd}$, where $\widetilde{\text{Mod}}_U^{fd}$ is the strict monoidal category associated to Mod_U^{fd} by Mac Lane's coherence theorem. In passing from Mod_U^{fd} to $\widetilde{\text{Mod}}_U^{fd}$, we essentially get rid of the associativity constraint by fixing a particular paranthesis order in multiple tensor products. By definition the objects in $\widetilde{\text{Mod}}_U^{fd}$ are the finite sequences $S = (M_1, M_2, \dots, M_k)$ of objects M_j in Mod_U^{fd} ($k \in \mathbb{Z}_{\geq 0}$). For such a sequence S we define

$$F(S) = ((\dots (M_1 \otimes M_2) \otimes \dots) \otimes M_{k-1}) \otimes M_k \in \text{Ob}(\text{Mod}_U^{fd}),$$

which should be read as the unit object $L(0) = \mathbb{K}$ if $k = 0$. The morphisms $S \rightarrow T$ for two sequences S and T in $\widetilde{\text{Mod}}_U^{fd}$ are by definition the morphisms $F(S) \rightarrow F(T)$ in Mod_U^{fd} . Clearly $\widetilde{\text{Mod}}_U^{fd}$ becomes a category in this way, and the map F , defined as the identity on morphisms, defines an equivalence $F : \text{Mod}_U^{fd} \rightarrow \widetilde{\text{Mod}}_U^{fd}$ of categories. The next step is to turn $\widetilde{\text{Mod}}_U^{fd}$ in a strict tensor category. On objects, the tensor product is concatenation with \emptyset serving as unit object,

$$S * S' = (M_1, \dots, M_k, M'_1, \dots, M'_l)$$

for $S = (M_1, \dots, M_k)$ and $S' = (M'_1, \dots, M'_l)$. On morphisms, the tensor product is more elaborate to construct: one uses the left/right unit constraint and the associativity constraint to define natural isomorphisms $\varphi(S, S') : F(S) \otimes F(S') \rightarrow F(S * S')$ (essentially setting the parantheses in the right order), which then allows one to define

$$f * f' = \varphi(T, T') \circ (f \otimes f') \circ \varphi(S, S')^{-1}$$

for morphisms $f : S \rightarrow T$ and $f' : S' \rightarrow T'$. Doing it carefully, we obtain a strict monoidal category $\widetilde{\text{Mod}}_U^{fd}$, and F can be extended to an equivalence of tensor categories (see [3, Section XI.5] for details). We abuse notations and write $M_1 \otimes \dots \otimes M_k$ for $S =$

$(M_1, \dots, M_k) \in \text{Ob}(\widetilde{\text{Mod}}_U^{fd})$ as well as for $F(S) = ((\cdots (M_1 \otimes M_2) \otimes \cdots \otimes M_{k-1}) \otimes M_k \in \text{Ob}(\text{Mod}_U^{fd})$.

Recall the two-dimensional vector representation V of U with basis v_+ and v_- and action defined by

$$Kv_{\pm} = q^{\pm 1}v_{\pm}, \quad Ev_+ = 0 = Fv_-, \quad Ev_- = v_+, \quad Fv_+ = v_-.$$

We have $V \simeq L(1)$ as U -modules, with the isomorphism given by the identification $v_+ \leftrightarrow \bar{m}_0(q)$ and $v_- \leftrightarrow \bar{m}_1(q)$ of the associated distinguished bases. The 4-dimensional type 1 U -module $V \otimes V$ thus is isomorphic to a direct sum of simple modules. To find out the simple components we look for highest weight vectors in $V \otimes V$. Observe that

$$\begin{aligned} (V \otimes V)_{q^2} &= \text{span}_{\mathbb{K}}\{v_+ \otimes v_+\}, \\ (V \otimes V)_1 &= \text{span}_{\mathbb{K}}\{v_+ \otimes v_+, v_- \otimes v_+\}, \\ (V \otimes V)_{q^{-2}} &= \text{span}_{\mathbb{K}}\{v_- \otimes v_-\}. \end{aligned}$$

It is easy to check that $w_2 = v_+ \otimes v_+$ (respectively $w_0 = v_+ \otimes v_- - qv_- \otimes v_+$) is a highest weight vector in $V \otimes V$ of highest weight q^2 (respectively 1). Thus $V_2 = Uw_2$ (respectively $V_0 = Uw_0$) is a U -submodule of $V \otimes V$, isomorphic to $L(2)$ (respectively $L(0)$). Thus $V_0 \cap V_2 = \{0\}$ and $V_0 + V_2 = V \otimes V$ (by a dimension count), hence $V \otimes V \simeq L(0) \oplus L(2)$ as U -modules.

