LECTURE NOTES ON QUIVER REPRESENTATIONS

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These are lecture notes that I have developed in the course of several years for the first year master course "Quivers" at the University of Amsterdam. The goal of the master course is to give first-year master students a first taste of various core topics in algebra and geometry within the concrete setting of quiver representations. These topics include representation theory, category theory, homological algebra, Lie theory and algebraic geometry. In these lecture notes I assume the following foreknowledge:

(1) Basics on linear algebra, rings and fields (typically the material covered in undergraduate courses on these topics).
(2) Some basic facts on category theory. We will give precise references, mainly to the lecture notes [7] of the undergraduate course "Modules and Categories" developed by Lenny Taelman.
(3) In the last section of the lecture notes (Section 11), when discussing the proof of Gabriel's Theorem, some results from algebraic geometry are needed, for which we refer to the classical text book [5]. The key ideas and steps in the proof of Gabriel's Theorem will though be clear without foreknowledge of algebraic geometry.

Initially the notes started as supplements to parts of the books by Schiffler [4] and Assem, Simson and Skowronski [1], but gradually grew out to become lecture notes covering the material of the whole master course. I tried to make the lecture notes as self-contained as possible, with references restricted mainly to the original sources [4, 1, 7] for the master course. If a proof is not given in detail, then a precise reference is given and these proofs can be understood without further additional foreknowledge unless explicitly stated otherwise.

The main topics of the lecture notes are: (projective) quiver representations, Krull-Schmidt Theorem, (standard) projective resolutions, Jacobson's Theorem for finite dimensional algebras, the presentation of basic finite dimensional algebras as bounded quiver algebras, induction and restriction, and Gabriel's Theorem on the classification of quivers of finite representation type.

These lecture notes are updated on a regular basis. Comments on the text are welcome.

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1. Introduction

A central problem in the course "Quivers" is how to unravel the structure of modules over finite dimensional algebras. Important tools come from rather abstract homological algebraic techniques, as we shall see later, but substantial parts of the theory become very concrete by reinterpreting modules as representations of quivers (oriented graphs). In this first section I will explain the basic idea how to relate modules to quivers. It corresponds to Section 1.1, Section 4.1 (up to Prop. 4.4), Section 4.3 (up to Lemma 4.9) and Section 5.2 (special case of Theorem 5.4) in [4].

1.1. **Basic definitions.** Let k be a field.

Definition 1.1. *An (associative k-)algebra A is a unital ring as well as a k-vector space, such that the ring addition is the same as the vector space addition and the ring multiplication map $A \times A \to A$ $(a, b) \mapsto ab$ is k-bilinear.*

We write 1 for the unit element of A. We often will omit the adjective "associative", since all algebras that we encounter will be associative.

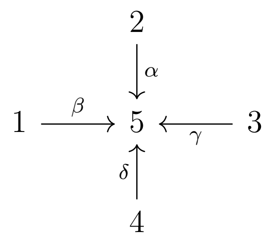
Example 1.2. a. *The field k itself is an associative k-algebra.*
b. *The ring $k[x_1, \ldots, x_n]$ of polynomials in $x_1, \ldots, x_n$ with coefficients in k is a commutative associative k-algebra with the natural scalar multiplication of polynomials.*
c. *Let V be a k-vector space. The k-vector space $\mathrm{End}_k(V)$ of linear endomorphisms of V is an associative k-algebra with ring multiplication the composition of maps. The unit element is Id_V.*
d. *Let G be a group and $k[G]$ the k-vector space with linear basis the group elements. It becomes an associative k-algebra with multiplication the k-bilinear extension of the group multiplication. The unit element is the neutral element of G.*

Path algebras of quivers form a class of associative k-algebras which are of central importance in this course. We start with the definition of a quiver.

Definition 1.3. *A quiver $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 of vertices, a set Q_1 of arrows, and a pair of maps $s, t : Q_1 \to Q_0$ called the source and target maps.*

Quivers are synonymous to oriented graphs, with the source and target maps indicating the starting and ending vertex of the oriented arrows, respectively. For instance, the oriented graph

(1.1)



corresponds to $Q_0 = \{1, 2, 3, 4, 5\}$, $Q_1 = \{\alpha, \beta, \gamma, \delta\}$ and source and target maps

$$s(\alpha) = 2, \qquad s(\beta) = 1, \qquad s(\gamma) = 3, \qquad s(\delta) = 4$$

and $t(\alpha) = t(\beta) = t(\gamma) = t(\delta) = 5$. We will always assume that quivers are finite, meaning that the number of vertices and arrows are finite.

Definition 1.4. *Let Q be a quiver and $i, j \in Q_0$. A path c of length $r \geq 1$ with starting point i and endpoint j is a r-tuple $\alpha_1\alpha_2 \cdots \alpha_r$ of vertices $\alpha_k \in Q_1$ such that $s(\alpha_1) = i$, $t(\alpha_k) = s(\alpha_{k+1})$ $(1 \leq k < r)$ and $t(\alpha_r) = j$.*

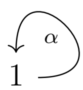
We write $s(c)$ and $t(c)$ for the starting point and endpoint of the path c. For $i \in Q_0$ we write e_i for the path of length zero starting and ending in i.

Let $\mathcal{P} = \mathcal{P}_Q$ be the set of paths of Q. For example, for the quiver (1.1) we have

$$\mathcal{P} = \{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta\}.$$

Here are two examples with paths of lengths > 1.

Example 1.5. a. *For the quiver*



we have $\mathcal{P} = \{e_1, \alpha, \alpha^2, \ldots\}$.
b. *For the quiver*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$$

we have $\mathcal{P} = \{c_{ij} \mid 1 \leq i \leq j \leq n\}$ with $c_{ii} := e_i$ and for $1 \leq i < j \leq n$,

$$c_{ij} := \alpha_i\alpha_{i+1} \cdots \alpha_{j-1}.$$

A cycle is a path c of length ≥ 1 with $s(c) = t(c)$.

Exercise 1.6. *Q has no cycles $\Leftrightarrow \#\mathcal{P}_Q < \infty$.*

1.2. **The path algebra.** Let kQ be the k-vector space with basis the set $\mathcal{P}_Q$ of paths of Q.

Proposition 1.7. *The k-vector space kQ is an associative k-algebra with multiplication the k-bilinear extension of the multiplication rule*

$$c \cdot c' := \begin{cases} cc' \text{ (concatenation of paths)} & \text{if } t(c) = s(c') \\ 0 & \text{otherwise} \end{cases}$$

for paths $c, c' \in \mathcal{P}$. The unit element is $1 = \sum_{i \in Q_0} e_i$.

kQ is called the *path algebra* of the quiver Q.

Proof. Associativity of the product rule is obvious. The element $\sum_{i \in Q_0} e_i$ serves as the unit element since

$$e_i c = \delta_{i,s(c)} c, \qquad c e_i = \delta_{i,t(c)} c$$

for $i \in Q_0$ and $c \in \mathcal{P}$ with $\delta_{i,j}$ the Kronecker delta function, which is 1 if $i = j$ and 0 otherwise. □

Example 1.8. *Consider example 1.5***b**. *For $1 \leq i \leq j \leq n$ and $1 \leq k \leq \ell \leq n$ we have*

$$c_{ij} c_{k\ell} = \delta_{j,k} c_{i\ell}.$$

Exercise 1.9. *Let $\mathrm{Mat}_{n,n}(k)$ be the associative k-algebra of $n \times n$ matrices with coefficients in k and matrix multiplication as ring structure. Write $\mathfrak{b}$ for the k-subalgebra of upper-triangular matrices in $\mathrm{Mat}_{n,n}(k)$.*

*Let Q be the quiver from Example 1.5***b**. *Show that the path algebra kQ is isomorphic to $\mathfrak{b}$ as algebras.*

The set $\{e_i\}_{i \in Q_0}$ is a so-called complete set of orthogonal idempotents of kQ. The definition of a complete set of orthogonal idempotents for an arbitrary k-algebra A is as follows.

Definition 1.10. *Let A be a finite dimensional associative k-algebra. A set $\{e_1, \ldots, e_n\}$ of elements $e_i \in A$ is called a complete set of orthogonal idempotents of A if:*

- **a.** $e_i^2 = e_i$ *(idempotent property),*
- **b.** $e_i e_j = 0$ *if $i \neq j$ (orthogonality),*
- **c.** $1 = \sum_{i=1}^n e_i$ *(completeness).*

The set $\{1\}$ is the trivial complete set of orthogonal idempotents of A.

Lemma 1.11. *Let A be a finite dimensional associative k-algebra and $\{e_1, \ldots, e_n\}$ a complete set of orthogonal idempotents of A.*

- **a.** $A = \bigoplus_{i,j=1}^n e_i A e_j$ *as k-vector spaces.*
- **b.** $e_i A e_i \subseteq A$ *is a k-subalgebra with unit element e_i.*

Proof. **a.** Let $a \in A$. Then

$$a = 1 \cdot a \cdot 1 = \sum_{i,j=1}^{n} e_i a e_j,$$

showing that $A = \sum_{i,j=1}^{n} e_i A e_j$. Suppose $a_{ij} \in e_i A e_j$ $(1 \leq i, j \leq n)$ and

$$\sum_{i,j=1}^{n} a_{ij} = 0.$$

Then for $1 \leq k, \ell \leq n$ we have

$$0 = e_k \big(\sum_{i,j}^{n} a_{ij}\big) e_\ell = \sum_{i,j=1}^{n} e_k e_i a_{ij} e_j e_\ell = a_{k\ell}$$

since $a_{ij} = e_i a_{ij} e_j$ and $e_k e_i = \delta_{i,k} e_k$, $e_j e_\ell = \delta_{j,\ell} e_\ell$. This proves **a**.
b. This is clear since e_i is an idempotent. □

Example 1.12. *For the path algebra kQ of a quiver Q and $i, j \in Q_0$,*

$$e_i(kQ)e_j = \mathrm{span}_k\{c \mid c \in \mathcal{P}_{ij}\}$$

with $\mathcal{P}_{ij}$ the set of paths starting at i and ending at j.

1.3. **The category of right kQ-modules.** Let R be a unital ring. A right R-module is a an abelian group M together with a map $M \times R \to M$, $(m, r) \mapsto m \cdot r$ satisfying

$$\begin{aligned}(m_1 + m_2) \cdot r &= m_1 \cdot r + m_2 \cdot r,\\ m \cdot (r_1 + r_2) &= m \cdot r_1 + m \cdot r_2,\\ m \cdot (r_1 r_2) &= (m \cdot r_1) \cdot r_2,\\ m \cdot 1 &= m\end{aligned}$$

for $m, m_1, m_2 \in M$ and $r, r_1, r_2 \in R$. In a similar fashion one defines left R-modules. See [7, Chapter 1] for some basic properties and examples of R-modules.

Note that if R is a k-algebra A and M is a right A-module, then the abelian group M becomes a k-vector space with scalar product $\lambda m := m \cdot (\lambda 1)$ for $\lambda \in k$ and $m \in M$, and the action map $M \times A \to M$ is k-bilinear.

Example 1.13. *Recall that a k-representation (π, V) of a group G is a k-vector space V together with a group homomorphism $\pi : G \to \mathrm{GL}(V)$. It endows V with the structure of a left $k[G]$-module by*

$$\big(\sum_{g \in G} \lambda_g g, v\big) \mapsto \sum_{g \in G} \lambda_g \pi(g) v, \qquad \lambda_g \in k.$$

Conversely, restricting the $k[G]$-action of a left $k[G]$-module M to G turns M in a k-representation of the group G. This is a bijective correspondence. In fact, with *the obvious notions of morphisms, it extends to an equivalence of categories between the category* Rep_G *of* k*-representations of* G *and the category of left* $k[G]$*-modules.*

The interpretation of modules over kG as representations of the group G provides elementary and powerful (group theoretic) tools to understand the structure of $k[G]$-modules (think of the undergraduate course on representation theory of finite groups). For right kQ-modules it turns out to be fruitful to interpret them as so-called quiver representations. The construction is as follows.

Let Q be a quiver and M a right kQ-module. Consider the subspaces $M_i := M \cdot e_i$ of M $(i \in Q_0)$. As in the proof of Lemma 1.11 one shows that

$$M = \bigoplus_{i \in Q_0} M_i$$

as k-vector spaces. For $\alpha \in Q_1$ define a k-linear map $\phi_\alpha \in \mathrm{Hom}_k(M_{s(\alpha)}, M_{t(\alpha)})$ by

$$\phi_\alpha(m) := m \cdot \alpha, \qquad m \in M_{s(\alpha)}. \tag{1.2}$$

Note that the image of ϕ_α is indeed contained in $M_{t(\alpha)}$ since $\alpha = \alpha e_{t(\alpha)}$ in kQ. Hence the right kQ-module structure on M gives rise to the data $\big(M_i, \phi_\alpha\big)_{i \in Q_0; \alpha \in Q_1}$ consisting of k-vector spaces M_i $(i \in Q_0)$ naturally attached to the vertices of Q and corresponding k-linear maps $\phi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ $(\alpha \in Q_1)$, naturally attached to the edges of Q.

Definition 1.14. *A representation* $M = \big(M_i, \phi_\alpha\big)_{i \in Q_0, \alpha \in Q_1}$ *of a quiver* Q *consists of a collection of* k*-vector spaces* M_i $(i \in Q_0)$ *together with a collection of* k*-linear maps* $\phi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ $(\alpha \in Q_0)$*. We say that* M *is finite dimensional if* $\dim(M_i) < \infty$ *for all* $i \in Q_0$*. We call*

$$\dim(M) := \big(\dim(M_i)\big)_{i \in Q_0}$$

the dimension vector of M*.*

We will assume quiver representations to be finite dimensional unless explicitly stated otherwise.

We picture quiver representations as decorated oriented graphs by attaching the k-vector space M_i to the vertex $i \in Q_0$ and attaching the k-linear maps ϕ_α to the edge $\alpha \in Q_1$. For instance, let $M = \big(\{M_i\}_{i=1}^5, \{\phi_\alpha, \phi_\beta, \phi_\gamma, \phi_\delta\}\big)$ be a representation of the quiver (1.1). Then we depict it as

$$\begin{array}{ccccc} & & M_2 & & \\ & & \big\downarrow{\scriptstyle \phi_\alpha} & & \\ M_1 & \xrightarrow{\phi_\beta} & M_5 & \xleftarrow[\phi_\gamma]{} & M_3 \\ & & {\scriptstyle \phi_\delta}\big\uparrow & & \\ & & M_4 & & \end{array} \tag{1.3}$$

Definition 1.15. *Let* $M := \big(M_i, \phi_\alpha\big)_{i\in Q_0,\alpha\in Q_1}$ *and* $N := \big(N_i, \psi_\alpha\big)_{i\in Q_0,\alpha\in Q_1}$ *be two representations of the quiver* Q. *A morphism* $f : M \to N$ *of quiver representations is a collection* $f = \big(f_i\big)_{i\in Q_0}$ *of linear maps* $f_i : M_i \to N_i$ $(i \in Q_0)$ *such that for all* $\alpha \in Q_1$, *the diagram*

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{\phi_\alpha} & M_{t(\alpha)} \\ \big\downarrow{\scriptstyle f_{s(\alpha)}} & & \big\downarrow{\scriptstyle f_{t(\alpha)}} \\ N_{s(\alpha)} & \xrightarrow{\psi_\alpha} & N_{t(\alpha)}. \end{array}$$

commutes.

We write $\mathrm{Hom}(M, N)$ for the class of morphisms $M \to N$ and $\mathrm{End}(M) = \mathrm{Hom}(M, M)$ for the endomorphisms of M. Note that

$$\mathrm{Hom}(M, N) \subset \prod_{i\in Q_0} \mathrm{Hom}_k(M_i, N_i),$$

from which $\mathrm{Hom}(M, N)$ inherits the structure of a k-vector space. Concretely, it is given by componentwise addition and scalar multiplication. We define a composition map

$$\mathrm{Hom}(M, N) \times \mathrm{Hom}(N, P) \to \mathrm{Hom}(M, P), \qquad (f, g) \mapsto g \circ f$$

with $(g\circ f)_i := g_i \circ f_i$ componentwise composition of linear maps (convince yourself that $g \circ f$ indeed is a morphism $M \to P$ of representations). We write $1_M = \big(\mathrm{Id}_{M_i}\big)_{i\in Q_0} \in \mathrm{End}(M)$, which serves as the identity morphism with respect to the composition rule. We obtain in this way a category Rep_Q, the category Rep_Q of finite dimensional representations of Q over k (see [7, Chapter 4 & 5] for the basics on categories).

Note that a morphism $f : M \to N$ is an isomorphism if $f_i : M_i \to N_i$ are k-linear isomorphisms for all $i \in Q_0$. The *isoclass* of the quiver representation M is the collection of representations N for which $\mathrm{Hom}(M, N)$ contains an isomorphism. In this case we write $M \simeq N$ (it is clearly an equivalence relation).

Example 1.16. a. *Consider the quiver representations* M *and* N *given by*

respectively, with matrices $B, C \in \mathrm{Mat}_{2,2}(k)$. *In this case a morphism* $M \to N$ *is a* 2×2*-matrix* D *such that* $DB = CD$. *Hence* $M \simeq N$ *iff* B *and* C *are in the same conjugation class.*
b. *Consider now the quiver representations* M *and* N *given by*

$$k^2 \xrightarrow{B} k^2, \qquad k^2 \xrightarrow{C} k^2$$

respectively, with $B, C \in \mathrm{Mat}_{2,2}(k)$. In this case a morphism $f \in \mathrm{Hom}(M,N)$ is a a pair $f = (D,E)$ of matrices $D, E \in \mathrm{Mat}_{2,2}(k)$ such that $EB = CD$. In this case $M \simeq N$ iff the matrices B and C have the same rank.

Write Mod_{kQ} for the category of finite dimensional right kQ-modules.

Proposition 1.17. *The categories Rep_Q and Mod_{kQ} are equivalent.*

Proof. We define a functor $\mathcal{F} : \mathrm{Mod}_{kQ} \to \mathrm{Rep}_Q$ as follows. For M a right kQ-module we define $\mathcal{F}\big(M\big) = \big(M_i, \phi_\alpha\big)_{i \in Q_0, \alpha \in Q_1}$ with $M_i = M \cdot e_i$ and ϕ_α given by (1.2). For a morphism $f : M \to N$ of right kQ-modules we define $\mathcal{F}f \in \mathrm{Hom}(\mathcal{F}M, \mathcal{F}N)$ by

$$\big(\mathcal{F}f\big)_i := f|_{M_i}.$$

We first verify that $\mathcal{F}f$ is a well defined morphism. For $m \in M_i$ we have $f(m) = f(m \cdot e_i) = f(m) \cdot e_i$, hence $\big(\mathcal{F}f\big)_i \in \mathrm{Hom}_k(M_i, N_i)$; furthermore, for $\alpha \in Q_1$ and $m \in M_{s(\alpha)}$, writing $\mathcal{F}N = (N_i, \psi_\alpha)_{i \in Q_0, \alpha \in Q_1}$,

$$\begin{aligned}\big(\psi_\alpha \circ (\mathcal{F}f)_{s(\alpha)}\big)(m) &= \psi_\alpha(f(m)) \\ &= f(\phi_\alpha(m)) = \big((\mathcal{F}f)_{t(\alpha)} \circ \phi_\alpha\big)(m),\end{aligned}$$

completing the proof that $\mathcal{F}f$ is well defined. Clearly $\mathcal{F}$ is a (covariant) functor.

A quasi-inverse $\mathcal{G}$ of $\mathcal{F}$ is constructed as follows. It maps a representation $(M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q to the external direct sum

$$M := \bigoplus_{i \in Q_0} M_i, \tag{1.4}$$

regarded as right kQ-module as follows: e_i acts on M as projection onto M_i along the direct sum decomposition (1.4), while a path $c = \alpha_1 \alpha_2 \cdots \alpha_r$ of length $r \geq 1$ acts on $m \in M_{s(c)}$ by

$$m \cdot c := \phi_{\alpha_r}(\cdots \phi_{\alpha_2}(\phi_{\alpha_1}(m)) \cdots)$$

while c acts as zero on M_i with $i \neq s(c)$. We leave the straightforward verifications that this turns the k-vector space M into a right kQ-module as an exercise.

If $f := (f_i)_{i \in Q_0}$ is a morphism from (M_i, ϕ_α) to (N_i, ψ_α) then $\mathcal{G}f$ is f, regarded as linear map $\bigoplus_{i \in Q_0} M_i \to \bigoplus_{i \in Q_0} N_i$. Note that $\mathcal{G}f$ is a morphism of kQ-modules, and $\mathcal{G}$ is a well defined functor $\mathcal{G} : \mathrm{Rep}_Q \to \mathrm{Mod}_{kQ}$. It is easy to check that $\mathcal{G}$ is a quasi-inverse of $\mathcal{F}$. □

2. Rep_Q as an abelian category

In the previous section we introduced the category Rep_Q of representations of a quiver Q, and showed that it is equivalent to the category Mod_{kQ} of right modules over the path algebra kQ. In this section we show that the category Rep_Q is an example of an abelian category. We furthermore give some basic facts on general abelian categories.

2.1. **Additive categories.** A category $\mathcal{C}$ is called **preadditive** if the hom-sets are abelian groups and the composition is bilinear with respect to the group addition. A preadditive category $\mathcal{C}$ is called k-**linear** category (or simply a k-category) if the hom-sets are k-vector spaces and the composition is k-bilinear. From Section 1 it is clear that Rep_Q is k-linear.

A preadditive category $\mathcal{C}$ is called **additive** if there exists an object 0 such that End(0) is the trivial group and if finite direct sums exist. The existence of finite direct sums means the following: for a finite set $\{X_i\}_{i\in I}$ of objects in $\mathcal{C}$ there exists an object $\bigoplus_{i\in I} X_i$ and morphisms $j_k : X_k \to \bigoplus_{i\in I} X_i$ ($k \in I$) satisfying the following universal property: if $Y \in \mathrm{Ob}(\mathcal{C})$ and $f_k \in \mathrm{Hom}(X_k, Y)$ ($k \in I$) then there exists a unique $f \in \mathrm{Hom}(\bigoplus_{i\in I} X_i, Y)$ such that $f \circ j_k = f_k$ for all $k \in I$.