A basis of V_2 is given by $\{w_2, Fw_2, F^2w_2\}$. Rescaling, we obtain the concrete \mathbb{K} -basis

$$\{u_0, u_1, u_2\} = \{v_+ \otimes v_+, v_+ \otimes v_- + q^{-1}v_- \otimes v_+, v_- \otimes v_-\}.$$

We now fix two morphisms

$$\phi \in \text{Hom}_U(V \otimes V, \mathbb{K}), \quad \psi \in \text{Hom}_U(\mathbb{K}, V \otimes V)$$

such that $\phi \circ \psi = -(q + q^{-1})\text{Id}_{\mathbb{K}}$. The two morphisms are determined up to a nonzero constant: other possible choices are $(\phi', \psi') = (c\phi, c^{-1}\psi)$ for $c \in \mathbb{K}^\times$. An explicit choice for the morphisms are

$$\psi(1) = -(q + q^{-1})(v_+ \otimes v_- - qv_- \otimes v_+),$$

and

$$\phi|_{V_2} \equiv 0, \quad \phi(v_+ \otimes v_- - qv_- \otimes v_+) = 1.$$

Corollary 3.3. *We have a \mathbb{K} -linear strict tensor functor $F_V^{sk} : \mathcal{S}(q^{-\frac{1}{2}}) \rightarrow \widetilde{\text{Mod}}_U^{fd}$ defined by $F_V^{sk}(1) = V$ and*

$$F_V^{sk}(\cup) = \psi, \quad F_V^{sk}(\cap) = \phi.$$

Proof. By Proposition 3.1 it suffices to show that

$$(3.1) \quad (\text{Id}_V \otimes \phi) \circ (\psi \otimes \text{Id}_V) = \text{Id}_V = (\phi \otimes \text{Id}_V) \circ (\text{Id}_V \otimes \psi),$$

which can be verified by a direct computation. \square

Exercise 3.4. (i) Prove (3.1).

(ii) Define $t = F_V^{sk}(\cup \circ \cap) = \psi \circ \phi \in \text{End}_U(V^{\otimes 2})$. Show that

$$\begin{aligned} t(v_+ \otimes v_+) &= 0 = t(v_- \otimes v_-), \\ t(v_+ \otimes v_-) &= -q^{-1}v_+ \otimes v_- + v_- \otimes v_+, \\ t(v_- \otimes v_+) &= v_+ \otimes v_- - qv_- \otimes v_+. \end{aligned}$$

Denote P_V for the associated strict tensor functor

$$P_V := F_V^{sk} \circ G : \mathcal{T} \rightarrow \widetilde{\text{Mod}}_U^{fd}.$$

Then the endomorphism $P_V(X^+) \in \text{End}_U(V^{\otimes 2})$ is a solution of the Yang-Baxter equation.

Proposition 3.5. We have $P_V(X^+) = c_{V,V}$ with respect to the braiding c of Mod_U^{fd} .

Proof. With the notations from Exercise 3.4 we have

$$P_V(X^+) = q^{-\frac{1}{2}}\text{id}_{V \otimes V} + q^{\frac{1}{2}}t$$

by the skein relation applied to $G(X^+) \in E_{1,1}(q^{-\frac{1}{2}})$. The endomorphism t is explicitly computed in Exercise 3.4. Comparing the resulting formulas for $P_V(X^+)$ with the explicit formula for $c_{V,V}$ constructed before, we see that $P_V(X^+) = c_{V,V}$. \square

At a later stage we shall construct for any $M \in \text{Ob}(\text{Mod}_U^{fd})$ a strict tensor functor $\mathcal{T}_{or} \rightarrow \widetilde{\text{Mod}}_U^{fd}$, with \mathcal{T}_{or} the category of oriented ribbon tangles, such that X^+ (with the proper orientation) is mapped to $c_{M,M}$. It is called the Reshetikhin-Turaev functor. Only for $M = V$ the restriction of the Reshetikhin-Turaev functor to a suitable full sub-category of \mathcal{T}_{or} factors through the skein category $\mathcal{S}(q^{-\frac{1}{2}})$.