Exercise 2.1. *Let $\mathcal{C}$ be an additive category and $\{X_i\}_{i\in I}$ a finite set of objects. Suppose that $Z \in \mathrm{Ob}(\mathcal{O})$ and $i_k \in \mathrm{Hom}(X_k, Z)$ ($k \in I$) also solve the above universal property. In other words, if $Y \in \mathrm{Ob}(\mathcal{C})$ and $g_k \in \mathrm{Hom}(X_k, Y)$ ($k \in I$) then there exists a unique $g \in \mathrm{Hom}(Z, Y)$ such that $g \circ i_k = g_k$ for all $k \in I$. Show that there exists an isomorphism $\phi : \bigoplus_{i\in I} X_i \xrightarrow{\sim} Z$ such that $\phi \circ j_k = i_k$ ($k \in I$).*

The representation category Rep_Q is an additive category. The zero element is the zero representation, (M_i, ϕ_α) with $M_i = \{0\}$ ($i \in Q_0$) and $\phi_\alpha \equiv 0$ ($\alpha \in Q_1$). The direct sum is defined as follows (we only explain the construction for two quiver representations, the extension to a finite number of quiver representations is analogous).

Let $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two representations of Q. Then $M \oplus N$ is the quiver representation $(M_i \oplus N_i, \phi_\alpha \oplus \psi_\alpha)$, with $M_i \oplus N_i$ the vector space direct sum of M_i and N_i for $i \in Q_0$ and

$$(\phi_\alpha \oplus \psi_\alpha)(m_{s(\alpha)}, n_{s(\alpha)}) := \big(\phi_\alpha(m_{s(\alpha)}), \psi_\alpha(n_{s(\alpha)})\big)$$

for $\alpha \in Q_1$. The corresponding morphisms $j^M = (j^M_i)_i \in \mathrm{Hom}(M, M \oplus N)$ and $j^N = (j^N_i)_i \in \mathrm{Hom}(N, M \oplus N)$ are

$$j^M_i(m_i) = (m_i, 0), \qquad j^N_i(n_i) = (0, n_i)$$

for $m_i \in M_i$ and $n_i \in N_i$ (check the details!).

Lemma 2.2. *Let $\mathcal{C}$ be an additive category and $0 \in \mathrm{Ob}(\mathcal{C})$ a zero object.*
a. *For all $M \in \mathrm{Ob}(\mathcal{C})$, the hom-set $\mathrm{Hom}(0, M)$ is the trivial group.*

b. *For all* $M \in \mathrm{Ob}(\mathcal{C})$*, the hom-set* $\mathrm{Hom}(M, 0)$ *is the trivial group.*
c. *If* $0'$ *is another zero object, then* $0 \simeq 0'$*.*

Proof. **a.** By assumption, $\mathrm{End}(0)$ is the trivial abelian group, but it also contains the identity morphism 1_0. Hence 1_0 is also the neutral element of the abelian group $\mathrm{End}(0)$, and $\mathrm{End}(0) = \{1_0\}$. Write 0_M for the neutral element of $\mathrm{Hom}(0, M)$ and fix $f \in \mathrm{Hom}(0, M)$. Let $\ell_f : \mathrm{End}(0) \to \mathrm{Hom}(0, M)$ be the group homomorphism $\ell_f(g) := f \circ g$ for $g \in \mathrm{End}(0)$. Then

$$f = f \circ 1_0 = \ell_f(1_0) = 0_M,$$

where we use for the third equality that 1_0 is the neutral element of $\mathrm{End}(0)$.
b. Similar to **a**.
c. By part **a** the hom-sets $\mathrm{Hom}(0, 0') = \{0_{0'}\}$ and $\mathrm{Hom}(0', 0) = \{0'_0\}$ are trivial abelian groups. Then

$$0'_0 \circ 0_{0'} \in \mathrm{End}(0) = \{1_0\}$$

hence $0'_0 \circ 0_{0'} = 1_0$, and similarly $0_{0'} \circ 0'_0 = 1_{0'}$. □

Additive categories have finite direct products. We give the construction for two objects.

Lemma 2.3. *Let* $\mathcal{C}$ *be an additive category and* $M, N \in \mathrm{Ob}(\mathcal{C})$*.*
a. *There exist unique* $\pi_M \in \mathrm{Hom}(M \oplus N, M)$ *and* $\pi_N \in \mathrm{Hom}(M \oplus N, N)$ *such that* $\pi_M \circ j_M = 1_M$*,* $\pi_M \circ j_N = 0$*,* $\pi_N \circ j_M = 0$ *and* $\pi_N \circ j_N = 1_N$*, where* 0 *stands for the neutral element in the appropriate hom-set.*
b. $(M \oplus N, \pi_M, \pi_N)$ *has the following universal property: for* $Y \in \mathrm{Ob}(\mathcal{C})$ *and* $f_M \in \mathrm{Hom}(Y, M)$*,* $f_N \in \mathrm{Hom}(Y, N)$ *there exists a unique* $f \in \mathrm{Hom}(Y, M \oplus N)$ *such that* $\pi_M \circ f = f_M$ *and* $\pi_N \circ f = f_N$*.*

Proof. **a.** Consider the morphisms $1_M : M \to M$ and $0 : N \to M$. By the existence of direct sums there exists a unique morphism $\pi_M : M \oplus N \to M$ such that $\pi_M \circ j_M = 1_M$ and $\pi_M \circ j_N = 0$. In a similar way one shows the existence and uniqueness of π_N.
b. Set $f := j_M \circ f_M + j_N \circ f_N \in \mathrm{Hom}(Y, M \oplus N)$. Then clearly $\pi_M \circ f = f_M$ and $\pi_N \circ f = f_N$. It remains to prove uniqueness. It suffices to prove: if $g \in \mathrm{Hom}(Y, M \oplus N)$ such that $\pi_M \circ g = 0$ and $\pi_N \circ g = 0$, then $g = 0$.

Note that the equalities $h \circ j_M = j_M$ and $h \circ j_N = j_N$ are satisfied for both $h = j_M \circ \pi_M + j_N \circ \pi_N \in \mathrm{End}(M \oplus N)$ and $h = 1_{M \oplus N} \in \mathrm{End}(M \oplus N)$. By the uniqueness of the direct sum we thus have

$$j_M \circ \pi_M + j_N \circ \pi_N = 1_{M \oplus N}$$

in $\mathrm{End}(M \oplus N)$. Hence

$$g = 1_{M \oplus N} \circ g = j_M \circ (\pi_M \circ g) + j_N \circ (\pi_N \circ g) = 0.$$

□

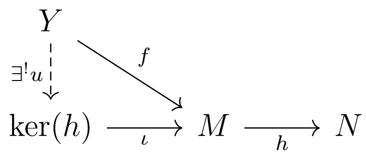
Exercise 2.4. *Prove that in an additive category $\mathcal{C}$ we have*

$$0 \oplus M \simeq M$$

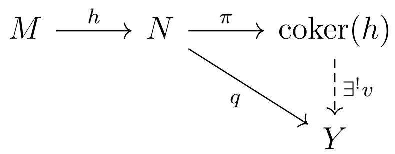
for all $M \in \mathrm{Ob}(\mathcal{C})$.

2.2. Abelian categories.

Definition 2.5. *Let $\mathcal{C}$ be a preadditive category and $h \in \mathrm{Hom}(M, N)$.*
a. *A kernel of h is a pair $(\ker(h), \iota)$ with $\ker(h) \in \mathrm{Ob}(\mathcal{C})$ and ι a morphism $\iota : \ker(h) \to M$ such that $h \circ \iota = 0$ and such that the following universal property is satisfied: if $f : Y \to M$ is a morphism such that $h \circ f = 0$, then there exists a unique $u \in \mathrm{Hom}(Y, \ker(h))$ such that $f = \iota \circ u$, i.e.*



with the triangle commuting.
b. *A cokernel of h is a pair $(\mathrm{coker}(h), \pi)$ with $\mathrm{coker}(h) \in \mathrm{Ob}(\mathcal{C})$ and π a morphism $\pi : N \to \mathrm{coker}(h)$ such that $\pi \circ h = 0$ and such that the following universal property holds true: if $q : N \to Y$ is a morphism such that $q \circ h = 0$, then there exists a unique $v \in \mathrm{Hom}(\mathrm{coker}(h), Y)$ such that $q = v \circ \pi$, i.e.*



with the triangle commuting.

One easily shows that if the kernel of h exists, then it is unique up to isomorphism (cf. Exercise 2.1). We will denote the kernel of h often with $\ker(h)$ (suppressing the morphism ι from the notations, which usually is obvious from the context).

Similar remarks and conventions apply to the cokernel of h.

Definition 2.6. *An additive category $\mathcal{C}$ is called preabelian if every morphism has a kernel and a cokernel.*

Proposition 2.7. *The category Rep_Q is preabelian. Concretely, a kernel and cokernel of a morphism $h = (h_i)_{i \in Q_0} \in \mathrm{Hom}(M, N)$ of quiver reprentations $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ is given as follows.*
a. *The kernel of h is*

$$\ker(h) := \big(\mathrm{Ker}(h_i), \phi_\alpha|_{\mathrm{Ker}(h_{s(\alpha)})}\big)$$

with $\mathrm{Ker}(h_i)$ the kernel of the linear map $h_i : M_i \to N_i$ and with the associated morphism $\iota = (\iota_i)_i : \ker(h) \to M$ the natural inclusion maps

$$\iota_i : \mathrm{Ker}(h_i) \hookrightarrow M_i.$$

b. *The cokernel of h is*

$$\operatorname{coker}(h) := \big(N_i/\operatorname{Im}(h_i), \overline{\psi}_\alpha\big)$$

with $\operatorname{Im}(h_i) = h_i(M_i) \subseteq N_i$ *the image of the linear map* $h_i : M_i \to N_i$, *with the connecting maps* $\overline{\psi}_\alpha : N_{s(\alpha)}/\operatorname{Im}(h_{s(\alpha)}) \to N_{t(\alpha)}/\operatorname{Im}(h_{t(\alpha)})$ *defined by*

$$\overline{\psi}_\alpha\big(n + \operatorname{Im}(h_{s(\alpha)})\big) := \psi_\alpha(n) + \operatorname{Im}(h_{t(\alpha)}), \qquad n \in N_{s(\alpha)},$$

and with the associated morphism $\pi = (\pi_i)_i : N \to \operatorname{coker}(h)$ *the natural projection maps*

$$\pi_i : N_i \twoheadrightarrow N_i/\operatorname{Im}(h_i), \qquad n \mapsto n + \operatorname{Im}(h_i).$$

Proof. **a.** We first need to show that $\ker(h)$ is a representation of Q. It suffices to show that $\phi_\alpha(\operatorname{Ker}(h_{s(\alpha)})) \subseteq \operatorname{Ker}(h_{t(\alpha)})$. Let $m \in \operatorname{Ker}(h_{s(\alpha)})$. Then

$$h_{t(\alpha)}\big(\phi_\alpha(m)\big) = \psi_\alpha\big(h_{s(\alpha)}(m)\big) = \psi_\alpha(0) = 0$$

showing that $\phi_\alpha(m) \in \operatorname{Ker}(h_{t(\alpha)})$.

By construction $h \circ \iota = 0$. Suppose that $f = (f_i) \in \operatorname{Hom}(Y, M)$ is a morphism with quiver representation $Y = (Y_i, \varphi_\alpha)$ such that $h \circ f = 0$. Then the image $f_i(Y_i)$ of the k-linear map $f_i : Y_i \to M_i$ is contained in $\operatorname{Ker}(h_i)$ for all $i \in Q_0$. Hence restricting the codomain of f_i we may view f_i as k-linear map $Y_i \to \operatorname{Ker}(h_i)$, in which case we denote it by u_i. By construction $\iota_i \circ u_i = f_i$ for $i \in Q_0$, so it remains to show that $u = (u_i)_i$ defines a morphism $Y \to M$. Let $\alpha \in Q_1$ and $y \in Y_{s(\alpha)}$, then we have

$$\begin{aligned} u_{t(\alpha)}\big(\varphi_\alpha(y)\big) = f_{t(\alpha)}\big(\varphi_\alpha(y)\big) &= \phi_\alpha\big(f_{s(\alpha)}(y)\big) \\ &= \phi_\alpha(u_{s(\alpha)}(y)) = \big(\phi_\alpha|_{\operatorname{Ker}(h_{s(\alpha)})}\big)(f_{s(\alpha)}(y)), \end{aligned}$$

as desired. Clearly u is unique.

b. The proof is similar and is left as an exercise. □

Let $\mathcal{C}$ be an arbitrary category and $Y, M, N \in \operatorname{Ob}(\mathcal{C})$. We call $f \in \operatorname{Hom}(M, N)$ a *monomorphism* if $f \circ g_1 = f \circ g_2$ for morphisms $g_1, g_2 \in \operatorname{Hom}(Y, M)$ implies that $g_1 = g_2$ (we write $f : M \hookrightarrow N$). We call f a *epimorphism* if $h_1 \circ f = h_2 \circ f$ for $h_1, h_2 : N \to Y$ implies $h_1 = h_2$ (we write $h : M \twoheadrightarrow N$).

Exercise 2.8. *Let* $h = (h_i)_i \in \operatorname{Hom}(M, N)$ *be a morphism in the category* Rep_Q.
a. *Show that h is a monomorphism iff h_i is injective for all $i \in Q_0$.*
b. *Show that h is an epimorphism iff h_i is surjective for all $i \in Q_0$.*

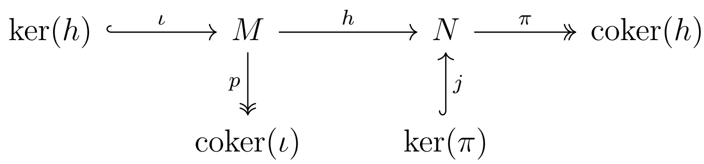
From the exercise it is clear that $\iota : \ker(h) \hookrightarrow M$ and $\pi : N \twoheadrightarrow \operatorname{coker}(h)$ for a morphism $h \in \operatorname{Hom}(M, N)$ in Rep_Q. This is a general fact in preabelian categories:

Exercise 2.9. *Let $\mathcal{C}$ be a preabelian category and* $f \in \operatorname{Hom}(X, Y)$ *a morphism.*
a. *Show that* $\iota : \ker(f) \to X$ *is a monomorphism and* $\pi : Y \to \operatorname{coker}(f)$ *is an epimorphism.*
b. *Prove: f is a monomorphism* $\Leftrightarrow \ker(f) \simeq 0$.
c. *Prove: f is an epimorphism* $\Leftrightarrow \operatorname{coker}(f) \simeq 0$.

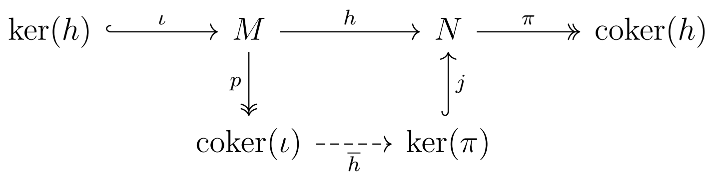
We now define the parallel of a morphism $h \in \operatorname{Hom}(M, N)$ in a preabelian category $\mathcal{C}$. By the existence of a kernel and cokernel of h we have the diagram of morphisms

$$\ker(h) \overset{\iota}{\hookrightarrow} M \overset{h}{\to} N \overset{\pi}{\twoheadrightarrow} \operatorname{coker}(h)$$

in $\mathcal{C}$. But the monomorphism $\iota : \ker(h) \hookrightarrow M$ has a cokernel $\operatorname{coker}(\iota)$, and the epimorphism $\pi : N \twoheadrightarrow \operatorname{coker}(h)$ has a kernel $\ker(\pi)$. For the moment we write $p : M \twoheadrightarrow \operatorname{coker}(\iota)$ and $j : \ker(\pi) \hookrightarrow N$ for the corresponding morphisms. We thus obtain the diagram

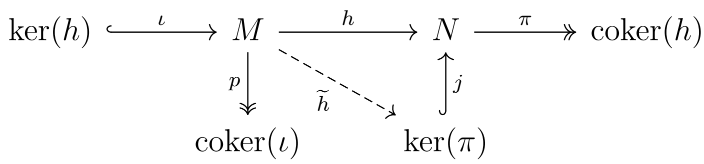


Proposition 2.10. *Let $\mathcal{C}$ be a preabelian category and $h : M \to N$ a morphism. There exists a unique morphism $\overline{h} \in \operatorname{Hom}\big(\operatorname{coker}(\iota), \ker(\pi)\big)$ such that $j \circ \overline{h} \circ p = h$, i.e., such that the square in the diagram*

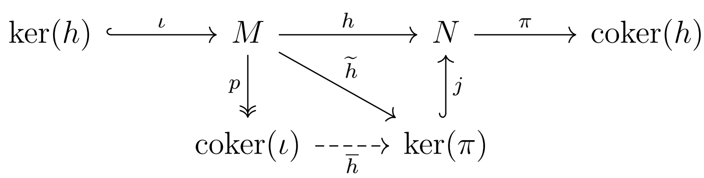


commutes.

Proof. This is by applying the universal property of the kernel and cokernel. Indeed, since $\pi \circ h = 0$ it follows from the universal property of $\ker(\pi)$ that there exists a unique morphism $\widetilde{h} : M \to \ker(\pi)$ such that



with the triangle commuting. Next we have $h \circ \iota = 0$, hence $j \circ \widetilde{h} \circ \iota = 0$. Since j is a monomorphism, this implies that $\widetilde{h} \circ \iota = 0$. By the universal property of $\operatorname{coker}(\iota)$ there exists a unique morphism $\overline{h} : \operatorname{coker}(\iota) \to \ker(\pi)$ such that



with the lower triangle commuting. This proves the existence of $\overline{h}$. The proof that $\overline{h}$ is unique is left as an exercise. $\square$

Definition 2.11. *Let $\mathcal{C}$ be a preabelian category and $h \in \mathrm{Hom}(M, N)$.*
a. *The kernel $j : \ker(\pi) \hookrightarrow N$ of the cokernel $\pi : N \twoheadrightarrow \mathrm{coker}(h)$ of h is called the image of h. We denote it by $\mathrm{im}(h)$.*
b. *The cokernel $p : M \twoheadrightarrow \mathrm{coker}(\iota)$ of the kernel $\iota : \ker(h) \hookrightarrow M$ of h is called the coimage of h. We denote it by $\mathrm{coim}(h)$.*

We usually suppress the morphisms j and p and write $\mathrm{im}(h)$ and $\mathrm{coim}(h)$ for the associated objects $\ker(\pi)$ and $\mathrm{coker}(\iota)$. So given a morphism $h \in \mathrm{Hom}(M, N)$ in a preabelian category $\mathcal{C}$, we have the extended diagram

$$\begin{array}{ccccccc} \ker(h) & \hookrightarrow & M & \xrightarrow{h} & N & \twoheadrightarrow & \mathrm{coker}(h) \\ & & \big\downarrow\!\!\!\downarrow & & \big\uparrow & & \\ & & \mathrm{coim}(h) & \xrightarrow[\overline{h}]{} & \mathrm{im}(h) & & \end{array}$$

with the square commuting.

In Rep_Q, with $h = (h_i)_i \in \mathrm{Hom}(M, N)$ and $M = (M_i, \phi_\alpha)$, $N = (N_i, \psi_\alpha)$, the image and cokernel of h is explicitly given by

$$\mathrm{im}(h) = \big(\mathrm{Im}(h_i), \psi_\alpha|_{\mathrm{Im}(h_{s(\alpha)})}\big),$$
$$\mathrm{coim}(h) = \big(M_i/\mathrm{Ker}(h_i), \overline{\phi}_\alpha\big)$$

with $\overline{\phi}_\alpha(m_{s(\alpha)} + \mathrm{Ker}(h_{s(\alpha)})) = \phi_\alpha(m_{s(\alpha)}) + \mathrm{Ker}(h_{t(\alpha)})$. The morphism $j = (j_i)$ of $\mathrm{im}(h)$ consists of the natural inclusion maps $j_i : \mathrm{Im}(h_i) \hookrightarrow N_i$, and the morphism $p = (p_i)$ of $\mathrm{coim}(h)$ consists of the natural projection maps $p_i : M_i \twoheadrightarrow M_i/\mathrm{Ker}(h_i)$.

Definition 2.12. *A preabelian category $\mathcal{C}$ is an abelian category if for all morphisms h of $\mathcal{C}$, its parallel $\overline{h} : \mathrm{coim}(h) \to \mathrm{im}(h)$ is an isomorphism.*

Informally a preabelian category is abelian if the first isomorphism theorem holds true for all morphisms. The relation to the first isomorphism theorem becomes clear from the example Rep_Q.

Theorem 2.13. Rep_Q *is an abelian category.*

Proof. Let $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two representations of Q and $h = (h_i)_i \in \mathrm{Hom}(M, N)$. Then $\overline{h} = (\overline{h}_i)_i$ with

$$\overline{h}_i : M_i/\mathrm{Ker}(h_i) \to \mathrm{Im}(h_i)$$

given by $\overline{h}_i(m + \mathrm{Ker}(h_i)) = h_i(m)$ for $m \in M_i$. By the first isomorphism theorem for linear maps between vector spaces, $\overline{h}_i$ is a linear isomorphism for all $i \in Q_0$. Hence $\overline{h}$ is an isomorphism. □

Remark 2.14. The category Mod_R of right R-modules over a ring R is an abelian category, with the obvious constructions of direct sums, kernels and cokernels. In particular, Mod_{kQ} is an abelian category. The equivalence $\mathcal{F} : \mathrm{Rep}_Q \to \mathrm{Mod}_{kQ}$ of categories from Section 1 is in fact an equivalence of abelian categories (it respects the additive structures). In particular, it respects (co)kernels.

3. Indecomposable representations

In this section we start with the detailed study of quivers representations. We discuss two useful results in this context: the splitting lemma and the Krull-Schmidt Theorem. We construct the simple quiver representations and give examples of indecomposable representations.

Since we focus on Rep_Q, we fix some objects which we defined earlier up to isomorphism. Concretely, if $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ are representations of Q and $f = (f_i)_i \in \mathrm{Hom}(M, N)$ is a morphism, then we set

$$\begin{aligned} \ker(f) &:= \big(\mathrm{Ker}(f_i), \phi_\alpha|_{\mathrm{Ker}(f_{s(\alpha)})}\big), \\ \mathrm{coker}(f) &:= \big(N_i/\mathrm{Im}(f_i), \overline{\psi}_\alpha\big), \\ \mathrm{coim}(f) &:= \big(M_i/\mathrm{Ker}(f_i), \overline{\phi}_\alpha\big), \\ \mathrm{im}(f) &:= \big(\mathrm{Im}(f_i), \psi_\alpha|_{\mathrm{Im}(f_{s(\alpha)})}\big) \end{aligned}$$

with $\overline{\psi}_\alpha(n+\mathrm{Im}(f_{s(\alpha)})) := \psi_\alpha(n)+\mathrm{Im}(f_{t(\alpha)})$ for $n \in N_{s(\alpha)}$ and $\overline{\phi}_\alpha(m+\mathrm{Ker}(f_{s(\alpha)})) := \phi_\alpha(m) + \mathrm{Ker}(f_{t(\alpha)})$ for $m \in M_{s(\alpha)}$. In addition we fix the direct sum of two quiver representations as in Section 2.

Recall that quivers are assumed to be finite without cycles, and that quiver representations are finite dimensional unless specified otherwise.

3.1. Simple and indecomposable modules.

Definition 3.1. *Let M be a representation of the quiver Q.*
(i) *M is called indecomposable if $M \not\simeq 0$ and $M \simeq N \oplus L$ as quiver representations implies $N \simeq 0$ or $L \simeq 0$.*
(ii) *The quiver representation L is said to be a subrepresentation of M if there exists a monomorphism $L \hookrightarrow M$.*
(iii) *M is simple if $M \not\simeq 0$ and $L \hookrightarrow M$ implies $L \simeq 0$ or $L \simeq M$.*

A simple representation is indecomposable. A representation which is not indecomposable is said to be decomposable.

A *sink* (resp. *source*) of the quiver Q is a vertex i without outgoing (resp. incoming) arrows. Since Q is a finite quiver without cycles, Q has at least one sink (check this!).

For $i \in Q_0$ define the representation $S(i) = (S(i)_j, \phi_\alpha)$ of Q by

$$S(i)_j = \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i \end{cases}$$

and $\phi_\alpha = 0$ for $\alpha \in Q_1$.

Proposition 3.2. *$S(i)$ $(i \in Q_0)$ is a complete set of representatives of the iso-classes of simple representations of Q.*

Proof. Looking at the dimension vectors of $S(i)$ it is clear that the $S(i)$ $(i \in Q_0)$ are inequivalent simple representations of Q.

Let $M = (M_i, \varphi_\alpha) \not\simeq 0$ be a quiver representation and define the subquiver $Q^M = (Q_0^M, Q_1^M, s, t)$ by $Q_0^M := \{i \in Q_0 \mid M_i \neq \{0\}\}$,

$$Q_1^M := \{\alpha \in Q_1 \mid s(\alpha), t(\alpha) \in Q_0^M\},$$

and source and target maps the restrictions of the source and target maps of the quiver Q to Q_1^M. Note that $Q_0^M \neq \emptyset$ since $M \not\simeq 0$. Let $i \in Q_0^M$ be a sink. Let $f_i : S(i)_i = k \to M_i$ be some nonzero linear map (which exists since $M_i \neq \{0\}$) and take $f_j : S(i)_j = 0 \to M_j$ to be the zero map for $j \neq i$. We claim that $f = (f_j)_j : S(i) \to M$ is a morphism.

The only nontrivial commutative diagrams are the ones involving $\alpha \in Q_1$ with $s(\alpha) = i$ or $t(\alpha) = i$. If $s(\alpha) = i$ then $t(\alpha) = j$ with $j \neq i$ and $j \notin Q_0^M$ since i is a sink of Q^M. Hence $M_j = \{0\}$ and

$$\begin{array}{ccc} S(i)_i = k & \xrightarrow{\phi_\alpha = 0} & S(i)_j = \{0\} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_j = 0} \\ M_i & \xrightarrow{\varphi_\alpha = 0} & M_j = \{0\} \end{array}$$

commutes. If $t(\alpha) = i$ then $s(\alpha) = j$ with $j \neq i$ and the diagram

$$\begin{array}{ccc} S(i)_j = \{0\} & \xrightarrow{\phi_\alpha = 0} & S(i)_i = k \\ \downarrow{\scriptstyle f_j = 0} & & \downarrow{\scriptstyle f_i} \\ M_j & \xrightarrow{\varphi_\alpha} & M_i \end{array}$$

commutes. Hence f is a monomorphism $f : S(i) \hookrightarrow M$.

If M in addition is a simple representation, then $f : S(i) \hookrightarrow M$ necessarily is an isomorphism. This completes the proof of the proposition. □

We have the direct sum of right kQ-modules

$$kQ = \bigoplus_{i \in Q_0} e_i kQ$$

with

$$e_i kQ = \mathrm{span}_k\{c \mid c \in \mathcal{P} : \ s(c) = i\}.$$

Recall the equivalence $\mathcal{F} : \mathrm{Mod}_{kQ} \to \mathrm{Rep}_Q$ of categories from Section 1. We write

$$P(i) = (P(i)_j, \phi_\alpha)_{j \in Q_0, \alpha \in Q_1} := \mathcal{F}\big(e_i kQ\big).$$

Note that $P(i)_j = e_i(kQ)e_j = \mathrm{span}_k\{c \mid c \in \mathcal{P} : \ s(c) = i \ \& \ t(c) = j\}$ with connecting map $\phi_\alpha : P(i)_{s(\alpha)} \to P(i)_{t(\alpha)}$ the linear map satisfying $\phi_\alpha(c) := c\alpha$ for paths $c \in \mathcal{P}$ with $s(c) = i$ and $t(c) = s(\alpha)$.

Example 3.3. **(i)** *Consider the quiver*

$$\begin{array}{ccccccccc} 1 & \longleftarrow & 2 & \longleftarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 \\ & & \uparrow & & & & & & \\ & & 3 & & & & & & \end{array}$$

Then $P(4)$ is isomorphic to

$$\begin{array}{ccccccccc} k & \xleftarrow{1} & k & \xleftarrow{1} & k & \xrightarrow{1} & k & \xrightarrow{1} & k \\ & & {\scriptstyle 0}\uparrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

where 1 stands for the identity map.
(ii) *Consider the quiver*

$$1 \rightrightarrows 2$$

In this case $P(1)$ is isomorphic to

$$k \overset{\binom{1}{0}}{\underset{\binom{0}{1}}{\rightrightarrows}} k^2$$

Exercise 3.4. *Show that $P(i) \simeq S(i) \Leftrightarrow i \in Q_0$ is a sink.*

Proposition 3.5. *$P(i)$ is indecomposable.*

Proof. Suppose M and N are subrepresentations of $P(i)$ such that $P(i) = M \oplus N$. Since $P(i)_i = ke_i$ is one-dimensional, we have $e_i \in M_i$ or $e_i \in N_i$. Without loss of generality we assume that $e_i \in M_i$.

Let $c = \alpha_1 \cdots \alpha_r \in \mathcal{P}$ be a path starting at the vertex i. Then

$$c = \phi_{\alpha_r}\big(\cdots \phi_{\alpha_2}\big(\phi_{\alpha_1}(e_i)\big)\cdots\big)$$

by the definition of the connecting maps ϕ_α of the representation $P(i)$. Since $M \subseteq P(i)$ is a subrepresentation, i.e. $\phi_\alpha\big(M_{s(\alpha)}\big) \subseteq M_{t(\alpha)}$ for all $\alpha \in Q_1$, we conclude that $c \in M_{t(c)}$. This shows that $P(i)_j \subseteq M_j$ for all $j \in Q_0$, hence $M = P(i)$ and $N = 0$. □

We have the following dual version of the representation $P(i)$. Let $i \in Q_0$, then

$$I(i) := \big(I(i)_j, \varphi_\alpha\big)_{j \in Q_0, \alpha \in Q_1}$$

is defined by

$$I(i)_j := e_j(kQ)e_i = \operatorname{span}_k\{c \mid c \in \mathcal{P} :\ s(c) = j \ \&\ t(c) = i\}$$

with $\varphi_\alpha : I(i)_{s(\alpha)} \to I(i)_{t(\alpha)}$ the linear map defined by

$$\varphi_\alpha(c) := \begin{cases} \alpha_2 \cdots \alpha_r & \text{if } \alpha_1 = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

for paths $c = \alpha_1\alpha_2 \cdots \alpha_r \in \mathcal{P}$ starting at $s(\alpha)$ and ending at i (if the edge α starts at i, then the connecting map $\varphi_\alpha : I(i)_i = ke_i \to I(i)_{t(\alpha)}$ is defined by $\varphi_\alpha(e_i) := 0$).

Exercise 3.6. *Show that $I(i)$ is an indecomposable representation of Q.*

3.2. **Splitting lemma.** Exact sequences and the splitting lemma have been discussed in [7, Chpt. 2] for the abelian category of modules over a ring R. In this section we give the definitions and results for Rep_Q, which are obtainable from the constructions and results for module categories through the equivalence $\mathcal{F} : \mathrm{Mod}_{kQ} \xrightarrow{\sim} \mathrm{Rep}_Q$ of categories from Section 1. In fact, this is a general fact for any abelian category by a well-known theorem of Freyd and Mitchell stating that an abelian category is equivalent to a module category over some ring.

With the equivalence to Mod_{kQ} in mind, let us introduce some convenient notations for quiver representations. Let $M = (M_i, \phi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two quiver representations and $f = (f_i)_i \in \mathrm{Hom}(M, N)$ a morphism. Then we call

$$m := (m_i)_{i\in Q_0} \in \bigoplus_{i\in Q_0} M_i$$

an element of the representation M, and we write $m \in M$. We write $f(m) := (f_i(m_i))_i \in \bigoplus_{i\in Q_0} N_i$ and call it the evaluation of f at m. In this way we can verify identities between morphisms of quiver representations "pointwise".

Definition 3.7. *Let L, M, N be quiver representations and fix two morphisms $f \in \mathrm{Hom}(L, M)$ and $g \in \mathrm{Hom}(M, N)$.*

(i) *$L \xrightarrow{f} M \xrightarrow{g} N$ is said to be exact at M if $\mathrm{im}(f) = \ker(g)$.*

(ii) *An arbitrary (finite or (semi-)infinite) sequence*

$$\cdots \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow \cdots$$

of morphisms is called exact if it is exact at all M_j's in the bulk of the sequence.

(iii) *An exact sequence of the form $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is called short exact. Concretely, this means that it is exact at L (f is a monomorphism), at M ($\mathrm{im}(f) = \ker(g)$) and at N (g is an epimorphism). We then call M the extension of N by L.*

For $g \in \mathrm{Hom}(M, N)$ we have the exact sequence

$$0 \longrightarrow \ker(g) \hookrightarrow M \xrightarrow{g} N \twoheadrightarrow \mathrm{coker}(g) \longrightarrow 0$$

and short exact sequences

$$0 \longrightarrow \ker(g) \hookrightarrow M \twoheadrightarrow \mathrm{coim}(g) \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{im}(g) \hookrightarrow N \twoheadrightarrow \mathrm{coker}(g) \longrightarrow 0.$$

The direct sum construction of representations provides another important example of short exact sequences. Let $M = (M_i, \phi_\alpha), N = (N_i, \psi_\alpha)$ be representations of Q and let $(M \oplus N, \iota_M, \iota_N)$ be its direct sum. Recall that $M \oplus N := (M_i \oplus N_i, \phi_\alpha \oplus \psi_\alpha)$ and $\iota_M \in \operatorname{Hom}(M, M \oplus N)$, $\iota_N \in \operatorname{Hom}(N, M \oplus N)$ are given by

$$\iota_{M,i}(m) := (m, 0), \qquad \iota_{N,i}(n) := (0, n)$$

for $m \in M_i$ and $n \in N_i$. Let $\pi_M \in \operatorname{Hom}(M \oplus N, M)$ and $\pi_N \in \operatorname{Hom}(M \oplus N, N)$ be the corresponding canonical epimorphisms, characterized by $\pi_M \circ \iota_M = 1_M$, $\pi_M \circ \iota_N = 0$ and $\pi_N \circ \iota_M = 0$, $\pi_N \circ \iota_N = 1_N$ (see Section 2). Then

$$0 \longrightarrow M \overset{\iota_M}{\hookrightarrow} M \oplus N \overset{\pi_N}{\twoheadrightarrow} N \longrightarrow 0 \tag{3.1}$$

is a short exact sequence.

Note that in this short exact sequence the morphism $\iota_M \in \operatorname{Hom}(M, M \oplus N)$ has a left inverse and $\pi_N \in \operatorname{Hom}(M \oplus N, N)$ has a right inverse. Indeed, $\pi_M \in \operatorname{Hom}(M \oplus N, M)$ and $\iota_N \in \operatorname{Hom}(N, M \oplus N)$ are the left-inverse of ι_M and the right-inverse of π_N, respectively ($\pi_M \circ \iota_M = 1_M$ and $\pi_N \circ \iota_N = 1_N$). In other words, ι_M is a so-called section and π_N a retraction:

Definition 3.8. *Let M, N be quiver representations and $f \in \operatorname{Hom}(M, N)$.*
(i) *f is called a section if there exists a $p \in \operatorname{Hom}(N, M)$ such that $p \circ f = 1_M$ (i.e., if f has a left inverse).*
(ii) *f is called a retraction if there exists a $\iota \in \operatorname{Hom}(N, M)$ such that $f \circ \iota = 1_N$ (i.e., if f has a right inverse).*

Note that a section (resp. retraction) is necessarily a monomorphism (resp. epimorphism). The splitting lemma gives criteria for short exact sequences to be of the form of (3.1).

Theorem 3.9 (Splitting lemma)**.** *Suppose that*

$$0 \longrightarrow L \overset{f}{\hookrightarrow} M \overset{g}{\twoheadrightarrow} N \to 0$$

is a short exact sequence in Rep_Q. The following statements are equivalent.

- **a.** *f is a section.*
- **b.** *g is a retraction.*
- **c.** *There exists an isomorphism $\beta : M \xrightarrow{\sim} L \oplus N$ such that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xhookrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_L} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & L & \xhookrightarrow{\iota_L} & L \oplus N & \xrightarrow{\pi_N} & N & \longrightarrow & 0 \end{array}$$

with the two squares commuting.

Proof. For each implication we give the construction of the desired map leaving the remaining details to the reader (which are analogous to [7, Proof of Thm. 2.3]).
(iii)⇒(i): $p := \pi_L \circ \beta \in \operatorname{Hom}(M, L)$ is a left-inverse of f.
(iii)⇒(ii): $\iota := \beta^{-1} \circ \iota_N \in \operatorname{Hom}(N, M)$ is a right-inverse of g.
(i)⇒(iii): $\beta \in \operatorname{Hom}(M, L \oplus N)$ is defined by $\beta(m) := (p(m), g(m))$ for $m \in M$, with $p \in \operatorname{Hom}(M, L)$ a left-inverse of f.
(ii)⇒(i): let $\iota \in \operatorname{Hom}(N, M)$ be a right-inverse of g. Define $p \in \operatorname{Hom}(M, L)$ by $p(m) = \ell$ with $\ell \in L$ such that $f(\ell) = m - \iota(g(m))$. Convince yourself that f is well defined and a morphism. Then it immediately follows that p is a left-inverse of f. □

If the equivalent conditions of the theorem holds true then we say that the short exact sequence splits.

Exercise 3.10. *Consider the quiver $Q = 1 \longrightarrow 2$ and its representation M given by*

$$k \xrightarrow{1} k.$$

Show that

$$0 \longrightarrow S(2) \xrightarrow{f} M \xrightarrow{g} S(1) \longrightarrow 0$$

with $f = (f_1, f_2) = (0, 1)$ and $g = (g_1, g_2) = (1, 0)$ is a well defined short exact sequence. Determine whether the short exact sequence splits.

Exercise 3.11. *Let M be a quiver representation and $e \in \operatorname{End}(M)$ an idempotent. Show that the short exact sequence*

$$0 \to \ker(e) \hookrightarrow M \twoheadrightarrow \operatorname{coim}(e) \to 0$$

splits.

3.3. **The Krull-Schmidt Theorem.** An endomorphism $f \in \operatorname{End}(N)$ is said to be nilpotent if $f^n = 0$ for some $n \in \mathbb{N}$.

Lemma 3.12 (Fitting lemma)**.** *Let N be an indecomposable quiver representation and $f \in \operatorname{End}(N)$. Then f is an isomorphism or f is nilpotent.*

Proof. We have inclusions of subrepresentations

$$\ker(f) \subseteq \ker(f^2) \subseteq \cdots \subseteq N,$$
$$\mathrm{N} \supseteq \operatorname{im}(f) \supseteq \operatorname{im}(f^2) \supseteq \cdots$$

Considering the dimensions of the subrepresentations in these sequences, only finitely many inclusions can be strict. Hence there exists an $n \in \mathbb{N}$ such that $\ker(f^n) = \ker(f^{n+1}) = \dots$ and $\operatorname{im}(f^n) = \operatorname{im}(f^{n+1}) = \dots$

Let $x \in N$ and choose $y \in N$ such that $f^n(x) = f^{2n}(y)$ (which is possible since $\operatorname{im}(f^n) = \operatorname{im}(f^{2n})$). Then

$$x = (x - f^n(y)) + f^n(y)$$

is a decomposition of x as a sum of an element in $\ker(f^n)$ and $\mathrm{im}(f^n)$. Hence

$$\mathrm{Ker}(f_i^n) + \mathrm{Im}(f_i^n) = N_i \qquad \forall\, i \in Q_0.$$

Let $x_i \in \mathrm{Ker}(f_i^n) \cap \mathrm{Im}(f_i^n)$. Let $y_i \in N_i$ such that $x_i = f_i^n(y_i)$. Then $0 = f_i^n(x_i) = f_i^{2n}(y_i)$, hence $y_i \in \mathrm{Ker}(f_i^{2n}) = \mathrm{Ker}(f_i^n)$. Consequently $x_i = f_i^n(y_i) = 0$. It follows that

$$\mathrm{Ker}(f_i^n) \oplus \mathrm{Im}(f_i^n) = N_i \qquad \forall\, i \in Q_0.$$

Since $\ker(f^n)$ and $\mathrm{im}(f^n)$ are subrepresentations of N, it follows that

$$N = \ker(f^n) \oplus \mathrm{im}(f^n)$$

as quiver representations. N is indecomposable, hence $\ker(f^n) = 0$ or $\mathrm{im}(f^n) = 0$. If $\mathrm{im}(f^n) = 0$ then $f^n = 0$ and f is nilpotent.

If $\ker(f^n) = 0$ then $\ker(f) = 0$, i.e. $f : N \hookrightarrow N$ is a monomorphism. In this case we also have $\mathrm{im}(f^n) = N$, hence f is an isomorphism. □

Remark 3.13. Fitting's lemma is valid in a module category Mod_R over a ring R if the indecomposable module N is noetherian and artinian. Noetherian means that any increasing sequence $N_1 \subseteq N_2 \subseteq \cdots \subseteq N$ of subrepresentations of N eventually stabilizes, and artinian means that any decreasing sequence $N \supseteq N_1 \supseteq N_2 \supseteq \cdots$ of subrepresentations eventually stabilizes.

For a ring R we write $R^\times$ for its group of units.

Definition 3.14. *A ring R is a local ring if $0 \neq 1$ and if $a, b \in R$ and $a + b \in R^\times$ imply that $a \in R^\times$ or $b \in R^\times$.*

Corollary 3.15. *If N is an indecomposable quiver representation then* $\mathrm{End}(N)$ *is a local ring.*

Proof. Let $f, g \in \mathrm{End}(N)$ with $f + g \in \mathrm{End}(N)^\times$ (i.e., $f + g$ is an isomorphism). Let $h \in \mathrm{End}(N)^\times$ be the inverse of $f + g$. Suppose that g is not an isomorphism. Then $g \circ h$ is not an isomorphism, hence $(g \circ h)^n = 0$ for some $n \in \mathbb{N}$ by Fitting's lemma. Since $f \circ h = 1_N - (g \circ h)$ we conclude that $f \circ h \in \mathrm{End}(N)^\times$ with inverse

$$1_N + (g \circ h) + \cdots + (g \circ h)^{n-1}.$$

Hence $f \in \mathrm{End}(N)^\times$. □

Exercise 3.16. *Prove the converse statement: if N is a quiver representation and* $\mathrm{End}(N)$ *is a local ring, then N is indecomposable.*

Lemma 3.17. *Let V_1, V_2, W_1, W_2 be quiver representations and $\phi : V_1 \oplus V_2 \to W_1 \oplus W_2$ an isomorphism. Write $\iota_i : V_i \hookrightarrow V_1 \oplus V_2$, $\pi_i : V_1 \oplus V_2 \twoheadrightarrow V_i$ for the canonical inclusion and projection morphisms, and similarly for $\iota'_i : W_i \hookrightarrow W_1 \oplus W_2$ and $\pi'_i : W_1 \oplus W_2 \twoheadrightarrow W_i$. If $\pi'_1 \circ \phi \circ \iota_1 : V_1 \to W_1$ is an isomorphism, then $V_2 \simeq W_2$.*

Proof. Write $\phi_{ij} := \pi_i' \circ \phi \circ \iota_j : V_j \to W_i$, and view ϕ as a 2×2-matrix with entries ϕ_{ij},

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : V_1 \oplus V_2 \to W_1 \oplus W_2$$

(the composition of morphisms becomes matrix multiplication). By assumption ϕ_{11} is an isomorphism. Consider the morphism

$$\psi := \begin{pmatrix} 1_{W_1} & 0 \\ -\phi_{21}\phi_{11}^{-1} & 1_{W_2} \end{pmatrix} : W_1 \oplus W_2 \to W_1 \oplus W_2.$$

Clearly ψ is an isomorphism, with inverse

$$\psi^{-1} := \begin{pmatrix} 1_{W_1} & 0 \\ \phi_{21}\phi_{11}^{-1} & 1_{W_2} \end{pmatrix}.$$

Hence

$$\psi \circ \phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} - \phi_{22}\phi_{11}^{-1}\phi_{12} \end{pmatrix} \in \mathrm{End}(V_1 \oplus V_2, W_1 \oplus W_2)$$

is an isomorphism. Write

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} : W_1 \oplus W_2 \to V_1 \oplus V_2$$

for the inverse of $\psi \circ \phi$. Then $\phi_{22} - \phi_{22}\phi_{11}^{-1}\phi_{12} \in \mathrm{End}(V_2, W_2)$ is an isomorphism with inverse α_{22}. □

Theorem 3.18 (Krull-Schmidt)**.** *Let $M = (M_i, \phi_\alpha)$ be a quiver representation. There exists indecomposable quiver representations $N_1, \ldots, N_r$ such that*

$$M \simeq N_1 \oplus \cdots \oplus N_r.$$

The decomposition is unique up to order and isomorphism. We call r the number of indecomposable factors of M.