Remark 3.6. Let \mathcal{C}_V be the full sub-category of $\widetilde{\text{Mod}}_U^{fd}$ with objects $V^{\otimes n}$ ($n \geq 0$), where $V^{\otimes 0}$ is the unit object \mathbb{K} . Then \mathcal{C}_V is canonically a \mathbb{K} -linear strict tensor category and the functor F_V^{sk} is an equivalence $F_V^{sk} : \mathcal{S}(q^{-\frac{1}{2}}) \xrightarrow{\sim} \mathcal{C}_V$ of \mathbb{K} -linear strict tensor categories.

4. APPENDIX A: GENERATORS AND RELATIONS FOR THE SKEIN CATEGORY

A strict tensor category \mathcal{M} is called \mathbb{K} -linear if for all objects $U, V \in \text{Ob}(\mathcal{M})$ the set $\text{Hom}_{\mathcal{M}}(U, V)$ of morphisms $U \rightarrow V$ is a \mathbb{K} -vector space and if the composition and tensor product maps

$$\begin{aligned} \circ : \text{Hom}_{\mathcal{M}}(V, W) \times \text{Hom}_{\mathcal{M}}(U, V) &\rightarrow \text{Hom}_{\mathcal{M}}(U, W), \\ \otimes : \text{Hom}_{\mathcal{M}}(U, V) \times \text{Hom}_{\mathcal{M}}(U', V') &\rightarrow \text{Hom}_{\mathcal{M}}(U \otimes U', V \otimes V') \end{aligned}$$

are \mathbb{K} -bilinear for all objects U, U', V, V', W in \mathcal{M} . The skein category $\mathcal{S}(q^{-\frac{1}{2}})$ is an example of a \mathbb{K} -linear strict tensor category.

Suppose we have a \mathbb{K} -linear strict monoidal category \mathcal{M} with the objects and morphisms forming sets (the latter requirement is usually referred to by saying that \mathcal{M} is small). Let G be a set of morphisms in \mathcal{M} such that the associated set \mathcal{A}_G of elementary morphisms with respect to G . We write $\mathcal{A}_G^{\mathbb{K}}(U, V)$ for the \mathbb{K} -linear span of $\mathcal{A}_G(U, V)$ in $\text{Hom}_{\mathcal{M}}(U, V)$.

We say that G generates \mathcal{M} if any morphism in \mathcal{M} can be written as \mathbb{K} -linear combination of admissible compositions of elementary morphisms.

For two objects U and V in \mathcal{M} we set $W_G^{\mathbb{K}}(U, V)$ for the \mathbb{K} -vector space with basis the set $w = w_1 * w_2 * \cdots * w_m$ of admissible words in the alphabet \mathcal{A}_G such that the source of w_m equals U and the target of w_1 equals V . We now extend this by defining \mathbb{K} -bilinear maps

$$* : W_G^{\mathbb{K}}(V, W) \times W_G^{\mathbb{K}}(U, V) \rightarrow W_G^{\mathbb{K}}(U, W)$$

by requiring it to be concatenation (with respect to $*$) on admissible words in the alphabet \mathcal{A}_G . The $*$ is associative in the usual sense.

Let $W_G^{\mathbb{K}}$ be the set of elements $w \in W_G^{\mathbb{K}}(U, V)$ ($U, V \in \text{Ob}(\mathcal{M})$). As before we have a canonical map $w \mapsto [w]$ from $W_G^{\mathbb{K}}$ to the set of morphisms of \mathcal{M} , which restricts to a \mathbb{K} -linear map $W_G^{\mathbb{K}}(U, V) \rightarrow \text{Hom}_{\mathcal{M}}(U, V)$ for all objects U and V in \mathcal{M} and which maps an admissible word $w = w_1 * w_2 * \cdots * w_m$ with the w_i in the alphabet \mathcal{A}_G to $[w] = w_1 \circ w_2 \circ \cdots \circ w_m$. Observe that $W_G^{\mathbb{K}}(U, V) \rightarrow \text{Hom}_{\mathcal{M}}(U, V)$ is surjective for all objects U and V if G generates \mathcal{M} . The equivalence relation $\sim_{\mathcal{M}}$ on $W_G^{\mathbb{K}}$ is now defined as $r \sim_{\mathcal{M}} r'$ iff $[r] = [r']$.