Proof. The existence of a decomposition as a direct sum of indecomposable subrepresentations follows by induction to $\sum_{i \in Q_0} \mathrm{Dim}_k(M_i)$. Suppose that

$$N_1 \oplus \cdots \oplus N_r = N_1' \oplus \cdots \oplus N_s'$$

are two decompositions of M as direct sums of indecomposable subrepresentations. Let $\iota_i \in \mathrm{Hom}(N_i, M)$ (resp. $\iota_j' \in \mathrm{Hom}(N_j', M)$) be the inclusion of N_i (resp. N_j') in M along the first (resp. second) decomposition. Similarly, let $\pi_i \in \mathrm{Hom}(M, N_i)$ (resp. $\pi_j' \in \mathrm{Hom}(M, N_j')$) be the projection on N_i (resp. N_j') along the first (resp. second) decomposition. Write

$$v_i := \pi_1 \circ \iota_i' \in \mathrm{Hom}(N_i', N_1), \qquad w_i := \pi_i' \circ \iota_1 \in \mathrm{Hom}(N_1, N_i').$$

Then $v_i \circ w_i \in \mathrm{End}(N_1)$ and

$$\sum_{i=1}^{s} v_i \circ w_i = \pi_1 \circ \big(\sum_{i=1}^{s} \iota_i' \circ \pi_i'\big) \circ \iota_1 = \pi_1 \circ 1_M \circ \iota_1 = 1_{N_1}$$

in $\mathrm{End}(N_1)$. Since $\mathrm{End}(N_1)$ is a local ring, $v_i \circ w_i \in \mathrm{End}(N_1)^\times$ for at least one i. We may assume that $i = 1$ without loss of generality.

From $v_1 \circ w_1 \in \mathrm{End}(N_1)^\times$ it follows that v_1 is an epimorphism and w_1 a monomorphism. We will show now first that v_1 is in fact an isomorphism.

Write $\alpha := (v_1 \circ w_1)^{-1}$. Then $v_1 \circ (w_1 \circ \alpha) = 1_{N_1}$, hence v_1 is a retraction. By the splitting lemma the short exact sequence

$$0 \longrightarrow \ker(v_1) \hookrightarrow N_1' \overset{v_1}{\twoheadrightarrow} N_1 \longrightarrow 0$$

splits. Hence $N_1' \simeq \ker(v_1) \oplus N_1$. But N_1' is indecomposable and $N_1 \not\simeq 0$, so $\ker(v_1) = 0$. Hence $v_1 : N_1' \to N_1$ is an isomorphism.

By Lemma 3.17 we then have

$$N_2 \oplus \cdots \oplus N_r \simeq N_2' \oplus \cdots \oplus N_s'.$$

By induction to $\min(r, s)$ we conclude that $r = s$ and that $N_i \simeq N'_{\sigma(i)}$ for some permutation σ of $\{2, \ldots, r\}$. This concludes the proof. $\square$

Exercise 3.19. **(i)** *Show that the quiver representation*

$$M = k^2 \overset{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}{\longrightarrow} k^2 \overset{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\longleftarrow} k$$

is decomposable.

(ii) *What is the number of indecomposable factors of M?*

4. Projective representations

Understanding how arbitrary quiver representations decompose in indecomposable representations is a difficult task. As a first step we analyze the problem for a subcategory of representations, the so-called *projective* representations. The text is based on [4, §1.4, §2.1].

4.1. **Definition of projective quiver representations.** Let Vect_k be the abelian category of finite dimensional vector spaces over k (i.e. the objects are finite dimensional k-vector spaces, and morphisms are k-linear maps). We recall from e.g. [7, Example 5.4] the definition of the covariant hom-functor in our present context.

Definition 4.1. *Let $P \in \mathrm{Ob}(\mathrm{Rep}_Q)$ be a finite dimensional representation of the quiver Q. The associated (covariant) hom-functor* $\mathrm{Hom}(P,-) : \mathrm{Rep}_Q \to \mathrm{Vect}_k$ *is defined by:*

$$M \mapsto \mathrm{Hom}(P, M)$$

for objects $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ and

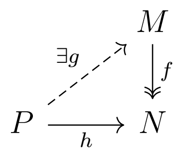
$$f \mapsto f_*$$

for morphisms $f \in \mathrm{Hom}(M,N)$, where $f_* : \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N)$ is defined by $f_*(g) := f \circ g$ (f_* is called the push-forward of f).

Note that $\mathrm{Hom}(P,-) : \mathrm{Rep}_Q \to \mathrm{Vect}_k$ is well defined since Rep_Q is a k-linear category.

Definition 4.2. *A quiver representation P is projective if the following property is true: if f is epimorphism in Rep_Q then f_* is an epimorphism in Vect_k.*

In other words, P is projective if the following *lifting property* holds true: if $h : P \to N$ is a morphism and $f : M \twoheadrightarrow N$ is an epimorphism, then there exists a morphism $g : P \to M$ such that $f_*(g) = f \circ g = h$ (i.e. the codomain of h is lifted to M through the epimorphism f). In a diagram,



with the triangle commuting. Note that the zero representation 0 is projective. We will next show that the indecomposable representations $P(i)$ ($i \in Q_0$) are projective. It is a consequence of the following description of the hom-spaces $\mathrm{Hom}(P(i), M)$.

Theorem 4.3. *Let $i \in Q_0$ and $M = (M_j, \varphi_\alpha)$ a quiver representation. The map*

$$\mathrm{Hom}(P(i), M) \to M_i, \qquad f = (f_j)_j \mapsto f_i(e_i) \tag{4.1}$$

is a k-linear isomorphism.

Proof. Write $P(i) = (P(i)_j, \phi_\alpha)$ and denote $\psi_i(f) := f_i(e_i)$ ($f \in \mathrm{Hom}(P(i), M)$) for the linear map (4.1). If $c = \alpha_1 \cdots \alpha_r \in P(i)_j$ is a path from i to j then

$$f_j(c) = f_j\big((\phi_{\alpha_r} \cdots \phi_{\alpha_1})(e_i)\big) = \big(\varphi_{\alpha_r} \cdots \varphi_{\alpha_1}\big)(f_i(e_i)) = \big(\varphi_{\alpha_r} \cdots \varphi_{\alpha_1}\big)(\psi_i(f)). \tag{4.2}$$

Hence $\psi_i(f) = 0$ implies that $f = 0$. Formula (4.2) also shows how to construct the pre-image of $m_i \in M_i$ under ψ_i: it is given by the linear extension of the assignment

$$f_j(c) := \big(\varphi_{\alpha_r} \cdots \varphi_{\alpha_1}\big)(m_i)$$

for paths $c \in P(i)_j$ from i to j. □

Corollary 4.4. *Let $i \in Q_0$. Then $P(i)$ is projective.*

Proof. Let $M = (M_j, \varphi_\alpha), N = (N_j, \psi_\alpha) \in \mathrm{Ob}(\mathrm{Rep}_Q)$ and $f = (f_j)_j : M \twoheadrightarrow N$ an epimorphism. Suppose that $h = (h_j)_j \in \mathrm{Hom}(P(i), N)$. We need to show that h can be lifted to a morphism $g = (g_j)_j : P(i) \to M$ such that $f \circ g = h$.

Let $m_i \in M_i$ such that $f_i(m_i) = h_i(e_i)$. Note that m_i exists since $f_i : M_i \twoheadrightarrow N_i$ is surjective. Take $g = (g_j)_j \in \mathrm{Hom}(P(i), M)$ such that $g_i(e_i) = m_i$, which exists by Theorem 4.3. Then $f_i(g_i(e_i)) = h_i(e_i)$. Hence, by Theorem 4.3 again, $f \circ g = h$. □

Exercise 4.5. *Show that $P(i) \simeq P(j) \Leftrightarrow i = j$.*

4.2. **Direct sums and decompositions of projective representations.** We next study how projective quiver representations behave with respect to direct sums. First we show that projective quiver representations only have trivial extensions.

Proposition 4.6. *Let P be a projective representation and $g : M \twoheadrightarrow P$ an epimorphism. Then g is a retraction and $M \simeq \ker(g) \oplus P$.*

Proof. Since P is projective, $g_* : \mathrm{Hom}(P, M) \to \mathrm{End}(P)$ is surjective. In particular there exists a morphism $f : P \to M$ such that $g_*(f) = 1_P$. Hence g is a retraction. The second statement now follows from the splitting lemma applied to the short exact sequence

$$0 \longrightarrow \ker(g) \hookrightarrow M \overset{g}{\twoheadrightarrow} P \to 0.$$

□

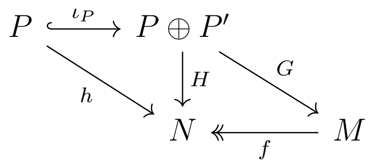
The next proposition show that the full subcategory of projective quiver representations is an additive subcategory of Rep_Q.

Proposition 4.7. *Let P and P' be two quiver representations. The following two statements are equivalent.*

- **a.** *$P \oplus P'$ is projective.*
- **b.** *P and P' are projective.*

Proof. **a** $\Rightarrow$ **b**. We show that P is projective (the argument for P' is the same). Fix a morphism $h : P \to N$ and an epimorphism $f : M \twoheadrightarrow N$.

Let $H : P \oplus P' \to N$ be the unique morphism such that $H \circ \iota_P = h$ and $H \circ \iota_{P'} = 0$ (with $\iota_P : P \hookrightarrow P \oplus P'$ and $\iota_{P'} : P' \hookrightarrow P \oplus P'$ the canonical monomorphisms). Since $P \oplus P'$ is projective by assumption, there exists a morphism $G : P \oplus P' \to M$ such that $f \circ G = H$. In other words, we have the diagram

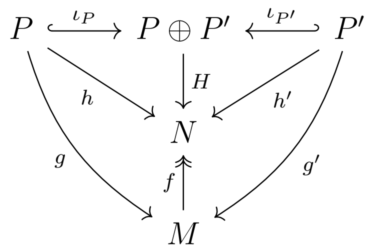


with the two triangles commuting. then $g := G \circ \iota_P \in \operatorname{Hom}(P, M)$ provides the lift of h along f,

$$f_*(g) = f \circ g = f \circ G \circ \iota_P = h.$$

b $\Rightarrow$ **a**. Fix a morphism $H : P \oplus P' \to N$ and an epimorphism $f : M \twoheadrightarrow N$.

Let $h : P \to N$ and $h' : P' \to N$ be the morphisms $h := H \circ \iota_P$ and $h' := H \circ \iota_{P'}$. Since P and P' are projective by assumption, there exist morphisms $g : P \to M$ and $g' : P' \to M$ such that $f_*(g) = h$ and $f_*(g') = h'$. In other words, we have the diagram



with the four triangles commuting. Let $G : P \oplus P' \to M$ be the unique morphism such that $G \circ \iota_P = g$ and $G \circ \iota_{P'} = g'$. Then

$$f_*(G) = f \circ G : P \oplus P' \to N$$

satisfies

$$f_*(G) \circ \iota_P = f \circ g = h = H \circ \iota_P,$$
$$f_*(G) \circ \iota_{P'} = f \circ g' = h' = H \circ \iota_{P'}.$$

By the universal property of the direct sum, this implies that $f_*(G) = H$. □

The proposition not only shows that direct sums of projective representations are projective, but also direct summands of projective representations are projective (we say that M is a *direct summand* of P if there exists a quiver representation N such that $P \simeq M \oplus N$).

Define the quiver representation A by

$$A := \bigoplus_{i \in Q_0} P(i).$$

Note that, under the equivalence $\mathcal{G} : \mathrm{Rep}_Q \to \mathrm{Mod}_{kQ}$ of categories from Section 1,

$$\mathcal{G}(A) = kQ$$

viewed as right kQ-module by right multiplication in the path algebra.

Definition 4.8. *A quiver representation M is called free if $M \simeq A \oplus \cdots \oplus A$.*

By convention we consider the zero representation 0 to be free (isomorphic to the empty direct sum of A's).

Proposition 4.9. *A free quiver representation is projective.*

Proof. This follows from Proposition 4.7 and the fact that the $P(i)$'s are projective by Corollary 4.4. □

Next we derive a convenient characterization of projective quiver representations in terms of free modules. For its proof we need the following exercise.

Exercise 4.10. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories and $f : M \twoheadrightarrow N$ an epimorphism in $\mathcal{C}$. Show that $\mathcal{F}f : \mathcal{F}M \twoheadrightarrow \mathcal{F}N$ is an epimorphism in $\mathcal{D}$.*

Theorem 4.11. *Let P be a quiver representation. The following two statements are equivalent:*

- **a.** *P is projective.*
- **b.** *P is a direct summand of a free quiver representation.*

The quiver representation N such that $P \oplus N$ is a free quiver representation is also projective.

Proof. **b** $\Rightarrow$ **a**. By assumption $P \oplus N$ is free, hence projective by the previous proposition. By Proposition 4.4 we conclude that both P and N are projective.
a $\Rightarrow$ **b**. By Proposition 4.6 it suffices to show that there exists an epimorphism $g : A^{\oplus r} \twoheadrightarrow P$ for some $r \in \mathbb{Z}_{\geq 0}$ (then $P \oplus \ker(g) \simeq A^{\oplus r}$).

By Exercise 4.10, applied to the equivalence $\mathcal{G} : \mathrm{Rep}_Q \to \mathrm{Mod}_{kQ}$ of categories from Section 1, it suffices to show that for each finite dimensional right kQ-module M there exists an epimorphism

$$G : kQ^{\oplus r} \twoheadrightarrow M$$

of right kQ-modules for some $r \in \mathbb{Z}_{\geq 0}$.

Take $r := \mathrm{Dim}_k(M)$ and choose a k-linear basis $\{m_s\}_{s=1}^r$ of M. Then

$$G : kQ^{\oplus r} \twoheadrightarrow M$$

defined by $G((a_1, \ldots, a_r)) := \sum_{i=1}^r m_i \cdot a_i$ for $a_i \in kQ$ is such an epimorphism of right kQ-modules. □

Corollary 4.12. a. *$\{P(i)\}_{i \in Q_0}$ is a complete set of representatives of the isoclasses of the indecomposable projective quiver representations.*

b. *If P is a projective quiver representation then there exist unique $m(i) \in \mathbb{Z}_{\geq 0}$ $(i \in Q_0)$ such that*

$$P \simeq \bigoplus_{i \in Q_0} P(i)^{\oplus m(i)}.$$

Proof. Let P be a projective quiver representation. By Theorem 4.11 there exists a quiver representation P' such that

$$P \oplus P' \simeq A^{\oplus r} \simeq \bigoplus_{i \in Q_0} P(i)^{\oplus r}$$

for some $r \in \mathbb{Z}_{\geq 0}$, where the second isomorphism is by the definition of A. On the other hand, by the Krull-Schmidt theorem, both P and P' decomposes as a finite direct sum of indecomposable quiver representations P_j and P'_ℓ,

$$P \simeq P_1 \oplus \cdots \oplus P_s, \qquad P' \simeq P'_1 \oplus \cdots \oplus P'_t. \tag{4.3}$$

Hence we have now two decompositions of $P \oplus P'$ as direct sums of indecomposable quiver representations,

$$\bigoplus_{i \in Q_0} P(i)^{\oplus r} \simeq P \oplus P' \simeq P_1 \oplus \cdots \oplus P_s \oplus P'_1 \oplus \cdots \oplus P'_t.$$

By the Krull-Schmidt theorem again (uniqueness of the indecomposable decomposition factors up to isomorphism and permutation), each P_j is isomorphic to some $P(i)$ $(i \in Q_0)$. The proof of the corollary can now be completed as follows.

b. Since each indecomposable summand P_j of P in (4.3) is isomorphic to some $P(i)$ $(i \in Q_0)$, we have a decomposition in indecomposable quiver representations of the form

$$P \simeq \bigoplus_{i \in Q_0} P(i)^{\oplus m(i)}$$

for some $m(i) \in \mathbb{Z}_{\geq 0}$. The uniqueness of the $m(i)$'s follows from the Krull-Schmidt theorem and Exercise 4.5.

a. Assume that P is projective and indecomposable. Then $s = 1$ in (4.3), hence $P \simeq P(i)$ for some $i \in Q_0$. Furthermore, $P(i) \not\simeq P(j)$ if $i \neq j$ by Exercise 4.5. □

5. Projective resolutions

In this section we introduce the notion of a projective resolution. The idea of projective resolutions is to "resolve" a quiver representation M in projective ones by constructing an exact sequence of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with the P_i's being projective. It allows to study M in terms of the projective quiver representations occurring in the resolution. We also construct the so called standard projective resolution of a quiver representation and we show that Rep_Q is hereditary, meaning that sub-representations of projective quiver representations are projective again (not only direct summands!).

5.1. **The hom-functor.** We first make a short digression and discuss properties of the hom-functor.

Let M be a finite dimensional quiver representation. Recall the hom-functor $\mathrm{Hom}(M,-) : \mathrm{Rep}_Q \to \mathrm{Vect}_k$ mapping a finite dimensional quiver representation N to the vector space $\mathrm{Hom}(M,N)$ and a morphism $f \in \mathrm{Hom}(N,L)$ to its push-forward $f_* : \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,L)$, defined by $g \mapsto f \circ g$.

A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between abelian categories is called left-exact if

$$0 \longrightarrow \mathcal{F}(X) \stackrel{\mathcal{F}(f)}{\longrightarrow} \mathcal{F}(Y) \stackrel{\mathcal{F}(g)}{\longrightarrow} \mathcal{F}(Z)$$

is exact in $\mathcal{D}$ when

$$0 \longrightarrow X \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Z$$

is exact in $\mathcal{C}$.

Proposition 5.1. *The hom-functor* $\mathrm{Hom}(M,-) : \mathrm{Rep}_Q \to \mathrm{Vect}_k$ *is left exact.*

Proof. Let

$$0 \longrightarrow L \stackrel{f}{\longrightarrow} N \stackrel{g}{\longrightarrow} R \tag{5.1}$$

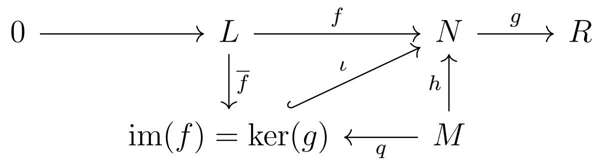
be an exact sequence.

1. $f_* : \mathrm{Hom}(M,L) \to \mathrm{Hom}(M,N)$ is injective: suppose $h \in \mathrm{Hom}(M,L)$ and $f_*(h) = f \circ h = 0$. Since f is a monomorphism, this implies that $h = 0$. Hence f_* is injective.

2. $\mathrm{Im}(f_*) \subseteq \mathrm{Ker}(g_*)$: this follows from the fact that $g_* \circ f_* = (g \circ f)_* = 0$ since $g \circ f = 0$.

3. $\mathrm{Ker}(g_*) \subseteq \mathrm{Im}(f_*)$: let $h \in \mathrm{Hom}(M,N)$ such that $g_*(h) = g \circ h = 0$.

Write $\iota : \ker(g) = \mathrm{im}(f) \hookrightarrow N$ for the monomorphism associated to $\ker(g) = \mathrm{im}(f)$. Since $g \circ h = 0$ there exists a morphism $q : M \to \ker(g)$ such that $h = \iota \circ q$. Furthermore, since f is a monomorphism, the first isomorphism theorem for f shows that $f = \iota \circ \overline{f}$ with $\overline{f} : L \stackrel{\sim}{\longrightarrow} \mathrm{im}(f)$ the parallel of f. Hence we have the diagram



with commuting triangles. Now set $r := \overline{f}^{-1} \circ q \in \mathrm{Hom}(M, L)$. Then $f_*(r) = f \circ r = h$, hence $\mathrm{Ker}(g_*) \subseteq \mathrm{Im}(f_*)$. □

Exercise 5.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ a left-exact functor. Show that $\mathcal{F}(\ker(f)) \simeq \ker(\mathcal{F}(f))$.*

A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between abelian categories is called exact if

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Z) \longrightarrow 0$$

is short exact in $\mathcal{D}$ when

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is short exact in $\mathcal{C}$.

Corollary 5.3. *If P is a finite dimensional projective quiver representation then $\mathrm{Hom}(P, -) : \mathrm{Rep}_Q \to \mathrm{Vect}_k$ is exact.*

Proof. Let

$$0 \longrightarrow L \overset{f}{\hookrightarrow} N \overset{g}{\twoheadrightarrow} R \longrightarrow 0$$

be a short exact sequence in Rep_Q. By the previous proposition

$$0 \longrightarrow \mathrm{Hom}(P, L) \xrightarrow{f_*} \mathrm{Hom}(P, N) \xrightarrow{g_*} \mathrm{Hom}(P, R) \longrightarrow 0$$

will be a short exact sequence if $g_* : \mathrm{Hom}(P, N) \to \mathrm{Hom}(P, R)$ is surjective. But this is the case since P is projective and g is an epimorphism. □

The hom-functor can also be used to verify that a sequence in Rep_Q is exact:

Proposition 5.4. *Let*

$$0 \longrightarrow L \xrightarrow{f} N \xrightarrow{g} R \tag{5.2}$$

be a sequence in Rep_Q such that

$$0 \longrightarrow \mathrm{Hom}(M, L) \xrightarrow{f_*} \mathrm{Hom}(M, N) \xrightarrow{g_*} \mathrm{Hom}(M, R)$$

is an exact sequence in Vect_k for all finite dimensional quiver representations M. Then (5.2) is exact in Rep_Q.

Proof. See the proof of [4, Thm. 1.10]. □

5.2. **Resolutions.**

Definition 5.5. *Let M be a quiver representation of the quiver Q.*
A projective resolution of M is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{5.3}$$

with the P_i's being projective quiver representations.

A projective resolution of the form

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \tag{5.4}$$

with $P_n \neq 0$ is said to have length n.

Every quiver representation M has projective resolutions. In fact, M admits *free* resolutions, i.e. exact sequences

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with the F_i's being free (hence projective) quiver representations. Indeed, write $r_0 := \sum_{i \in Q_0} \mathrm{Dim}_k(M_i)$ for the total dimension of M, then there exists an epimorphism

$$f_0 : F_0 := A^{\oplus r_0} \twoheadrightarrow M$$

by the proof of Theorem 4.11. Write $\iota_0 : \ker(f_0) \hookrightarrow F_0$ for the canonical monomorphism associated to the kernel of f_0 and set r_1 for the total dimension of $\ker(f_0)$. By the same argument there exists an epimorphism $\widetilde{f_1} : F_1 := A^{\oplus r_1} \twoheadrightarrow \ker(f_0)$, and we set

$$f_1 := \iota_0 \circ \widetilde{f_1} : F_1 \to F_0.$$

By construction $\mathrm{im}(f_1) = \ker(f_0)$, hence we have built the second step of the free resolution of M,

$$F_1 = A^{\oplus r_1} \xrightarrow{f_1} F_0 = A^{\oplus r_0} \xrightarrow{f_0} M.$$

Continuing inductively, we obtain a free resolution of M.