For U an object of \mathcal{M} there are well-defined maps $\text{id}_U \otimes \cdot : W_G^{\mathbb{K}} \rightarrow W_G^{\mathbb{K}}$ and $\cdot \otimes \text{id}_U : W_G^{\mathbb{K}} \rightarrow W_G^{\mathbb{K}}$, defined as follows: restricted to $W_G^{\mathbb{K}}(V, W)$, they are the \mathbb{K} -linear maps

$$\begin{aligned} \text{id}_U \otimes \cdot & : W_G^{\mathbb{K}}(V, W) \rightarrow W_G^{\mathbb{K}}(U \otimes V, U \otimes W), \\ \cdot \otimes \text{id}_U & : W_G^{\mathbb{K}}(V, W) \rightarrow W_G^{\mathbb{K}}(V \otimes U, W \otimes U) \end{aligned}$$

determined by

$$(4.1) \quad \begin{aligned} \text{id}_U \otimes (w_1 * \cdots * w_m) & = (\text{id}_U \otimes w_1) * \cdots * (\text{id}_U \otimes w_m), \\ (w_1 * \cdots * w_m) \otimes \text{id}_U & = (w_1 \otimes \text{id}_U) * \cdots * (w_m \otimes \text{id}_U) \end{aligned}$$

where $w = w_1 * \cdots * w_m$ is an admissible word in $W_G^{\mathbb{K}}(V, W)$ in the alphabet \mathcal{A}_G (note that the right hand sides in (4.1) are again admissible words in the alphabet \mathcal{A}_G). The above two maps on $W_G^{\mathbb{K}}$ commute.

With these maps at hand, the procedure to incorporate relations on $W_G^{\mathbb{K}}$ is as follows. We start with a set $R \subset W_G^{\mathbb{K}} \times W_G^{\mathbb{K}}$ of elements (r, r') having the same source and target: $r, r' \in W_G^{\mathbb{K}}(U, V)$ for some objects U and V . We enlarge R to a subset R_1 of $W_G^{\mathbb{K}} \times W_G^{\mathbb{K}}$ and from R_1 to \tilde{R} in the same way as in the syllabus of three weeks ago. Then we say that $w \sim_R w'$ for $w, w' \in W_G^{\mathbb{K}}$ iff $w, w' \in W_G^{\mathbb{K}}(U, V)$ for some objects U and V and if w' can be obtained from w by a finite sequence of steps of the following form:

- (1) Adding an element of the form $\lambda(w_1 * \text{id}_Y * w_2 - w_1 * w_2)$ with $\lambda \in \mathbb{K}$, $w_1 \in W_G^{\mathbb{K}}(Y, V)$ and $w_2 \in W_G^{\mathbb{K}}(U, Y)$ for some object Y .
- (2) Adding an element of the form $\lambda(w_1 * r' * w_2 - w_1 * r * w_2)$ for some $\lambda \in \mathbb{K}$, $w_1 \in W_G^{\mathbb{K}}(Z, V)$, $w_2 \in W_G^{\mathbb{K}}(U, Y)$ and $(r, r') \in \tilde{R}$ with r, r' having source Y and target Z .

Note: if $[r] = [r']$ for $(r, r') \in R$, then \sim_R is stronger than $\sim_{\mathcal{M}}$.

Definition 4.1. Let G be a set of morphisms of a \mathbb{K} -linear strict tensor category \mathcal{M} . Let $R \subset W_G^{\mathbb{K}} \times W_G^{\mathbb{K}}$ be a set of pairs (r, r') with r, r' having the same source and target. We say that (G, R) is a presentation of \mathcal{M} if G is a set of generators of \mathcal{M} and if \sim_R agrees with $\sim_{\mathcal{M}}$ on $W_G^{\mathbb{K}}$.