The minimal length of a projective resolution of M is called the *projective dimension* $\mathrm{pd}(M)$ of M.

Exercise 5.6. *Let M be a quiver representation.*
Prove: M is projective $\Leftrightarrow$ $\mathrm{pd}(M) = 0$.

We will see later that $\mathrm{pd}(M) \leq 1$ for all quiver representations M.

5.3. **The standard projective resolution.** We construct an explicit projective resolution for any quiver representation M (the so-called *standard projective resolution*). Denote $\otimes$ for the standard tensor product of vector spaces and k-linear maps. For a quiver representation $M = (M_i, \phi_\alpha)$ and a finite dimensional k-vector space we define a new quiver representation M^V by

$$M^V := \big(M_i \otimes V, \phi_\alpha \otimes \mathrm{Id}_V\big).$$

Note that $M^V \simeq M^{\oplus \mathrm{Dim}_k(V)}$ (the advantage of M^V is that its construction does not depend on a choice of a linear basis of V).

For $j \in Q_0$, define a linear map

$$g_j : \bigoplus_{i\in Q_0} P(i)_j \otimes M_i \to M_j$$

as follows: for $c = \alpha_1 \cdots \alpha_r \in \mathcal{P}$ a path from i to j in Q and $m_i \in M_i$ we set

$$g_j(c \otimes m_i) := \big(\phi_{\alpha_r} \cdots \phi_{\alpha_1}\big)(m_i).$$

Furthermore, for $j \in Q_0$ we construct a linear map

$$f_j : \bigoplus_{\alpha\in Q_1} P(t(\alpha))_j \otimes M_{s(\alpha)} \to \bigoplus_{i\in Q_0} P(i)_j \otimes M_i$$

as follows. For $\alpha \in Q_1$,

$$f_j^\alpha := f_j|_{P(t(\alpha))_j \otimes M_{s(\alpha)}} : P(t(\alpha))_j \otimes M_{s(\alpha)} \to \bigoplus_{i\in Q_0} P(i)_j \otimes M_i$$

is the linear map satisfying

$$f_j^\alpha(c \otimes m_{s(\alpha)}) := \alpha c \otimes m_{s(\alpha)} - c \otimes \phi_\alpha(m_{s(\alpha)})$$

for $c = \alpha_1 \cdots \alpha_r \in \mathcal{P}$ a path from $t(\alpha)$ to j and $m_{s(\alpha)} \in M_{s(\alpha)}$ (note that the image of f_j^α is contained in $(P(s(\alpha))_j \otimes M_{s(\alpha)}) \oplus (P(t(\alpha))_j \otimes M_{t(\alpha)})$).

Theorem 5.7 (Standard projective resolution)**.** *We keep the notations and setup as above. Consider the projective quiver representations*

$$P_0 := \bigoplus_{i\in Q_0} P(i)^{M_i}, \qquad P_1 := \bigoplus_{\alpha\in Q_1} P(t(\alpha))^{M_{s(\alpha)}}.$$

Then

- **a.** $f = (f_j)_j : P_0 \to P_1$ *and* $g = (g_j)_j : P_0 \to M$ *define morphisms of quiver representations.*
- **b.** *The sequence*

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

is exact.

Proof. **a.** Let $\gamma \in Q_1$, let $c = \alpha_1 \cdots \alpha_r$ be a path from i to $s(\gamma)$, and let $m_i \in M_i$. Then

$$\phi_\gamma\big(g_{s(\gamma)}(c \otimes m_i)\big) = \big(\phi_\gamma \phi_{\alpha_r} \cdots \phi_{\alpha_1}\big)(m_i) = g_{t(\gamma)}\big(c\gamma \otimes m_i\big),$$

which proves that $g = (g_j)_j$ is a morphism.

Fix $\alpha, \gamma \in Q_1$. Let $c = \alpha_1 \cdots \alpha_r$ be a path from $t(\alpha)$ to $s(\gamma)$ and $m_{s(\alpha)} \in M_{s(\alpha)}$. Then

$$\begin{aligned}\big(\phi_\gamma \otimes \mathrm{Id}\big)(f^\alpha_{s(\gamma)}(c \otimes m_{s(\alpha)})) &= \alpha c\gamma \otimes m_{s(\alpha)} - c\gamma \otimes \phi_\alpha(m_{s(\alpha)}) \\ &= f^\alpha_{t(\gamma)}(c\gamma \otimes m_{s(\alpha)} - c\gamma \otimes m_{s(\alpha)}),\end{aligned}$$

which shows that $f = (f_j)_j$ is a morphism.
b. This splits up in several steps.
Step 1: f is a monomorphism. Suppose that $\eta_j \in \bigoplus_{\alpha \in Q_1} P(t(\alpha))_j \otimes M_{s(\alpha)}$ and $f_j(\eta_j) = 0$. Write $\eta_j = \sum_{\alpha\in Q_1} \eta_j^\alpha$ with $\eta_j^\alpha \in P(t(\alpha))_j \otimes M_{s(\alpha)}$, and write

$$\eta_j^\alpha = \sum_{c \in \mathcal{P}_{t(\alpha),j}} c \otimes m_c^\alpha$$

with $\mathcal{P}_{t(\alpha),j}$ the set of paths starting at $t(\alpha)$ and ending at j and with $m_c^\alpha \in M_{s(\alpha)}$. Then

$$\begin{aligned} 0 = f_j(\eta_j) &= \sum_{\alpha\in Q_1} f_j^\alpha(\eta_j^\alpha) \\ &= \sum_{\alpha \in Q_1} \sum_{c\in\mathcal{P}_{t(\alpha),j}} \big(\alpha c \otimes m_c^\alpha - c \otimes \phi_\alpha(m_c^\alpha)\big) \in \bigoplus_{i\in Q_0} P(i)_j \otimes M_i. \end{aligned} \tag{5.5}$$

Hence, for each $i \in Q_0$,

$$\sum_{\alpha\in Q_1 : s(\alpha)=i} \ \sum_{c \in \mathcal{P}_{t(\alpha),j}} \alpha c \otimes m_c^\alpha = \sum_{\alpha \in Q_1 : t(\alpha)=i} \ \sum_{c\in\mathcal{P}_{t(\alpha),j}} c \otimes \phi_\alpha(m_c^\alpha) \tag{5.6}$$

in $P(i)_j \otimes M_i$. Consider the set $\mathcal{S}$ consisting of paths $c \in \mathcal{P}$ for which there exists an $\alpha \in Q_1$ such that $c \in \mathcal{P}_{t(\alpha),j}$ and $m_c^\alpha \neq 0$. If $\mathcal{S} \neq \emptyset$ then choose an $c' \in \mathcal{S}$ of maximal length. Let $\alpha' \in Q_1$ such that $c' \in \mathcal{P}_{t(\alpha'),j}$ and $m_{c'}^{\alpha'} \neq 0$. Suppose that $s(\alpha') = i$. Picking out on both sides of (5.6) the term with the basis element $\alpha' c'$ in the first tensor component we get $m_{c'}^{\alpha'} = 0$, which is a contradiction. Hence $\mathcal{S} = \emptyset$, i.e., $\eta_j = 0$.
Step 2: g is an epimorphism. For $m_j \in M_j$ we have $g_j(e_j \otimes m_j) = m_j$, hence each g_j is surjective.
Step 3: $g \circ f = 0$. Let $\alpha \in Q_1$, let $c = \alpha_1 \cdots \alpha_r \in \mathcal{P}$ a path from $t(\alpha)$ to j and $m_{s(\alpha)} \in M_{s(\alpha)}$. Then

$$\begin{aligned} (g_j \circ f_j^\alpha)(c \otimes m_{s(\alpha)}) &= g_j\big(\alpha c \otimes m_{s(\alpha)} - c \otimes \phi_\alpha(m_{s(\alpha)})\big) \\ &= (\phi_{\alpha_r} \cdots \phi_{\alpha_1}\phi_\alpha)(m_{s(\alpha)}) - (\phi_{\alpha_r}\cdots\phi_{\alpha_1})(\phi_\alpha(m_{s(\alpha)})) = 0. \end{aligned}$$

Step 4: $\mathrm{im}(f) = \ker(g)$. We have $\mathrm{Im}(f_j) \subseteq \mathrm{Ker}(g_j)$ by step 3. Let $\eta_j \in \mathrm{Ker}(g_j) \subseteq \bigoplus_{i\in Q_0} P(i)_j \otimes M_i$ and write $\eta_j = \sum_c c \otimes m_c$ with the sum over paths c ending at j and $m_c \in M_{s(c)}$. Let $\deg(\eta_j)$ be the length of the longest path c with $m_c \neq 0$ (it is set to be zero if $\eta_j = 0$). If $\deg(\eta_j) = 0$ then $\eta_j = e_j \otimes m$ with $m \in M_j$ and $0 = g_j(\eta_j) = m$, hence $\eta_j = 0$. If $r := \deg(\eta_j) > 0$ and $c = \alpha_1 \cdots \alpha_r$ is a path of length r ending at j with $m_c \neq 0$. Then

$$c \otimes m_c - (\alpha_2 \cdots \alpha_r) \otimes \phi_{\alpha_1}(m_c) = f_j^{\alpha_1}(\alpha_2 \cdots \alpha_r \otimes m_c) \in \mathrm{Im}(f_j),$$

hence the term $c \otimes m_c$ can be removed in η_j modulo $\mathrm{Im}(f_j)$ by a term involving a path of smaller length in the first tensor component. It follows from this observation that $\eta_j \in \mathrm{Im}(f_j)$, using induction to $\deg(\eta_j)$. □

Remark 5.8. Pushing the results to Mod_{kQ} through the equivalence $\mathrm{Rep}_Q \simeq \mathrm{Mod}_{kQ}$, the maps are given by

$$g\big(\sum_{c\in\mathcal{P}} c\otimes n_c\big) = \sum_{c\in\mathcal{P}} n_c\cdot c, \qquad n_c\in M_{s(c)},$$

and for $\alpha\in Q_1$,

$$f(c\otimes m_{s(\alpha)}) = \alpha c\otimes m_{s(\alpha)} - c\otimes(m_{s(\alpha)}\cdot\alpha), \qquad c\in P(t(\alpha)),\ \ m_{s(\alpha)}\in M_{s(\alpha)}.$$

Remark 5.9. Note that the standard projective resolution for $M = P(j)$ is *not* the trivial resolution

$$0\longrightarrow P_0 = P(j)\longrightarrow P(j)\longrightarrow 0.$$

But for quiver representations M that are *not* projective, the standard projective resolution does give a projective resolution of minimal length.

Example 5.10. *In this example we consider the standard projective resolution of $M = S(i)$. Since $S(i)_i = k$ and $S(i)_j = \{0\}$ for $j\neq i$ we have*

$$P_0 = P(i), \qquad P_1 = \bigoplus_{\alpha\in Q_1:s(\alpha)=i} P(t(\alpha)).$$

The morphisms $f = (f_j)_j$ and $g = (g_j)_j$ in the standard projective resolution

$$0\longrightarrow \bigoplus_{\alpha\in Q_1:s(\alpha)=i} P(t(\alpha)) \overset{f}{\hookrightarrow} P(i) \overset{g}{\twoheadrightarrow} S(i)\longrightarrow 0$$

are as follows: $g = (g_j)_j : P(i)\twoheadrightarrow S(i)$ is the epimorphism satisfying $g_i(e_i) = 1$ and $g_j\equiv 0$ for $j\neq i$, and $f = (f_j)_j : \oplus_{\alpha\in Q_1:s(\alpha)=i}P(t(\alpha))\hookrightarrow P(i)$ is the monomorphism satisfying

$$f_j^\alpha(c) = \alpha c$$

for $\alpha\in Q_1$ with $s(\alpha) = i$ and $c = \alpha_1\cdots\alpha_r\in P(t(\alpha))_j$ a path from $t(\alpha)$ to j.

5.4. $\mathbf{Rep}_Q$ is hereditary.

Definition 5.11. *Let*

$$f = (f_j)_j : \bigoplus_{\alpha\in Q_1:s(\alpha)=i} P(t(\alpha))\hookrightarrow P(i)$$

the monomorphism occurring in the standard projective resolution of $S(i)$ (see Example 5.10). The subrepresentation $\mathrm{im}(f)$ *of $P(i)$ is called the radical of $P(i)$. We denote it by* $\mathrm{rad}(P(i))$.

We have the following explicit description of $\mathrm{rad}(P(i))$. Write $P(i) = (P(i)_j,\phi_\alpha)$, then

$$\mathrm{rad}(P(i))_j = \begin{cases} 0 & \text{if } j = i,\\ P(i)_j & \text{if } j\neq i\end{cases}$$

(and the connecting maps of $\mathrm{rad}(P(i))$ are the restrictions of the connecting maps ϕ_α to $\mathrm{rad}(P(i))_{s(\alpha)}$). Indeed, if $j \neq i$ and $c' = \alpha c$ is a path from i to j starting with the edge $\alpha \in Q_1$, then $s(\alpha) = i$ and $c' = f_j^\alpha(c)$.

Note that $\mathrm{rad}(P(i)) = 0 \Leftrightarrow i \in Q_0$ is a sink $\Leftrightarrow P(i) = S(i)$.

Lemma 5.12. *Any proper subrepresentation of $P(i)$ is also a subrepresentation of* $\mathrm{rad}(P(i))$.

Proof. Let $M = (M_j, \psi_\alpha)$ be a representation and $h = (h_j)_j : M \hookrightarrow P(i)$ be a monomorphism. Then $h_i : M_i \hookrightarrow P(i)_i = ke_i$, hence $\mathrm{Dim}_k(M_i) = 0$ or $\mathrm{Dim}_k(M_i) = 1$. If $\mathrm{Dim}_k(M_i) = 0$ then $\mathrm{Im}(h_i) = \{0\}$ hence $\mathrm{im}(h) \subseteq \mathrm{rad}(P(i))$. If $\mathrm{Dim}_k(M_i) = 1$ then h_i is a linear isomorphism. In particular, there exists an $m_i \in M_i$ such that $h_i(m_i) = e_i$. If $c = \alpha_1 \cdots \alpha_r$ is a path from i to j in the quiver Q then

$$h_j(\psi_{\alpha_r} \cdots \psi_{\alpha_1}(m_i)) = \phi_{\alpha_r} \cdots \phi_{\alpha_1}(h_i(m_i)) = e_i\alpha_1 \cdots \alpha_r = c,$$

so h_j is surjective for all $j \in Q_0$. Hence $h : M \hookrightarrow P(i)$ is an isomorphism if $\mathrm{Dim}_k(M_i) = 1$. This concludes the proof. □

Recall that every direct summand of a projective quiver representation is projective again (this holds in the general context of module categories). In Rep_Q the following stronger property holds true.

Theorem 5.13. *The abelian category Rep_Q of finite dimensional quiver representation is hereditary, meaning that a subrepresentation of a projective representation is projective again.*

Proof. Let P be a projective quiver representation and $u = (u_j)_j : M \hookrightarrow P$ a subrepresentation. Write $M = (M_i, \phi_\alpha)$ and $P = (P_i, \varphi_\alpha)$. We show that M is projective by induction to the total dimension $\sum_{j \in Q_0} \mathrm{Dim}_k(P_j)$ of P. If the total dimension of P is zero then there is nothing to prove. For the induction step, suppose that the total dimension of P is $r > 0$. Write $P = P(i) \oplus P'$ using the Krull-Schmidt theorem and the fact that a direct summand of a projective representation is projective again (hence, P' is also projective). Let $\pi : P \twoheadrightarrow P(i)$ be the canonical projection on the first component and $\pi' : P \twoheadrightarrow P'$ the canonical projection on the second component.

If $\pi \circ u : M \to P(i)$ is an epimorphism then

$$M \simeq P(i) \oplus \ker(\pi \circ u)$$

since projective representations only have nontrivial extensions (i.e. $\pi \circ u$ is a retraction). But $\ker(\pi \circ u) \hookrightarrow P'$ via the restriction of $\pi' \circ u$ to $\ker(\pi \circ u)$. By the induction hypothesis we conclude that $\ker(\pi \circ u)$ is projective, hence M as direct sum of projectives, is projective.

If $\pi \circ u : M \to P(i)$ is not an epimorphism, then its image lies in $\mathrm{rad}(P(i))$, which is projective since it is isomorphism to $\bigoplus_{\alpha \in Q_1 : s(\alpha) = i} P(t(\alpha))$. Hence M is a subrepresentation of the projective quiver representation $\mathrm{rad}(P(i)) \oplus P'$, which is of strict smaller total dimension than P. By the induction hypothesis it follows that M is projective. $\square$

6. Semisimple algebras

Good alternative sources for the material in this section are Chapters 4 and 5 of Schiffler's book [4], the book of Lang [3] (concerning semisimple modules) and the book of Assem, Simson and Skowronski [1].

In this section k is a field and A is a finite dimensional associative k-algebra with unit $1 \neq 0$. If I do not specify whether the ideal is a left, right or two-sided ideal, then the statement holds true for all three types of ideals. A right A-module M is finite dimensional as vector space over k unless explicitly specified otherwise.

6.1. **The radical.** For ideals $I, J \subseteq A$, we write $I + J \subseteq A$ for the smallest ideal containing the elements $a + b$ ($a \in I$, $b \in J$). Clearly

$$I + J = \{a + b \mid a \in I, \; b \in J\}.$$

An ideal I of A is said to be a maximal ideal if $I \neq A$ and if $J \subseteq A$ is an ideal containing I, then $J = I$ or $J = A$.

Definition 6.1. *Jacobson's radical* $\mathrm{rad}(A)$ *is*

$$\mathrm{rad}(A) := \bigcap \{I \mid I \subset A \text{ maximal right ideal}\}.$$

Since A has at least one maximal right ideal (Zorn's lemma), $\mathrm{rad}(A) \subset A$ is a proper right ideal of A. In this section we show that $\mathrm{rad}(A)$ tells something about the complete reducibility of right A-modules (Jacobson's theorem).

In the following lemma we show that $1 + \mathrm{rad}(A)$ is a "neighborhood" of 1 in the subgroup $A^{\times}$ of units.

Lemma 6.2. *Let $a \in A$. The following statements are equivalent.*

a. *$a \in \mathrm{rad}(A)$.*
b. *$\forall\, b \in A$: $1 - ab$ has a right inverse.*
c. *$\forall\, b \in A$: $1 - ab$ has a two-sided inverse.*

Proof. **a** $\Rightarrow$ **b**: let $a \in \mathrm{rad}(A)$ and suppose that $1 - ab$ does not have a right inverse for $b \in A$. Then the right ideal $(1 - ab)A$ generated by $1 - ab$ does not contain $1 \in A$, hence there exists (by Zorn's lemma) a maximal right ideal $I \subset A$ containing $(1 - ab)A$. It follows that $1 - ab \in I$ and also $a \in \mathrm{rad}(A) \subseteq I$. Since I is a right ideal, this implies that $1 \in I$. But this contradicts that $I \neq A$.
b $\Rightarrow$ **c**: Fix $b \in A$ and write $c \in A$ for a right inverse of $1 - ab$. Then

$$c = 1 + abc = 1 - a(-bc).$$

Hence c is of the form $1 - ab'$ with $b' \in A$, so it also has a right inverse $d \in A$. Then

$$1 = cd = (1 + abc)d = d + ab(cd) = d + ab,$$

hence $d = 1 - ab$. Then $1 = cd = c(1 - ab)$, so c is also a left inverse of $1 - ab$.
c $\Rightarrow$ **a**: Suppose $1 - ab$ has a two-sided inverse for all $b \in A$. Suppose that $a \notin \mathrm{rad}(A)$. Then there exists a maximal right ideal I of A such that $a \notin I$. So

$I + aA$ is a right ideal of A properly containing I. By the maximality of I, we conclude that $I + aA = A$. Hence there exists an $x \in I$ and an $b \in A$ such that $x + ab = 1$, i.e., such that $x = 1 - ab$. By assumption, $1 - ab$ has a right inverse, so $xA = A$. This contradicts the fact that x lies in the maximal right ideal I. □

Lemma 6.3. *The radical* $\operatorname{rad}(A)$ *is equal to the intersection of all maximal* left *ideals in* A. *In particular,* $\operatorname{rad}(A)$ *is a two-sided ideal.*

Proof. Write $\operatorname{rad}_\ell(A)$ for the intersection of all maximal *left* ideals in A. Interchanging the role of left/right in (the proof of) the previous lemma, we have $a \in \operatorname{rad}_\ell(A)$ $\Leftrightarrow$ $1 - ba$ has a two-sided inverse for all $b \in A$. Hence it suffices to show that

Claim: If $a, b \in A$ and $1 - ab$ has a two-sided inverse, then $1 - ba$ has a two-sided inverse.

It suffices to prove $\Rightarrow$. Suppose that $1 - ab$ has a two-sided inverse $c \in A$. We claim that $d := 1 + bca$ is the two-sided inverse of $1 - ba$. Indeed,

$$\begin{aligned}(1-ba)d &= (1-ba)(1+bca)\\ &= 1 - ba + bca - babca\\ &= 1 - ba + b(1-ab)ca\\ &= 1 - ba + ba = 1,\end{aligned}$$

and

$$\begin{aligned}d(1-ba) &= (1+bca)(1-ba)\\ &= 1 + bca - ba - bcaba\\ &= 1 + bc(1-ab)a - ba\\ &= 1 + ba - ba = 1.\end{aligned}$$

□

Since $\operatorname{rad}(A)$ is a two-sided ideal of A by the previous lemma, we can form the quotient algebra $A/\operatorname{rad}(A)$.

Corollary 6.4. $\operatorname{rad}\big(A/\operatorname{rad}(A)\big) = \{0\}$.

Proof. Let $\pi : A \twoheadrightarrow A/\operatorname{rad}(A)$ be the canonical map and write $\overline{a} := \pi(a)$ for $a \in A$. Suppose that $\overline{a} \in \operatorname{rad}(A/\operatorname{rad}(A))$. We show that $a \in \operatorname{rad}(A)$.

Fix $b \in A$. By Lemma 6.2 applied to $\overline{a} \in \operatorname{rad}(A/\operatorname{rad}(A))$ in $A/\operatorname{rad}(A)$, there exists an $\overline{c} \in A/\operatorname{rad}(A)$ such that

$$\overline{(1-ab)c} = \overline{1}.$$

Let $x \in \operatorname{rad}(A)$ such that $(1 - ab)c = 1 + x$. Then $1 + x$ has a right inverse $d \in A$ by Lemma 6.2 applied to $x \in \operatorname{rad}(A)$ in A, hence

$$(1-ab)cd = 1.$$

We conclude that $1-ab$ has a right inverse in A. Since $b \in A$ was chosen arbitrarily, it follows that $a \in \operatorname{rad}(A)$ by yet another application of Lemma 6.2. □

An ideal I of A is said to be nilpotent if $I^n = \{0\}$ for some $n \geq 1$, where I^n is the smallest ideal containing the elements $a_1 \dots a_n$ with $a_j \in I$. Note that

$$\dots \subseteq I^{n+1} \subseteq I^n \subseteq \dots \subseteq I \subseteq A,$$

and I^n is the set of elements in A that can be written as a finite sum of elements of the form $a_1 \dots a_n$ with $a_i \in I$.

Lemma 6.5. *If I is a nilpotent two-sided ideal in A, then $I \subseteq \mathrm{rad}(A)$.*

Proof. Let $n \in \mathbb{N}$ such that $I^n = \{0\}$. Let $a \in I$ and fix $b \in A$. Then

$$(1 - ab)(1 + ab + \dots + (ab)^{n-1}) = 1 - (ab)^n = 1$$

since $(ab)^n \in I^n = \{0\}$. Hence $1-ab$ has a right inverse for all $b \in A$. Consequently $a \in \mathrm{rad}(A)$ by Lemma 6.2. □

Lemma 6.6. *A has a unique largest two-sided nilpotent ideal with respect to inclusion.*

Proof. Let $I \subseteq A$ be a nilpotent two-sided ideal of maximal dimension as vector space over k. If $J \subseteq A$ is another nilpotent two-sided ideal then $I + J$ is nilpotent. Indeed, $I \cdot J, J \cdot I \subset I \cap J$, hence if $I^m = 0$ and $J^n = 0$, then $(I + J)^{m+n} = 0$. So $I + J$ is a nilpotent two-sided ideal containing I, hence $I + J = I$ by maximality of the dimension of I, and so $J \subseteq I$. Thus every nilpotent two-sided ideal J is contained in I, proving that I is the unique largest nilpotent two-sided ideal of A. □

We will see in a moment that $\mathrm{rad}(A)$ is the unique maximal two-sided nilpotent ideal of A (as part of Jacobson's theorem).

6.2. Semisimple modules.

Definition 6.7. *Let M be a right A-module.*

- **(a)** *M is called simple if $M \neq 0$ and if the only A-submodules of M are $\{0\}$ and M.*
- **(b)** *M is said to be semisimple if M is the direct sum of simple right A-modules.*
- **(c)** *The algebra A is said to be semisimple if every right A-module M is semisimple.*

Remark. We regard $M = \{0\}$ as semisimple module (being the empty direct sum of simple modules).

Lemma 6.8. *Let M be a right A-module. The following statements are equivalent.*

- **(a)** *M is semisimple.*
- **(b)** *If $N \subseteq M$ is an A-submodule then there exists an A-submodule $L \subseteq M$ such that $M = N \oplus L$.*
- **(c)** *M is a sum of simple right A-modules (here the sum is not required to be direct!).*

Proof. **(b)** $\Rightarrow$ **(a)**: by induction to $\text{Dim}_k(M)$. If $M = 0$ then there is nothing to prove. Otherwise, let $0 \neq S \subseteq M$ be a nontrivial submodule of minimal dimension. Then S is simple, and by the assumption there exists a right A-submodule $P \subseteq M$ such that $M = S \oplus P$. It remains to show that the right A-submodule P satisfies property **(b)**.

Let $N \subseteq P$ be a right A-submodule. Since N is also a right A-submodule of M, there exists a right A-submodule $T \subseteq M$ such that $M = N \oplus T$. Set $L := T \cap P$. Then $L \subseteq P$ is a right A-submodule and $N \oplus L \subseteq P$. If $p \in P$ and $p = n + t$ with $n \in N$ and $t \in T$, then $t = p - n \in T \cap P = L$, hence $P = N \oplus L$.
(a) $\Rightarrow$ **(c)**: trivial.
(c) $\Rightarrow$ **(b)**: Let $N \subseteq M$ be a right A-submodule. Let $L \subseteq M$ be a right A-submodule of largest possible dimension such that $L \cap N = \{0\}$. Then $L + N = L \oplus N$ is a right A-submodule of M. Suppose that $L \oplus N$ is strictly contained in M. Then the assumption **(c)** implies that there exists a simple right A-submodule S of M such that $L \oplus N$ is strictly contained in $L \oplus N + S$. Hence $(L \oplus N) \cap S$ is a right A-module strictly contained in S. But S is simple, so $(L \oplus N) \cap S = \{0\}$. Then $(L \oplus S) \cap N = \{0\}$, contradicting the choice of L. Hence $M = N \oplus L$, completing the proof. □

Corollary 6.9. *Let M be a semisimple right A-module.*

- **(a)** *Submodules and quotients of M are semisimple.*
- **(b)** *The associative k-algebra A is semisimple $\Leftrightarrow$ A, viewed as right A-module, is semisimple.*

Proof. **(a)** Let $L \subseteq M$ be a submodule and write $\pi : M \to M/L$ for the canonical morphism, $\pi(m) := m + L$.

We first show that M/L is a semisimple right A-module. Let $M = S_1 \oplus S_2 \oplus \cdots \oplus S_t$ be a decomposition in simple modules. Then

$$M/L = \pi(S_1) + \cdots + \pi(S_t)$$

with $\pi(S_i) \subseteq M/L$ right A-submodules. It suffices to show that $\pi(S_i)$ is $\{0\}$ or simple. Suppose that $0 \subseteq V_i \subseteq \pi(S_i)$ is a submodule. Then

$$U_i := \pi^{-1}(V_i) \cap S_i$$

is a submodule of S_i, hence $U_i = \{0\}$ or $U_i = S_i$. If $U_i = \{0\}$ then $V_i = \pi(U_i) = \{0\}$, if $U_i = S_i$ then $V_i = \pi(S_i)$.

Next we show that L is a semisimple right A-module. Since M is semisimple, there exists a right A-submodule $N \subseteq M$ such that $M = L \oplus N$. Then $L \simeq M/N$, hence L is semisimple by the first part of the proof of **(a)**.
(b) $\Rightarrow$ is trivial.
$\Leftarrow$: Let M be a right A-module. Fix a k-basis $\{m_1, \ldots, m_n\}$ of M. We then have

a surjective right A-module morphism $\phi : A^{\oplus n} \twoheadrightarrow M$, defined by

$$\phi((a_1, \ldots, a_n)) := \sum_{i=1}^{n} m_i a_i,$$

hence $M \simeq A^{\oplus n}/\ker(\phi)$. As quotient of the semisimple right A-module $A^{\oplus n}$, M is semisimple by **(a)**. □

6.3. **Jacobson's Theorem.** We need a preliminary lemma and exercise before we can formulate and prove Jacobson's Theorem.

Lemma 6.10. $\mathrm{rad}(A)$ *is the intersection of a finite number of maximal right A-ideals.*

Proof. Let $\mathcal{S}$ be the (nonempty) set

$$\mathcal{S} := \{I_1 \cap I_2 \cap \cdots \cap I_r \mid I_i \subsetneq A \text{ maximal right ideal}, r \in \mathbb{N}\}$$

and let $L := I_1 \cap \cdots \cap I_s \in \mathcal{S}$ be an element from $\mathcal{S}$ of minimal dimension. Clearly $\mathrm{rad}(A) \subseteq L$. On the other hand, if I is an arbitrary maximal right ideal, then $L \cap I \in \mathcal{S}$ must have the same dimension as L, hence $L \cap I = L$, i.e., $L \subseteq I$. It follows that $L \subseteq \mathrm{rad}(A)$. Hence $\mathrm{rad}(A) = L$, i.e., $\mathrm{rad}(A)$ is the intersection of a finite number of maximal ideals. □

Exercise 6.11. *Let M be a right A-module. Show that there exists a series*

$$\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

of right A-submodules such that M_i/M_{i-1} is simple for $i = 1, \ldots, n$ (such a series is called a composition series of M).

Theorem 6.12 (Jacobson)**.** *The following statements hold true.*

(a) $\mathrm{rad}(A) = \{a \in A \mid Ma = \{0\} \text{ if } M \text{ is a simple right } A-\text{module}\}$.
(b) $\mathrm{rad}(A)$ *is the largest two-sided nilpotent ideal of A.*
(c) $\mathrm{rad}(A)$ *is the smallest two-sided ideal such that* $A/\mathrm{rad}(A)$ *is semisimple.*

Proof. **(a)** Write

$$J := \{a \in A \mid Ma = \{0\} \quad \text{if } M \text{ is a simple right } A-\text{module}\},$$

which is a right ideal in A.

We first prove that $J \subseteq \mathrm{rad}(A)$. Let $a \in J$ and $I \subset A$ a maximal right ideal. It suffices to show that $a \in I$. Note that A/I is a simple right A-module since I is maximal, hence $(A/I)a = \{0\}$ since $a \in J$. In particular, $(1 + I)a = a + I = I$, i.e., $a \in I$.

Next we prove that $\mathrm{rad}(A) \subseteq J$. Suppose not. Then there exists a simple right A-module M and $0 \neq m \in M$ such that $m\mathrm{rad}(A) \neq \{0\}$. But $m\mathrm{rad}(A)$ is a right A-submodule of M, since $\mathrm{rad}(A)$ is a right ideal. Since M is simple, we have $m\mathrm{rad}(A) = M$. Hence there exists an element $x \in \mathrm{rad}(A)$ such that $mx = m$. Write $K := \{a \in A \mid ma = 0\}$ for the annihilator of m. Then $K \subset A$ is a right ideal not containing 1, and $1 - x \in K$. By Zorn's Lemma there exists a maximal right ideal I containing K, so in particular $1 - x \in I$. But also $x \in \mathrm{rad}(A) \subseteq I$, which would imply $1 \in I$. This is a contradiction. Hence $\mathrm{rad}(A) \subseteq J$.

(b) We know that $\mathrm{rad}(A)$ contains all two-sided nilpotent ideals of A, so it suffices to show that $\mathrm{rad}(A)$ is nilpotent. Take a composition series

$$\{0\} = A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset A_n = A$$

of A, viewed as right A-module. By **(a)** we have for all $a \in \mathrm{rad}(A)$ and for $1 \leq i \leq n$,

$$(A_i/A_{i-1})a = \{0\}$$

since A_i/A_{i-1} is simple. Hence $A_n\mathrm{rad}(A)^n \subseteq A_0 = \{0\}$. In particular, since $1 \in A_n = A$, we have $\mathrm{rad}(A)^n = \{0\}$.

(c) We first show that $A/\mathrm{rad}(A)$ is semisimple. Write $\mathrm{rad}(A) = I_1 \cap I_2 \cap \cdots \cap I_r$ with $I_i \subset A$ maximal right ideals. Then the morphism of right A-modules

$$\phi : A/\mathrm{rad}(A) \to A/I_1 \oplus \cdots \oplus A/I_{r-1} \oplus A/I_r$$

defined by $\phi(a + \mathrm{rad}(A)) := (a + I_1, a + I_2, \ldots, a + I_r)$ is injective. Hence $A/\mathrm{rad}(A)$, as right A-module, is isomorphic to a submodule of the semisimple right A-module $A/I_1 \oplus \cdots \oplus A/I_r$, hence it is semisimple. It follows that $A/\mathrm{rad}(A)$ is semisimple as right $A/\mathrm{rad}(A)$-module, hence $A/\mathrm{rad}(A)$ is semisimple as k-algebra.

Let $X \subseteq A$ be a two-sided ideal such that the quotient algebra A/X is semisimple. It remains to show that $\mathrm{rad}(A) \subseteq X$. Let $\pi : A \to A/X$ be the canonical map $\pi(a) := a + X$, which we view here as epimorphism of right A-modules. Since A/X is semisimple, we have a decomposition

$$A/X = S_1 \oplus S_2 \oplus \cdots \oplus S_t$$

with $S_i \subseteq A/X$ simple right A-submodules. Consider the right ideals $L_i := \pi^{-1}(S_i) \subset A$ and $X \subseteq M_i := \sum_{j \neq i} L_j \subset A$ for $1 \leq i \leq t$. Note that

$$\pi(M_i) = M_i/X = S_1 \oplus \cdots \oplus S_{i-1} \oplus S_{i+1} \oplus \cdots \oplus S_t. \tag{6.1}$$

Then by the third isomorphism theorem,

$$\begin{aligned} A/M_i &\simeq (A/X)/(M_i/X) \\ &\simeq (S_1 \oplus \cdots \oplus S_t)/(S_1 \oplus \cdots \oplus S_{i-1} \oplus S_{i+1} \oplus \cdots \oplus S_t) \simeq S_i \end{aligned}$$

hence $M_i \subset A$ is a maximal ideal for each $i = 1, \ldots, t$. In particular,

$$\mathrm{rad}(A) \subseteq M_1 \cap M_2 \cap \cdots \cap M_t.$$

Now note that $(M_1 \cap \cdots \cap M_t)/X = \{0\}$ by (6.1), i.e.

$$M_1 \cap \cdots \cap M_t = X,$$

hence $\mathrm{rad}(A) \subseteq X$. □

Corollary 6.13. *The algebra A is semisimple if and only if* $\mathrm{rad}(A) = \{0\}$.

Proof. This is by part **(c)** of Jacobson's theorem. □

Corollary 6.14. *Let Q be a finite quiver without oriented cycles. Let $R_Q \subseteq kQ$ be the two-sided ideal generated by the edges Q_1 of Q. Then* $\mathrm{rad}(kQ) = R_Q$*, and*

$$kQ/\mathrm{rad}(kQ) \simeq \bigoplus_{i \in Q_0} S(i)$$

as right kQ-modules.

Proof. Since Q has no oriented cycles and R_Q^m is spanned by paths of length $\geq m$, we have $R_Q^m = \{0\}$ for sufficiently large m. Hence $R_Q \subset kQ$ is a two-sided nilpotent ideal, implying that $R_Q \subseteq \mathrm{rad}(kQ)$ (another way of proving this is by noting that $S(i)R_Q = \{0\}$ for all $i \in Q_0$ and using that the $S(i)$'s ($i \in Q_0$) are, up to isomorphism, the only simple right kQ-modules).

View $S(i)$ as right kQ-module. It is one-dimensional. Write $m_i \in S(i)$ for a nonzero vector. Then we have a morphism $\phi : kQ/R_Q \to \bigoplus_{i \in Q_0} S(i)$ of right kQ-modules, defined by

$$\phi(a + R_Q) := (m_i a)_{i \in Q_0}.$$

Note that ϕ is an isomorphism, with inverse given by $(\lambda_i m_i)_{i \in Q_0} \mapsto \sum_{i \in Q_0} \lambda_i e_i + R_Q$ for $\lambda_i \in k$. Hence $kQ/R_Q \simeq \bigoplus_{i \in Q_0} S(i)$ as right kQ-modules, hence also as right kQ/R_Q-modules. It follows that kQ/R_Q is semisimple as right kQ/R_Q-module, hence kQ/R_Q is semisimple as k-algebra. Consequently, $\mathrm{rad}(kQ) \subseteq R_Q$. □

6.4. **The radical of a module.**

Definition 6.15. *Let M be a right A-module.*

1. *$N \subseteq M$ is called maximal if $N \neq M$ and the only submodules $L \subseteq M$ containing N are M and N.*
2. *The (Jacobson) radical* $\mathrm{rad}(M)$ *of M is the intersection of all maximal submodules of M.*

Remark 6.16. Let M be a right A-module. The following holds true:

i. $N \subseteq M$ is a maximal submodule $\Leftrightarrow$ M/N is simple.
ii. For $M = A$, viewed as right A-module in the canonical way, $\mathrm{rad}(M)$ is equal to the radical $\mathrm{rad}(A)$ of the algebra A.

Exercise 6.17. *Let M, N be right A-modules. Prove the following statements.*

a. $\mathrm{rad}(S) = \{0\}$ *if S is a simple right A-module.*
b. *$m \in M$ belongs to* $\mathrm{rad}(M)$ *if and only if $f(m) = 0$ for all morphisms* $f \in \mathrm{Hom}_A(M, S)$ *with S simple.*
c. *If* $f \in \mathrm{Hom}_A(M, N)$ *is a morphism of A-modules then* $f(\mathrm{rad}(M)) \subseteq \mathrm{rad}(N)$.
d. $\mathrm{rad}(M \oplus N) = \mathrm{rad}(M) \oplus \mathrm{rad}(N)$.
Hint: *apply* **c** *to the canonical embeddings* $\iota_M \in \mathrm{Hom}_A(M, M \oplus N)$ *and* $\iota_N \in \mathrm{Hom}_A(N, M \oplus N)$ *and to the canonical projections* $\pi_M \in \mathrm{Hom}_A(M \oplus N, M)$ *and* $\pi_N \in \mathrm{Hom}_A(M \oplus N, N)$.

e. *$M/M\mathrm{rad}(A)$ is a semisimple right A-module, where $M\mathrm{rad}(A) \subseteq M$ is the right A-submodule*

$$M\mathrm{rad}(A) := \mathrm{span}_k\{ma \mid m \in M,\ a \in \mathrm{rad}(A)\}.$$

f. $\mathrm{rad}(M/M\mathrm{rad}(A)) = \{0\}$.

g. $M\mathrm{rad}(A) = \mathrm{rad}(M)$.
Hint: *To prove $M\mathrm{rad}(A) \subseteq \mathrm{rad}(M)$, consider for $m \in M$ the map $f_m : A \to M$ defined by $f_m(a) := ma$.*

Definition 6.18. *Let M be a right A-module. The quotient module*

$$\mathrm{top}(M) := M/\mathrm{rad}(M)$$

is called the top of M.

In view of parts **e** and **g** of the exercise, $\mathrm{top}(M)$ is a semisimple right A-module.

Definition 6.19. *Let M be a right A-module. A right A-submodule $L \subseteq M$ is called superfluous if the following property holds true: if $N \subseteq M$ is a right A-submodule such that $L + N = M$, then $N = M$.*

Lemma 6.20. *If M is a right A-submodule and $L \subseteq M$ is a submodule that is contained in $\mathrm{rad}(M)$, then L is superfluous.*

Proof. Suppose N is a right A-submodule of M such that $N \neq M$ and $L+N = M$. By Zorn's Lemma, there exists a maximal right A-submodule $X \subset M$ containing N. But then $L \subseteq \mathrm{rad}(M) \subseteq X$ and $N \subseteq X$, hence $L + N \subseteq X$. This is a contradiction. □

7. Idempotents

The next goal is to associate to a finite dimensional k-algebra A a quiver Q_{A^b} and an appropriate quotient $kQ_{A^b}/\mathcal{I}$ of its path algebra (a so-called bounded quiver algebra) such that A and $kQ_{A^b}/\mathcal{I}$ are *Morita equivalent*, meaning that the associated categories of finite dimensional right modules are equivalent.

The approach will be as follows. We consider an appropriate subalgebra A^b of A whose decomposition in indecomposable right A-modules only contains the mutually inequivalent summands from the decomposition of A in indecomposables. The subalgebra A^b is called the basic subalgebra of A. We associate to A^b a quiver Q_{A^b} such that A^b is isomorphic to a quotient of kQ_{A^b}. We furthermore show that A^b and A are Morita equivalent via the so-called induction functor.

In this section we prepare the construction of the quiver Q_{A^b} by considering complete sets of primitive orthogonal idempotents. Primitive orthogonal idempotents will label the vertices of the quiver.

We use the following conventions: k is a field and A is a finite dimensional associative k-algebra with unit $1 \neq 0$. If I do not specify whether the ideal is a left, right or two-sided ideal, then the statement holds true for all three types of ideals. A right A-module M is finite dimensional as vector space over k unless explicitly specified otherwise.

Other sources for this section are Chapters 4 and 5 of Ralf Schiffler's book [4] and the book of Assem, Simson and Skowronski [1].

Remark: In this section we use the Krull-Schmidt theorem, the splitting lemma and basic facts on projective representations in the context of the category Mod_A of finite dimensional right A-modules. Earlier we formulated and proved these results for the category Rep_Q of quiver representations, but obvious adjustments to the proofs give the analogous statements for Mod_A.

7.1. Primitive orthogonal idempotents.

Recall from Section 1 the definition of orthogonal idempotents. If e_1 and e_2 are two idempotents that are orthogonal to each-other, $e_1e_2 = 0 = e_2e_1$, then $e_1 + e_2$ is again an idempotent.

We first show that a decomposition of an idempotent $e \in A$ as sum of orthogonal idempotents produces a direct sum decomposition of the right ideal eA in right A-submodules.

Proposition 7.1. *Let $e \in A$ be an idempotent and $e = e_1+e_2$ with e_1, e_2 orthogonal idempotents. Then $eA = e_1A \oplus e_2A$.*

Proof. For $a \in eA$ we have $a = ea = e_1a + e_2a$, hence $eA \subseteq e_1A + e_2A$. If $a_i \in e_iA$ then $a_i = e_ia_i = ee_ia_i \in eA$, where we use that e_1, e_2 are orthogonal for the second equality. Hence $e_1A + e_2A \subseteq eA$. If $a \in e_1A \cap e_2A$ then $e_1a = a = e_2a$, hence $a = e_1a = e_1(e_2a) = 0$. Consequently $eA = e_1A \oplus e_2A$. □

Conversely, we have

Proposition 7.2. *Let $e \in A$ be an idempotent. Suppose $eA = I_1 \oplus I_2$ with $I_i \subseteq eA$ right A-submodules. Let $e_i \in I_i$ $(i = 1, 2)$ such that $e = e_1 + e_2$. Then*

(a) *e_1, e_2 are orthogonal idempotents.*
(b) *$I_i = e_i A$ for $i = 1, 2$.*

Proof. Clearly $e_i A \subseteq I_i$. Let $a_i \in I_i$. Then $a_1 = ea_1 = e_1 a_1 + e_2 a_1$. Since $a_1, e_1 a_1 \in I_1$ and $e_2 a_1 \in I_2$ we have $e_2 a_1 = 0$ and $e_1 a_1 = a_1$. Similarly $e_1 a_2 = 0$ and $e_2 a_2 = a_2$. In particular $a_i \in e_i A$, hence $I_i = e_i A$ for $i = 1, 2$. Taking $a_i = e_i$ in the above computation shows that e_1, e_2 are orthogonal idempotents. □

We call an idempotent $e \in A$ *primitive* if $e \neq 0$ and if the only possible decompositions $e = e_1 + e_2$ as sum of two mutually orthogonal idempotents $e_1, e_2 \in A$ are the trivial ones: $0 + e = e = e + 0$.