Analogous to Theorem 4.22 in the syllabus of three weeks ago one can now prove

Theorem 4.2. The skein category $\mathcal{S}(q^{-\frac{1}{2}})$ is generated as \mathbb{K} -linear strict tensor category by $\{\cup, \cap\}$ with the set of defining relations being given by

$$R = \{((\text{id}_1 \otimes \cap) * (\cup \otimes \text{id}_1), \text{id}_1), ((\cap \otimes \text{id}_1) * (\text{id}_1 \otimes \cup), \text{id}_1), (\cap * \cup, -(q + q^{-1})\text{id}_0)\}.$$

Proof. We will use the following fact about the $E_{kl}(q^{-\frac{1}{2}})$. Let D_{kl} be the \mathbb{K} -vector space with \mathbb{K} -basis (representatives of) equivalence classes of planar isotopy classes of smooth (no crossings and no loops) (k, l) -tangle diagrams. It is well known (see the proof of Thm. 3.17 in the syllabus of three weeks ago) that D_{kl} is zero-dimensional if $k + l$ is odd, and its dimension is the n th Catalan number if $k + l = 2n$. We define the (Kauffman) diagram category $\mathcal{D}(q^{-\frac{1}{2}})$ as follows. The objects are $\mathbb{Z}_{\geq 0}$, and $\text{Hom}_{\mathcal{D}(q^{-\frac{1}{2}})}(k, l) = D_{kl}$. The composition is putting smooth tangle diagrams on top of each other, and removing the loops on the cost of a factor $-(q + q^{-1})$ for each removed loop.

It is clear that $\mathcal{D}(q^{-\frac{1}{2}})$ is a \mathbb{K} -linear strict tensor category again: unit object is 0 and tensor product is putting tangle diagrams next to each other. Furthermore, we have a \mathbb{K} -linear strict tensor functor $F : \mathcal{D}(q^{-\frac{1}{2}}) \rightarrow \mathcal{S}(q^{-\frac{1}{2}})$ which is the identity on objects and which maps a smooth (k, l) -tangle diagram to its corresponding class in $E_{kl}(q^{-\frac{1}{2}}) = \text{Hom}_{\mathcal{S}(q^{-\frac{1}{2}})}(k, l)$. It is clear that $F : D_{kl} \rightarrow E_{kl}(q^{-\frac{1}{2}})$ is surjective for all $k, l \in \mathbb{Z}_{\geq 0}$. The injectivity of F amounts to proving that the representatives of the planar isotopy classes of the smooth (k, l) -tangle diagrams, viewed as elements in $E_{kl}(q^{-\frac{1}{2}})$, are \mathbb{K} -linearly independent. This requires an argument to which we come back in the syllabus of next week (compare also with Theorem 3.17 in the syllabus of three weeks ago). Thus F is an equivalence of \mathbb{K} -linear strict tensor categories.

Thus the theorem can be alternatively formulated that (G, R) is a presentation of the diagram category $\mathcal{D}(q^{-\frac{1}{2}})$ as a \mathbb{K} -linear strict tensor category. Taking a smooth (k, l) -tangle diagram, it is planar isotopic to a generic smooth (k, l) -tangle diagram, which can be written as $[w]$ for an admissible word w in the elementary morphisms \mathcal{A}_G . Thus G generates $\mathcal{D}(q^{-\frac{1}{2}})$.

First note that \sim_R is stronger than $\sim_{\mathcal{D}}$ for $\mathcal{D} = \mathcal{D}(q^{-\frac{1}{2}})$. We show that $w \sim_{\mathcal{D}} w'$ (so $[w] = [w']$) implies $w \sim_R w'$ for $w, w' \in W_G^{\mathbb{K}}$. So suppose that $w, w' \in W_G^{\mathbb{K}}(k, l)$ satisfy $w \sim_{\mathcal{D}} w'$, so $[w] = [w']$ in D_{kl} . Write $\{[u_i]\}_{i \in I}$ for a set of representatives of planar isotopy classes of smooth (k, l) -tangle diagrams (which is a basis of D_{kl}), then we can uniquely write $[w] = \sum_i \lambda_i [u_i] = [w']$ for unique $\lambda_i \in \mathbb{K}$. On the other hand, let $\{v_j\}_{j \in J}$ be the canonical \mathbb{K} -basis of $W_G^{\mathbb{K}}(k, l)$ consisting of admissible words in the elementary morphisms \mathcal{A}_G . There exists a unique surjective map $\sigma : J \rightarrow I$ and unique nonnegative integers $m_j \in \mathbb{Z}_{\geq 0}$ such that