Corollary 7.3. *Let $e \in A$ be an idempotent. Then e is primitive if and only if eA is an indecomposable right A-module.*

Proof. $\Rightarrow$: if $eA = I_1 \oplus I_2$ is a decomposition in right A-submodules $I_i \subseteq eA$ then, with the notations as in Proposition 7.2, either $e_1 = 0$ or $e_2 = 0$. Hence $I_1 = 0$ or $I_2 = 0$.
$\Leftarrow$: if $e = e_1 + e_2$ with e_1, e_2 orthogonal idempotents then $eA = e_1 A \oplus e_2 A$ by Proposition 7.1. Hence $e_1 A = 0$ or $e_2 A = 0$, i.e. either $e_1 = 0$ or $e_2 = 0$. □

Definition 7.4. *We call $\{e_1, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents of A if*

(i) *The e_i's are primitive idempotents.*
(ii) *$e_i e_j = 0$ if $1 \leq i \neq j \leq n$ (orthogonality).*
(iii) *$1 = e_1 + e_2 + \cdots + e_n$ (completeness).*

The existence of a complete set of primitive orthogonal idempotents is guaranteed by the following theorem.

Theorem 7.5. *There exists a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents of A. If $\{e'_1, \ldots, e'_r\}$ is another complete set of primitive orthogonal idempotents of A, then $n = r$ and, after renumbering the e_i's,*

$$e_i A \simeq e'_i A$$

as right A-modules for all i.

Proof. **Existence:** By the Krull-Schmidt theorem we have a decomposition

$$A = I_1 \oplus \cdots \oplus I_n$$

of A as right A-module in indecomposable submodules $I_i \subseteq A$. Let $e_i \in I_i$ such that

$$1 = e_1 + \cdots + e_n.$$

By Proposition 7.2 and Corollary 7.3 it follows that $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A and $I_i = e_iA$ $(1 \leq i \leq n)$, in particular

$$A = e_1A \oplus \cdots \oplus e_nA.$$

Uniqueness: Suppose that $\{e'_1, \ldots, e'_r\}$ is another complete set of primitive orthogonal idempotents of A. By Proposition 7.1 and Corollary 7.3 we get another decomposition

$$A = e'_1A \oplus \cdots \oplus e'_rA$$

of A as right A-module in indecomposable submodules. By the uniqueness part of the Krull-Schmidt theorem it follows that $r = n$ and that there exists a permutation σ of $\{1, \ldots, n\}$ such that $e_iA \simeq e'_{\sigma(i)}A$ as right A-modules $(i = 1, \ldots, n)$. □

As remarked already at the start of this section, we are using now notions and results defined for the category Rep_Q for the category Mod_A of finite dimensional right A-modules. Projective modules in Mod_A are the topic of the following exercise.

Exercise 7.6. *A right A-module P is called projective if for all epimorphism $g : M \twoheadrightarrow N$ of right A-modules, the push-forward $g_* : \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N)$, defined by $g_*(f) := g \circ f$, is surjective. In case $A = kQ$ with Q a quiver without oriented cycles, this definition coincides with the definition of a projective representation of Q.*

- **a.** *Prove that $A^{\oplus r}$, viewed as right A-module by $(a_1, \ldots, a_r)\cdot a := (a_1a, \ldots, a_ra)$, is projective.*

If P and P' are right A-modules, then $P \oplus P'$ is projective if and only if both P and P' are projective. The proof is the same as the proof for quiver representations, see Proposition 4.7. In particular, direct summands of $A^{\oplus r}$ are projective.

- **b.** *Let $e \in A$ be an idempotent. Show that eA is a projective right A-module.*
- **c.** *Let $e \in A$ be a primitive idempotent. Show that there exists a complete set of primitive orthogonal idempotents containing e.*

The splitting lemma holds true in any abelian category, in particular in the abelian category of right A-modules, see [3]. *In particular, if*

$$0 \to N \hookrightarrow M \overset{f}{\twoheadrightarrow} L \to 0$$

is a short exact sequence of right A-modules and f a retraction, then L is a direct summand of M.

- **d.** *Let P be a projective right A-module. Show that there exists a right A-module P' such that $P \oplus P' \simeq A^{\oplus r}$ for some $r \geq 1$ (then P' is necessarily also projective).*
- **e.** *Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents. Let P be an indecomposable projective right A-module. Show that $P \simeq e_iA$ for some i.*

7.2. **Lifting idempotents.** For $a \in A$ we write $\overline{a} := a + \mathrm{rad}(A)$ for its corresponding class in the semisimple quotient $A/\mathrm{rad}(A)$ of A. We are now going to study how idempotents of A behave when projected down to $A/\mathrm{rad}(A)$. First note that if $0 \neq e \in A$ is a nonzero idempotent then $e \notin \mathrm{rad}(A)$ (since $\mathrm{rad}(A)$ is nilpotent), hence $\overline{0} \neq \overline{e} \in A/\mathrm{rad}(A)$ is a nonzero idempotent. Preservation of primitivity is more subtle, we need a couple of initial lemmas for that.

Lemma 7.7. *Let $e \in A$ be an idempotent. As right A-module we have*

$$\mathrm{top}(eA) \simeq \overline{e}(A/\mathrm{rad}(A)),$$

with the right hand side viewed as right A-module by $\overline{a} \cdot b := \overline{ab}$ for $a \in eA$ and $b \in A$.

Proof. Using Exercise 6.17,

$$\mathrm{top}(eA) = eA/\mathrm{rad}(eA) = eA/eA\mathrm{rad}(A) = eA/e\,\mathrm{rad}(A).$$

Let $\pi_e : eA \twoheadrightarrow \overline{e}(A/\mathrm{rad}(A))$ be the epimorphism $\pi_e(a) := \overline{a}$ of right A-modules. Its kernel is $\mathrm{Ker}(\pi_e) = eA \cap \mathrm{rad}(A) = e\,\mathrm{rad}(A)$. Hence $eA/e\,\mathrm{rad}(A) \simeq \overline{e}(A/\mathrm{rad}(A))$ as right A-modules by the first isomorphism theorem. □

The following exercise is an addendum to Exercise 6.17.

Exercise 7.8. *Let M be a right A-module.*

- **(a)** *Show that $\mathrm{rad}(M) = \{0\}$ if M is semisimple.*
- **(b)** *Suppose that $N \subseteq M$ is a right A-submodule such that M/N is semisimple. Show that $\mathrm{rad}(M) \subseteq N$ (in particular, $\mathrm{rad}(M)$ is the smallest submodule of M such that $M/\mathrm{rad}(M)$ is semisimple).*

An idempotent of the semisimple quotient $A/\mathrm{rad}(A)$ can be lifted to A as follows.

Lemma 7.9 (Lift lemma)**.** *Let $e \in A$ and assume that $\overline{e} \in A/\mathrm{rad}(A)$ is an idempotent. Then there exists an $\epsilon \in eA$ such that $\overline{\epsilon} = \overline{e}$ and $\epsilon^2 = \epsilon$ in A.*

Proof. Let $m \geq 1$ such that $\mathrm{rad}(A)^m = \{0\}$. Since $\overline{e}$ is an idempotent, $e - e^2 \in \mathrm{rad}(A)$, but $e - e^2$ is not necessarily zero. Its mth power is, since $\mathrm{rad}(A)^m = 0$. Hence

$$0 = (e - e^2)^m = e^m(1-e)^m = e^m g$$

with

$$g = \sum_{j=0}^{m} (-1)^j \binom{m}{j} e^j$$

by the binomium theorem. Writing $g = 1 - et$ with

$$t = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j} e^{j-1}$$

we get $e^m = e^{m+1}t$ in A, and clearly $et = te$. Now set $\epsilon := (et)^m \in A$. Then $\overline{et} = \overline{e^{m+1}t} = \overline{e^m} = \overline{e}$, where we use the fact that $\overline{e}$ is an idempotent for the first and third equality. Hence

$$\overline{\epsilon} = (\overline{et})^m = \overline{e}^m = \overline{e}.$$

Furthermore, using $et = te$ and $e^{m+1}r = e^m$ in A repeatedly, we compute in A,

$$\begin{aligned}\epsilon^2 = (et)^{2m} &= e^{2m}t^{2m} \\ &= e^{2m-1}t^{2m-1} = \cdots \\ &= e^m t^m \\ &= (et)^m = \epsilon.\end{aligned}$$

□

Now we are ready to show that a primitive orthogonal idempotent of A projects down to a primitive orthogonal idempotent of $A/\mathrm{rad}(A)$.

Proposition 7.10. *Let $e \in A$ be a primitive idempotent. Then $\overline{e} \in A/\mathrm{rad}(A)$ is a primitive idempotent.*

Proof. Suppose that

$$\overline{e}(A/\mathrm{rad}(A)) = M_1 \oplus M_2$$

is a decomposition with nonzero right $A/\mathrm{rad}(A)$-submodules M_i and with M_1 nonzero. Choose $f_i \in A$ such that $\overline{e} = \overline{f}_1 + \overline{f}_2$ and $\overline{f}_i \in M_i$. Then $M_i = \overline{f}_i(A/\mathrm{rad}(A))$, $\overline{f}_i^2 = \overline{f}_i$ and $\overline{f}_1\overline{f}_2 = \overline{0} = \overline{f}_2\overline{f}_1$ by Proposition 7.1. In particular, $\overline{f}_1 \neq \overline{0}$ and

$$\overline{f}_1 = \overline{f}_1^2 = (\overline{e} - \overline{f}_2)\overline{f}_1 = \overline{ef}_1.$$

We now apply the lift lemma to ef_1. This gives us an element $\epsilon_1 \in ef_1A \subseteq eA$ such that $\overline{\epsilon}_1 = \overline{ef}_1(= \overline{f}_1)$ and $\epsilon_1^2 = \epsilon_1$. We now first show that $eA = \epsilon_1 A$.

Note that $A = \epsilon_1 A \oplus (1 - \epsilon_1)A$ since $\epsilon_1, 1 - \epsilon_1$ are orthogonal idempotents, and $\epsilon_1 A \subseteq eA$. Hence

$$eA = \epsilon_1 A \oplus \{((1 - \epsilon_1)A) \cap eA\}.$$

But eA is indecomposable and $\epsilon_1 A$ is nonzero, so this forces $eA = \epsilon_1 A$.

By Lemma 7.7 we now have

$$\overline{e}(A/\mathrm{rad}(A)) \simeq \mathrm{top}(eA) = \mathrm{top}(\epsilon_1 A) \simeq \overline{\epsilon}_1(A/\mathrm{rad}(A)) = \overline{f}_1(A/\mathrm{rad}(A)) = M_1.$$

Hence $M_2 = 0$, as desired. □

Corollary 7.11. *Let $e \in A$ be a primitive idempotent. Then eA has a unique maximal right A-submodule, namely* $\mathrm{rad}(eA) = e\,\mathrm{rad}(A)$. *The quotient module* $\mathrm{top}(eA)$ *is simple.*

Proof. Lemma 7.7 and Proposition 7.10 show that

$$\text{top}(eA) \simeq \overline{e}(A/\text{rad}(A))$$

is an indecomposable right A-module. So $\text{top}(eA)$ is indecomposable and semisimple, hence simple. Consequently $\text{rad}(eA) \subset eA$ is a maximal right A-submodule. But $\text{rad}(eA)$ is the intersection of all maximal right A-submodules of eA, hence $\text{rad}(eA)$ is the unique maximal right A-submodule of eA. □

Exercise 7.12. *Let Q be a finite quiver without oriented cycles, and $P(i)$ the indecomposable right kQ-module associated to $i \in Q_0$.*

a. *Determine* $\text{rad}(P(i))$.
b. *Describe* $\text{top}(P(i))$.

A second immediate consequence of Proposition 7.10 is

Corollary 7.13. *Let $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents. Then $\{\overline{e}_1, \dots, \overline{e}_n\}$ is a complete set of primitive orthogonal idempotents of $A/\text{rad}(A)$.*

Proof. Since $e_i \in A$ is a nonzero idempotent and $\text{rad}(A)$ is nilpotent, we have $e_i \notin \text{rad}(A)$, hence $\overline{e}_i \neq \overline{0}$. Consequently $\{\overline{e}_1, \dots, \overline{e}_n\}$ is a set of orthogonal idempotents of $A/\text{rad}(A)$ with $\overline{1} = \overline{e}_1 + \cdots + \overline{e}_n$. It remains to show that $\overline{e}_i$ is primitive, which follows from Proposition 7.10. □

The following proposition shows that equivalences between indecomposable projective right A-modules are determined by equivalences of their tops.

Proposition 7.14. *Let $e, f \in A$ be two idempotents. Then*

$$eA \simeq fA \quad \Leftrightarrow \quad \text{top}(eA) \simeq \text{top}(fA).$$

Proof. $\Leftarrow$: suppose that $eA \simeq fA$ with isomorphism $\phi : eA \xrightarrow{\sim} fA$. Then

$$\phi(\text{rad}(eA)) = \text{rad}(fA)$$

by Exercise 6.17, hence we obtain an isomorphism $\xi : \text{top}(eA) \to \text{top}(fA)$ defined by

$$\xi(a + \text{rad}(eA)) = \phi(a) + \text{rad}(fA).$$

$\Rightarrow$: suppose that $\text{top}(eA) \simeq \text{top}(fA)$ with isomorphism $\xi : \text{top}(eA) \xrightarrow{\sim} \text{top}(fA)$. Let $p_e : eA \twoheadrightarrow \text{top}(eA)$ and $p_f : fA \twoheadrightarrow \text{top}(fA)$ be the canonical epimorphisms. Since eA, as direct summand of A as right A-module, is projective, we have that the push-forward

$$(p_f)_* : \text{Hom}_A(eA, fA) \twoheadrightarrow \text{Hom}_A(eA, \text{top}(fA))$$

of p_f is surjective. Hence there exists an $\phi \in \text{Hom}_A(eA, fA)$ such that $(p_f)_*(\phi) = \xi \circ p_e$, i.e. $p_f \circ \phi = \xi \circ p_e$. Now both $p_f : fA \twoheadrightarrow \text{top}(fA)$ and $p_f \circ \phi : eA \twoheadrightarrow \text{top}(fA)$ are epimorphisms, hence $\text{Im}(\phi) + \text{Ker}(p_f) = fA$. But $\text{Ker}(p_f) = \text{rad}(fA) \subseteq fA$ is superfluous (see Lemma 6.20), so we conclude that $\text{Im}(\phi) = fA$, hence ϕ is an epimorphism. Consequently $\dim_k(eA) \geq \dim_k(fA)$. Replacing the role of eA and fA we also have $\dim_k(fA) \geq \dim_k(eA)$, hence $\dim_k(eA) = \dim_k(fA)$ and ϕ is an isomorphism. □

Exercise 7.15. *Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents, and let*

$$\{j_1, \ldots, j_r\} \subseteq \{1, \ldots, n\}$$

such that $e_{j_i}A \not\simeq e_{j_\ell}A$ if $i \neq \ell$ and each e_sA is isomorphic to some $e_{j_i}A$. Prove that $\{\mathrm{top}(e_{j_i}A)\}_{i=1}^r$ is a complete set of representatives of the equivalence classes of the simple right A-modules.

Let $e \in A$ be an idempotent and M a right A-module. Recall that e is the unit element of the idempotent subalgebra eAe of A. The subspace $Me \subseteq M$ inherits from M a right eAe-module structure. On the other hand, eA is a (eAe, A)-bimodule, with eAe acting from the left and A from the right on A in the natural way (the left and right actions are compatible, in the sense that $(ba)a' = b(aa')$ for $b \in eAe$, $a \in eA$ and $a' \in A$). Consequently, the space $\mathrm{Hom}_A(eA, M)$ of A-module morphisms $eA \to M$ is a right eAe-module by

$$(\phi \cdot b)(a) := \phi(ba), \qquad \phi \in \mathrm{Hom}_A(eA, M),\ b \in eAe,\ a \in eA.$$

Lemma 7.16. *Let $e \in A$ be an idempotent, and M a right A-module. Then*

$$\theta_M : \mathrm{Hom}_A(eA, M) \to Me,$$

defined by $\theta_M(\phi) := \phi(e)$, is an isomorphism of right eAe-modules. If $M = eA$ then $\theta_{eA} : \mathrm{End}_A(eA) \to eAe$ is an isomorphism of algebras.

Proof. For $\phi \in \mathrm{Hom}_A(eA, M)$ we have $\phi(e) = \phi(e^2) = \phi(e)e \in Me$, hence θ_M is a well-defined k-linear map. For $b \in eAe$ and $\phi \in \mathrm{Hom}_A(eA, M)$ we have

$$\theta_M(\phi \cdot b) = (\phi \cdot b)(e) = \phi(be) = \phi(eb) = \phi(e)b = \theta_M(\phi)b$$

since e is the unit of the subalgebra eAe. Hence θ_M is a morphisms of right eAe-modules. Define $\theta'_M : Me \to \mathrm{Hom}_A(eA, M)$ by $\theta'_M(m)(a) := ma$ for $m \in Me$ and $a \in eA$. Then for $m \in Me$, $b \in eAe$ and $a \in eA$ we have

$$\theta'_M(mb)(a) = (mb)a = m(ba) = \theta'_M(m)(ba) = (\theta'_M(m) \cdot b)(a)$$

where we have used that $ba \in eA$, hence θ'_M is a morphism of right eAe-modules. For $\phi \in \mathrm{Hom}_A(eA, M)$ and $a \in eA$ we have

$$\theta'_M(\theta_M(\phi))(a) = \theta_M(\phi)a = \phi(e)a = \phi(ea) = \phi(a),$$

and for $m \in Me$ we have

$$\theta_M(\theta'_M(m)) = \theta'_M(m)(e) = me = m.$$

Hence θ_M is an isomorphism with inverse θ'_M.

For the second statement, note that $\theta_{eA}(\mathrm{Id}_{eA}) = e$ and if $\phi_1, \phi_2 \in \mathrm{End}_A(eA)$ then

$$\theta_{eA}(\phi_1 \circ \phi_2) = \phi_1(\phi_2(e)) = \phi_1(e\phi_2(e)) = \phi_1(e)\phi_2(e) = \theta_{eA}(\phi_1)\theta_{eA}(\phi_2).$$

□

8. Bounded quiver algebras

In this section we introduce and study basic, connected and bounded quiver algebras. We show that any finite dimensional basic associative k-algebra A is isomorphic to a bounded quiver algebra when k is algebraically closed.

We use the usual conventions: k is a field and A is a finite dimensional associative k-algebra with unit $1 \neq 0$. Modules over A are always finite dimensional as vector space over k. If I do not specify whether the ideal is a left, right or two-sided ideal, then the statement holds true for all three types of ideals.

Other sources for this section is the book of Assem, Simson and Skowronski [1].

8.1. Connected algebras.

Definition 8.1. *A is called connected if the following property holds true: if A' and A'' are k-algebras and $A \simeq A' \oplus A''$ as algebras, then $A' = \{0\}$ or $A'' = \{0\}$.*

Recall that the center $Z(A)$ of A is the commutative subalgebra

$$Z(A) := \{a \in A \mid ab = ba \quad \forall\, b \in A\}.$$

Lemma 8.2. *A is connected if and only if 0 and 1 are the only central idempotents in A.*

Proof. $\Rightarrow$: let $e \in Z(A)$ be a central idempotent. Since $e, 1-e$ are orthogonal idempotents we have

$$A = eA \oplus (1-e)A \tag{8.1}$$

as right ideals. But e and $1-e$ are central, so $eA, (1-e)A \subseteq A$ are subalgebras. Furthermore, if $a \in eA$ and $b \in (1-e)A$, then $a = ea$ and $b = (1-e)b$ hence $ab = e(1-e)ab = 0$. Similarly $ba = 0$. Hence (8.1) is a decomposition as direct sum of subalgebras. Since A is connected, this implies $eA = \{0\}$ or $(1-e)A = \{0\}$. But A has a unit, so this shows that $e = 0$ or $e = 1$.

$\Leftarrow$: suppose that A has only 0 and 1 as central idempotents. Let $A = A' \oplus A''$ be a decomposition of A as direct sum of two subalgebras $A', A'' \subseteq A$. In particular, $ab = 0 = ba$ for $a \in A'$ and $b \in A''$.

Write $1 = e + (1-e)$ with $e \in A'$ and $1 - e \in A''$. Then $e(1-e) = 0 = (1-e)e$ and consequently

$$e = 1 \cdot e = e^2 + (1-e)e = e^2,$$

so $e \in A$ is an idempotent. If $a \in A'$ then

$$a = 1 \cdot a = ea + (1-e)a = ea$$

In a similar way, $a = ae$ for $a \in A'$, hence $ea = ae$ for all $a \in A'$. Note also that $ea = 0 = ae$ for $a \in A''$, hence $e \in Z(A)$. So $e \in A$ is a central idempotent, implying that $e = 0$ or $e = 1$. If $e = 0$ then $A' = eA' = \{0\}$. If $e = 1$ then $A'' = (1-e)A'' = \{0\}$. Hence A is connected. ☐

Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. Let $I \subseteq A$ be a two-sided ideal. Then $I = 1 \cdot I \cdot 1 = \sum_{i,j=1}^n e_i I e_j$ and the sum is direct by orthogonality of the idempotents. Hence

$$I = \bigoplus_{i,j=1}^{n} e_i I e_j$$

as k-vector spaces. This is in particular the case for $I = A$ and $I = \mathrm{rad}(A)$. Note that

$$(e_i I e_j)(e_k A e_l) = \{0\} = (e_i A e_j)(e_k I e_l)$$

if $j \neq k$ and

$$(e_i I e_j)(e_j A e_l), (e_i A e_j)(e_j I e_l) \subseteq e_i I e_l.$$

Lemma 8.3. *Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. The following statements are equivalent.*

- **a.** *A is connected.*
- **b.** *If $\{1, \dots, n\} = J_1 \cup J_2$ is a disjoint union such that $e_i A e_j = 0 = e_j A e_i$ for $i \in J_1$ and $j \in J_2$, then $J_1 = \emptyset$ or $J_2 = \emptyset$.*

Proof. Let $\{1, \dots, n\} = J_1 \cup J_2$ be a nontrivial disjoint union (i.e., $J_1 \neq \emptyset \neq J_2$) such that $e_i A e_j = 0 = e_j A e_i$ for $i \in J_1$ and $j \in J_2$. Write

$$A_i := \bigoplus_{k,l \in J_i} e_k A e_l, \qquad i = 1, 2.$$

Then $A_i \subseteq A$ is a nontrivial subalgebra with unit element $1_i := \sum_{l \in J_i} e_l$, and $e_l \in A_i$ if $l \in J_i$. Since $A = \bigoplus_{i,j=1}^n e_i A e_j$, the assumptions imply that

$$A = A_1 \oplus A_2$$

as algebras, hence A is not connected.