$$[v_j] = d^{m_j} [u_{\sigma(j)}], \quad j \in J,$$

where $d = -q - q^{-1}$. We now write in $W_G^{\mathbb{K}}(k, l)$,

$$(4.2) \quad w = \sum_j \mu_j v_j, \quad w' = \sum_j \mu'_j v_j$$

for unique $\mu_j, \mu'_j \in \mathbb{K}$. The assumption is that

$$[w] = \sum_j \mu_j [v_j] = \sum_j \mu_j d^{m_j} [u_{\sigma(j)}]$$

in $\text{Hom}_{\mathcal{D}}(k, l)$ equals

$$[w'] = \sum_j \mu'_j [v_j] = \sum_j \mu'_j d^{m_j} [u_{\sigma(j)}],$$

hence, for all $i \in I$, we have

$$(4.3) \quad \sum_{j \in \sigma^{-1}(i)} \mu_j d^{m_j} = \sum_{j \in \sigma^{-1}(i)} \mu'_j d^{m_j}.$$

So for $w, w' \in W_G^{\mathbb{K}}(k, l)$ given by (4.2), such that the coefficients satisfy (4.3), we need to show that $w \sim_R w'$. For this it suffices to show that

$$\sum_{j \in \sigma^{-1}(i)} \mu_j v_j =: w_i \sim_R w'_i := \sum_{j \in \sigma^{-1}(i)} \mu'_j v_j$$

for all $i \in I$. For this it suffices to show that for $j, j' \in \sigma^{-1}(i)$ we have $d^{-m_j} v_j \sim_R d^{-m_{j'}} v_{j'}$. This is easy and goes by the same (but simpler) argument as for ribbon tangles, as in the proof of Theorem 4.22 in the syllabus of three weeks ago. \square

We call a strict tensor functor $F : \mathcal{M} \rightarrow \mathcal{N}$ between \mathbb{K} -linear strict tensor categories \mathcal{M} and \mathcal{N} \mathbb{K} -linear if $F : \text{Hom}_{\mathcal{M}}(U, V) \rightarrow \text{Hom}_{\mathcal{N}}(F(U), F(V))$ is \mathbb{K} -linear for $U, V \in \text{Ob}(\mathcal{M})$. We have now the direct analog of Theorem 4.24 in the syllabus of three weeks ago, with the extra assumption that the categories are \mathbb{K} -linear and that the strict tensor functor F is \mathbb{K} -linear. In other words, we conclude that Proposition 3.1 holds true.

5. APPENDIX B: TENSOR PRODUCT DECOMPOSITIONS

For $r, s \in \mathbb{Z}_{\geq 0}$ the tensor product module $L(r) \otimes L(s)$ is finite dimensional and of type 1, hence it decomposes as a direct sum of the simple U -modules $L(n)$ ($n \geq 0$). The following theorem (quantum Clebsch-Gordon decomposition) gives a more precise statement.

Theorem 5.1. *For $r, s \in \mathbb{Z}_{\geq 0}$ we have*

$$L(r) \otimes L(s) \simeq L(|r - s|) \oplus L(|r - s| + 2) \oplus \cdots \oplus L(r + s - 2) \oplus L(r + s)$$

as U -modules.

Proof. It suffices to prove the theorem for $r \geq s$ (indeed, $L(r) \otimes L(s) \simeq L(s) \otimes L(r)$ since Mod_U^{fd} is a braided category). In this case we are asked to prove that

$$L(r) \otimes L(s) \simeq \bigoplus_{l=0}^s L(r-s+2l).$$

The dimension over \mathbb{K} of the left hand side is $(r+1)(s+1)$, which agrees with the dimension of the right hand side since

$$\sum_{l=0}^s (r-s+2l+1) = (r+1)(s+1).$$

To prove the theorem it thus suffices to construct highest weight vectors $w_l \in L(r) \otimes L(s)$ of highest weight $r-s+2l+1$ for $l=0, \dots, s$. First note that

$$(L(r) \otimes L(s))_{q^{r-s+2l}} = \text{span}_{\mathbb{K}} \{ \bar{m}_n(q^r) \otimes \bar{m}_{s-l-n}(q^s) \}_{n=-l}^{s-l}$$

for $l=0, \dots, s$. Now a vector

$$w_l = \sum_{n=-l}^{s-l} c_n (\bar{m}_n(q^r) \otimes \bar{m}_{s-l-n}(q^s)) \in (L(r) \otimes L(s))_{q^{r-s+2l}}, \quad c_n \in \mathbb{K}$$

satisfies $Ew_l = 0$ if and only if

$$c_{n+1} = -q^{n-r-s} \frac{(s-l-n)_{q^2} (l+n+1)_{q^2}}{(n+1)_{q^2} (r-n)_{q^2}} c_n \quad (n = -l, \dots, s-l-1)$$

(note that the denominators are nonzero). There is a nontrivial solution to these recurrence relations, hence there exists a highest weight vector of highest weight q^{r-s+2l} in $L(r) \otimes L(s)$ for all $l \in \{0, \dots, s\}$. \square

The following result should be compared to Theorem 3.17 in the syllabus of three weeks ago.

Corollary 5.2. *For $k, l \in \mathbb{Z}_{\geq 0}$ we have*

$$\text{Dim}_{\mathbb{K}}(\text{Hom}_U(V^{\otimes k}, V^{\otimes l})) = \begin{cases} 0, & \text{if } k+l \text{ odd,} \\ \frac{1}{n+1} \binom{2n}{n}, & \text{if } k+l = 2n \text{ even.} \end{cases}$$

Proof. For $n \in \mathbb{Z}_{\geq 0}$ we have

$$(5.1) \quad V^{\otimes n} \simeq \bigoplus_{j \in \mathbb{Z}_{\geq 0}} L(n)^{\otimes c_j(n)}$$

for unique $c_j(n) \in \mathbb{Z}_{\geq 0}$. For $n=0$ we read the left hand side as $L(0) = \mathbb{K}$ (the unit object in Mod_U^{fd}), so that $c_0(0) = 1$ and $c_j(0) = 0$ for $j \geq 1$. By the above theorem we have the recurrence relations

$$(5.2) \quad c_0(n) = c_1(n-1), \quad c_j(n) = c_{j+1}(n-1) + c_{j-1}(n-1) \quad (j \geq 1)$$

for $n \geq 1$. These recurrence relations, together with the initial conditions $c_0(0) = 1$ and $c_j(0) = 0$ ($j \geq 1$), determine the coefficients $c_j(n)$ uniquely. Now define the polynomial $V_j(y)$ of degree $j \geq 0$ by

$$V_j(x + x^{-1}) = \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}} = \sum_{l=0}^j x^{j-2l}.$$

It is the character of the simple \mathfrak{sl}_2 -module of dimension $j + 1$ or, from the viewpoint of orthogonal polynomials, a Chebyshev polynomial of the second kind (up to a rescaling of the variable). The $V_j(y)$'s satisfy $V_0(y) = 1$ and the recurrence relation

$$yV_j(y) = V_{j+1}(y) + V_{j-1}(y), \quad j \geq 1.$$

By induction to n we conclude from the recurrence relations for the $c_j(n)$ and the $V_j(y)$ that

$$y^n = \sum_{j=0}^{\infty} c_j(n) V_j(y)$$

(this is in fact the equivalent reformulation of (5.1) for the characters of the associated \mathfrak{sl}_2 -modules). Since

$$V_j(2 \cos(\theta)) = \frac{\sin((j+1)\theta)}{\sin(\theta)}$$

we have the orthogonality relations

$$\frac{1}{\pi} \int_{-\pi}^{\pi} V_i(2 \cos(\theta)) V_j(2 \cos(\theta)) \sin^2(\theta) d\theta = \delta_{i,j}.$$

We conclude that

$$\begin{aligned} \text{Dim}_{\mathbb{K}}(\text{Hom}_U(V^{\otimes k}, V^{\otimes l})) &= \sum_{j=0}^{\infty} c_j(k) c_j(l) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} (2 \cos(\theta))^{k+l} \sin^2(\theta) d\theta \\ &= -\frac{1}{4\pi} \sum_{r=0}^{k+l} \binom{k+l}{r} \int_{-\pi}^{\pi} e^{i(2r-k-l)\theta} (e^{2i\theta} + e^{-2i\theta} - 2) d\theta. \end{aligned}$$

The latter expression is zero if $k + l$ is odd. If $k + l = 2n$ is even, then it equals

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$$

the n th Catalan number. □

Combining the results of section 3 in the syllabus of four weeks ago with Theorem 3.17 in the syllabus of three weeks ago, we conclude

Corollary 5.3. *The Temperley-Lieb algebra $TL_n(q^{\frac{1}{2}})$ is isomorphic to $\text{End}_U(L(1)^{\otimes n})$ as algebra over \mathbb{K} . The explicit isomorphism has been constructed in Exercise 3.14 of the syllabus of week 14.*

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