For the converse statement, suppose that the only disjoint union $\{1, \dots, n\} = J_1 \cup J_2$ satisfying $e_i A e_j = 0 = e_j A e_i$ for $i \in J_1$ and $j \in J_2$ is the trivial one, $J_1 = \emptyset$ or $J_2 = \emptyset$. Let $A = A_1 \oplus A_2$ be a decomposition of A as direct sum of subalgebras. Write $1 = f_1 + f_2$ with $f_i \in A_i$. Then $\{f_1, f_2\}$ are central orthogonal idempotents in A and $f_i \in A_i$ is the unit element of the algebra A_i. Define the disjoint subsets $J_1 := \{i \mid e_i f_1 e_i = e_i\}$ and $J_2 := \{i \mid e_i f_1 e_i = 0\}$ of $\{1, \dots, n\}$. Alternatively we can write $J_1 = \{i \mid e_i f_2 e_i = 0\}$ and $J_2 = \{i \mid e_i f_2 e_i = e_i\}$. Since the $A_i \subseteq A$ are two-sided ideals, we have $e_i \in A_1$ if $i \in J_1$ and $e_i \in A_2$ if $i \in J_2$.

We claim that $J_1 \cup J_2 = \{1, \dots, n\}$. Indeed, note that $e_i f_1 e_i$ and $e_i(1 - f_1)e_i$ are mutually orthogonal idempotents in A such that $e_i = e_i f_1 e_i + e_i(1 - f_1)e_i$, hence the primitivity of e_i implies that $e_i f_1 e_i = e_i$ or $e_i f_1 e_i = 0$.

Note furthermore that for $i \in J_1$ and $j \in J_2$ we have $e_i A e_j \subseteq A_1 \cap A_2 = \{0\}$ and $e_j A e_i \subseteq A_1 \cap A_2 = \{0\}$. But the only such decomposition $\{1, \dots, n\} = J_1 \cup J_2$ is the trivial one by assumption, hence $J_1 = \emptyset$ or $J_2 = \emptyset$. If $J_2 = \emptyset$ then $1 = \sum_{i \in J_1} e_i \in A_1$, hence $A_1 = A$. If $J_1 = \emptyset$ then $A_2 = A$. Hence A is connected. □

8.2. **Basic algebras.**

Definition 8.4. *A is called basic if A, as right A-module, has a decomposition*

$$A = I_1 \oplus I_2 \oplus \cdots \oplus I_n \tag{8.2}$$

in indecomposable right A-modules I_i with $I_i \not\simeq I_j$ if $i \neq j$.

Remark 8.5. By the Krull-Schmidt theorem the definition makes sense. Note that the $I_j \subseteq A$ in (8.2) are projective. Recall from Section 7 that if we decompose $1 = e_1 + \cdots + e_n$ with $e_i \in I_i$, then $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A and $I_i = e_i A$. Conversely, any choice of a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents gives rise to a decomposition of A as right A-module in indecomposable projective submodules $I_i = e_i A$. In particular, if $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A, then A is basic if and only if $e_i A \not\simeq e_j A$ as right A-modules when $i \neq j$.

Example 8.6. *Let Q be a finite quiver without oriented cycles. Then $\{e_i\}_{i \in Q_0}$ is a complete set of primitive orthogonal polynomials, and $e_i kQ$ corresponds to the indecomposable projective quiver representation $P(i)$ of Q. Since $P(i) \simeq P(j)$ iff $i = j$, the path algebra kQ is basic.*

Proposition 8.7. *A is basic if and only if $A/\mathrm{rad}(A)$ is basic.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. Since as right A-modules, $\overline{e}_i(A/\mathrm{rad}(A)) \simeq e_i A/\mathrm{rad}(e_i A) = \mathrm{top}(e_i A)$, we have to show that $e_i A \simeq e_j A$ iff $\mathrm{top}(e_i A) \simeq \mathrm{top}(e_j A)$. This is Proposition 7.14. □

Lemma 8.8 (Schur's lemma)**.** *Suppose that k is algebraically closed. Let M and N be inequivalent simple right A-modules. Then*

$$\mathrm{Hom}_A(M, N) = \{0\}$$

and $\mathrm{End}_A(M) = k\mathrm{Id}_M$.

Proof. If $\phi \in \mathrm{Hom}_A(M, N)$ then $\ker(\phi) \subseteq M$ and $\mathrm{im}(\phi) \subseteq N$ are right A-submodules. Since $M \not\simeq N$ it is not possible that both $\ker(\phi) = \{0\}$ and $\mathrm{im}(\phi) = N$. Hence $\ker(\phi) = M$ or $\mathrm{im}(\phi) = \{0\}$, i.e., $\phi = 0$.

For $\phi \in \mathrm{End}_A(M)$ choose $\lambda \in k$ an eigenvalue of ϕ (this is possible since k is algebraically closed). Let

$$\{0\} \neq M_\lambda := \{m \in M \mid \phi(m) = \lambda m\} \subseteq M$$

be the corresponding eigenspace. Then $\{0\} \neq M_\lambda$ is a right A-submodule of M, hence $M_\lambda = M$ and $\phi = \lambda \mathrm{Id}_M$. □

Proposition 8.9. *Suppose that k is algebraically closed. Let A be a basic and semisimple finite dimensional k-algebra. Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. Then*

$$A = ke_1 \oplus ke_2 \oplus \cdots \oplus ke_n.$$

In particular, $A \simeq k^{\oplus n}$ as algebras, A is commutative, and all simple A-modules are one-dimensional over k.

Proof. We have $A = e_1A \oplus \cdots \oplus e_nA$ with $e_iA \subseteq A$ simple right A-modules (since A is semisimple) and $e_iA \not\simeq e_jA$ if $i \neq j$ (since A is basic). By Lemma 7.16 and Lemma 8.8 we have for $i \neq j$,

$$e_jAe_i \simeq \mathrm{Hom}_A(e_iA, e_jA) = \{0\},$$

while

$$e_iAe_i \simeq \mathrm{End}_A(e_iA) = k\mathrm{Id}_{e_iA},$$

hence $e_iAe_i = ke_i$. Consequently

$$A = \bigoplus_{i,j=1}^{n} e_iAe_j = \bigoplus_{i=1}^{n} ke_i.$$

In particular, A is commutative. The simple right A-modules are e_iA ($i = 1, \ldots, n$). Since A is commutative and e_i is an idempotent, we have $e_iA = e_iAe_i = ke_i$, hence the simple right A-modules are one-dimensional. □

Remark 8.10. Let k be algebraically closed and A semisimple. In this case the algebraic structure of A is fully described by the Wedderburn-Artin Theorem:

$$A \simeq \mathrm{End}_k(k^{n_1}) \oplus \cdots \oplus \mathrm{End}_k(k^{n_s})$$

as algebras. Here the n_i's are the dimensions of the simple right A-modules (up to isomorphism).

Exercise 8.11. *Suppose that A is basic and k algebraically closed. Let M be an indecomposable projective right A-module. Show that* $\mathrm{rad}(M) \subset M$ *has co-dimension one over k.*

Example 8.12. *Let G be a finite group. Let $\mathbb{C}[G]$ be the group algebra of G over $\mathbb{C}$. By Maschke's Theorem we know that $\mathbb{C}[G]$ is a finite dimensional semisimple $\mathbb{C}$-algebra with unit, hence the Wedderburn-Artin Theorem applies to $\mathbb{C}[G]$ (see, e.g.,* [6, Thm. 5.5.6] *for a formulation in this context, where $\mathbb{C}[G]$ is identified with the convolution algebra $L(G)$ of $\mathbb{C}$-valued functions on G by the map $g \mapsto \delta_g$ for $g \in G$, where $\delta_g \in L(G)$ is the delta-function at $g \in G$).*

A formulation directly in terms of $\mathbb{C}[G]$ is as follows. Let $\widehat{G}$ be the dual of G (the equivalence classes of the irreducible representations of G). An irreducible representation $\pi \in \widehat{G}$ is represented by an algebra map $\pi : \mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V_\pi)$ such that V_π, viewed as left $\mathbb{C}[G]$-module via π, is simple. For $\pi \in \widehat{G}$ write $\chi_\pi(g) :=$ $\mathrm{Tr}_{V_\pi}(\pi(g))$ *for the trace of π and set*

$$p_\pi := \frac{\dim_{\mathbb{C}}(V_\pi)}{\#G} \sum_{g \in G} \overline{\chi_\pi(g)} g \in \mathbb{C}[G]$$

for $\pi \in \widehat{G}$. The p_π's ($\pi \in \widehat{G}$) form a $\mathbb{C}$-linear basis of $Z(\mathbb{C}[G])$. Then $\{p_\pi\}_{\pi\in\widehat{G}}$ is a set of central orthogonal idempotents in $\mathbb{C}[G]$ satisfying $1 = \sum_{\pi\in\widehat{G}} p_\pi$, and $\{p_\pi\}_{\pi\in\widehat{G}}$ is a $\mathbb{C}$-linear basis of $Z(\mathbb{C}[G])$. Hence $p_\pi\mathbb{C}[G] \subseteq \mathbb{C}[G]$ is a subalgebra with unit p_π and

$$\mathbb{C}[G] = \bigoplus_{\pi\in G} p_\pi\mathbb{C}[G]$$

as algebras. Finally, $p_\pi\mathbb{C}[G] \simeq \mathrm{End}_{\mathbb{C}}(V_\pi)$ as algebras by the map $a \mapsto \pi(a)$ for $a \in p_\pi\mathbb{C}[G]$.

8.3. **Admissible ideals.** Let Q be a finite quiver, possibly with oriented cycles. Let $\mathcal{P}_Q$ be the set of paths in Q. We write $Q_m \subseteq \mathcal{P}_Q$ for the paths of length m in Q. In particular Q_0 is the set of constant paths, which we identify with the set of vertices of Q.

Recall that $R_Q \subseteq kQ$ denotes the two-sided ideal of kQ generated by the arrows Q_1 of the quiver Q. For $m \geq 1$, the mth power R_Q^m of R_Q is the two-sided ideal of kQ generated by the paths of length m. Note that R_Q^m has a k-linear basis consisting of the paths in Q of length $\geq m$. In particular,

$$\cdots \subseteq R_Q^{m+1} \subseteq R_Q^m \subseteq \cdots \subseteq R_Q^2 \subseteq R_Q \subseteq kQ.$$

Let $\mathcal{I} \subseteq kQ$ be a two-sided ideal such that $R_Q^m \subseteq \mathcal{I}$ for some $m \geq 1$. Then the (nontrivial) quotient algebra $kQ/\mathcal{I}$ is finite dimensional as vector space over k since its dimension is bounded by the number of paths in Q of length $< m$.

Proposition 8.13. *Let $\mathcal{I} \subseteq kQ$ be a two-sided ideal such that $R_Q^m \subseteq \mathcal{I} \subseteq R_Q$ for some $m \geq 1$. For $a \in kQ$ write $\overline{a} := a + \mathcal{I}$ for its class in $kQ/\mathcal{I}$. Then $\{\overline{e}_i\}_{i\in Q_0}$ is a complete set of primitive orthogonal idempotents of $kQ/\mathcal{I}$.*

Proof. Clearly $\{\overline{e}_i\}_{i\in Q_0}$ is a complete set of orthogonal idempotents in $kQ/\mathcal{I}$. It remains to show that the idempotent $\overline{e}_i \in kQ/\mathcal{I}$ is primitive. Suppose that $\overline{e}_i = \overline{e} + \overline{e}'$ with $\{\overline{e}, \overline{e}'\}$ orthogonal idempotents in $kQ/\mathcal{I}$. Representatives e and e' of $\overline{e}$ and $\overline{e}'$ in kQ have unique expressions of the form

$$e = \sum_{c\in Q_0} \lambda_c c + f,$$
$$e' = \sum_{c\in Q_0} \lambda'_c c + f'$$

with $\lambda_c, \lambda'_c \in k$ and $f, f' \in R_Q$. Since $\mathcal{I} \subseteq R_Q$, the fact that $\{\overline{e}, \overline{e}'\}$ are orthogonal idempotents in $kQ/\mathcal{I}$ implies

$$\lambda_c^2 = \lambda_c, \quad (\lambda'_c)^2 = \lambda'_c, \quad \lambda_c\lambda'_c = 0$$

for all $c \in Q_0$. Hence

$$(\lambda_c, \lambda'_c) \in \{(0,0), (0,1), (1,0)\}$$

for all $c \in Q_0$. Furthermore, the identity $\overline{e}_i = \overline{e} + \overline{e}'$ in $kQ/\mathcal{I}$ yields

$$\lambda_{e_i} + \lambda'_{e_i} = 1, \qquad \lambda_c + \lambda'_c = 0$$

for $c \in Q_0 \setminus \{e_i\}$, where we again used that $\mathcal{I} \subseteq R_Q$. Hence $(\lambda_c, \lambda'_c) = (0,0)$ if $c \in Q_0 \setminus \{e_i\}$ and $(\lambda_{e_i}, \lambda'_{e_i}) \in \{(1,0),(0,1)\}$. Without loss of generality we may and will assume that $(\lambda_{e_i}, \lambda'_{e_i}) = (0,1)$, so that $\lambda_c = 0$ for all $c \in Q_0$.

We conclude that $\overline{e} = \overline{f}$ with $f \in R_Q$. Then

$$\overline{e} = \overline{e}^m = \overline{f^m} = \overline{0}$$

in $kQ/\mathcal{I}$, since $f^m \in R_Q^m \subseteq \mathcal{I}$. Hence $\overline{e}_i$ is primitive. □

Remark 8.14. Note that $R_Q^m = 0$ for some $m \geq 1$ is equivalent to Q not having no oriented cycles. In particular, if Q does not have oriented cycles then the condition $R_Q^m \subseteq \mathcal{I}$ for some $m \geq 1$ is void. Furthermore, if Q does not have oriented cycles then $\mathrm{rad}(kQ) = R_Q$ (see Corollary 6.14), hence Proposition 8.13 applied to $\mathcal{I} = \mathrm{rad}(kQ) = R_Q$ reestablishes Corollary 7.13 for $A = kQ$.

Proposition 8.15. *Let $\mathcal{I} \subseteq kQ$ be a two-sided ideal such that $R_Q^m \subseteq \mathcal{I} \subseteq R_Q$ for some $m \geq 1$. The radical of the finite dimensional k-algebra $kQ/\mathcal{I}$ is $R_Q/\mathcal{I}$.*

Proof. We have $R_Q^m \subseteq \mathcal{I} \subseteq R_Q$ for some $m \geq 1$, hence $\big(R_Q/\mathcal{I}\big)^m = \{0\}$. It follows that $R_Q/\mathcal{I} \subseteq \mathrm{rad}(kQ/\mathcal{I})$, since $\mathrm{rad}(kQ/\mathcal{I})$ is the unique largest nontrivial two-sided nilpotent ideal of $kQ/\mathcal{I}$ by Jacobson's Theorem.

On the other hand, note that

$$(kQ/\mathcal{I})/(R_Q/\mathcal{I}) \simeq kQ/R_Q \simeq k^{\oplus \# Q_0}$$

as algebras, with the isomorphism $k^{\oplus \# Q_0} \xrightarrow{\sim} kQ/R_Q$ given by

$$k^{\oplus \# Q_0} \ni (\lambda_i)_{i \in Q_0} \mapsto \sum_{i \in Q_0} \lambda_i (e_i + R_Q).$$

Clearly $k^{\oplus \# Q_0}$ is a semisimple algebra, hence $(kQ/\mathcal{I})/(R_Q/\mathcal{I})$ is semisimple. Again by Jacobson's Theorem, it follows that $\mathrm{rad}(kQ/\mathcal{I}) \subseteq R_Q/\mathcal{I}$. □

Remark 8.16. If Q does not have oriented cycles then we established the semisimplicity of kQ/R_Q before, see Corollary 6.14.

Remark 8.17. Let $\mathcal{I} \subseteq kQ$ be a two-sided ideal such that $R_Q^m \subseteq \mathcal{I} \subseteq R_Q$ for some $m \geq 1$. By Proposition 8.15 we have

$$\mathrm{rad}(kQ/\mathcal{I})/\mathrm{rad}(kQ/\mathcal{I})^2 \simeq R_Q/R_Q^2,$$

and the latter space has $\{\alpha + R_Q\}_{\alpha \in Q_1}$ as a k-linear basis. Since $\mathrm{rad}(kQ/\mathcal{I})$ and $\mathrm{rad}(kQ/\mathcal{I})^2$ are two-sided ideals in $kQ/\mathcal{I}$, we have a canonical left and right action of $kQ/\mathcal{I}$ on $\mathrm{rad}(kQ/\mathcal{I})/\mathrm{rad}(kQ/\mathcal{I})^2$. The same holds true for R_Q/R_Q^2. It now follows that for $i, j \in Q_0$, the k-vector space

$$e_i\big(\mathrm{rad}(kQ/\mathcal{I})/\mathrm{rad}(kQ/\mathcal{I})^2\big)e_j \simeq e_i\big(R_Q/R_Q^2\big)e_j$$

has dimension equal to the number of arrows in Q_1 with source i and target j.

Corollary 8.18. *Let $\mathcal{I} \subseteq kQ$ be a two-sided ideal such that $R_Q^m \subseteq \mathcal{I} \subseteq R_Q$ for some $m \geq 1$. Then $kQ/\mathcal{I}$ is a basic algebra.*

Proof. Write $A = kQ/\mathcal{I}$. It suffices to show that $A/\mathrm{rad}(A)$ is basic by Proposition 7.14. By Proposition 8.15 we have $A/\mathrm{rad}(A) \simeq kQ/R_Q$, so it suffices to show that kQ/R_Q is basic. Write $\overline{a} = a + R_Q \in kQ/R_Q$ for $a \in kQ$. Since $\{\overline{e}_i\}_{i \in Q_0}$ is a complete set of primitive orthogonal idempotents of kQ/R_Q by Proposition 8.13, it suffices to show that $\overline{e}_i(kQ/R_Q) \not\simeq \overline{e}_j(kQ/R_Q)$ if $i \neq j$.

Note that $\overline{e}_j(kQ/R_Q) = k\overline{e}_j$, hence for $i \neq j$,

$$\mathrm{Hom}_{kQ/R_Q}\big(\overline{e}_i(kQ/R_Q), \overline{e}_j(kQ/R_Q)\big) \simeq \overline{e}_j(kQ/R_Q)\overline{e}_i = \{0\},$$

where we used Lemma 7.16 for the first step, which gives the desired result. □

Definition 8.19. *A two-sided ideal $\mathcal{I} \subseteq kQ$ is called admissible if $R_Q^m \subseteq \mathcal{I} \subseteq R_Q^2$ for some $m \geq 2$. The associated quotient algebra*

$$kQ/\mathcal{I}$$

is called a bounded quiver algebra.

The condition $R_Q^m \subseteq \mathcal{I}$ is to ensure finite dimensionality of the bounded quiver algebra $kQ/\mathcal{I}$ as vector space over k. Although $\mathcal{I} \subseteq R_Q$ is sufficient to preserve primitivity of idempotents when projecting to the bounded quiver algebra (see Proposition 8.13), we need the more restrictive condition $\mathcal{I} \subseteq R_Q^2$ to ensure that projecting to $kQ/\mathcal{I}$ behaves well with respect to connectivity.

Lemma 8.20. *Let $\mathcal{I} \subseteq kQ$ be an admissible two-sided ideal. Let $Q_0 = J_1 \cup J_2$ be a decomposition of Q_0 as a disjoint union of subsets J_1 and J_2. Equivalent:*

- **a.** *The decomposition of Q_0 is in fact a decomposition of the quiver Q as disjoint union of sub-quivers. In other words, if $\alpha \in Q_1$ is an edge then either $s(\alpha), t(\alpha) \in J_1$ or $s(\alpha), t(\alpha) \in J_2$.*
- **b.** *$\overline{e}_i\big(kQ/\mathcal{I}\big)\overline{e}_j = \{\overline{0}\}$ if $(i,j) \in (J_1 \times J_2) \cup (J_2 \times J_1)$.*

Proof. **a** ⇒ **b**: Let $(i,j) \in (J_1 \times J_2) \cup (J_2 \times J_1)$. Then $\overline{e}_i\big(kQ/\mathcal{I}\big)\overline{e}_j = \{\overline{0}\}$ since $\overline{e}_i\big(kQ/\mathcal{I}\big)\overline{e}_j$ is spanned by the classes of the paths in Q from i to j.
b ⇒ **a**: Let $(i,j) \in (J_1 \times J_2) \cup (J_2 \times J_1)$ and suppose there exists an edge $\alpha \in Q_1$ from i to j. Then $\overline{\alpha} = \overline{e}_i\overline{\alpha}\overline{e}_j = \overline{0}$, hence $\alpha \in \mathcal{I}$. But $\mathcal{I} \subseteq R_Q^2$, hence $\mathcal{I}$ does not contain edges. This contradiction establishes the desired result. □

Proposition 8.21. *The bounded quiver algebra $kQ/\mathcal{I}$ is connected ⇔ Q is connected.*

Proof. Combine Lemma 8.3 and Lemma 8.20. □

8.4. **Presentations of basic algebras as bounded quiver algebras.** Recall that the radical $\mathrm{rad}(A)$ of A is a two-sided ideal. We will write $\mathrm{rad}^m(A) := \mathrm{rad}(A)^m$ for its mth power ($m \geq 1$). Since $\mathrm{rad}(A) \subseteq A$ is a two-sided ideal, it can be thought of as an (A, A)-bimodule having $\mathrm{rad}^2(A)$ as a sub (A, A)-bimodule. For idempotents $e, f \in A$ write

$$e\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big)f := \{e \cdot \overline{x} \cdot f \mid \overline{x} \in \mathrm{rad}(A)/\mathrm{rad}^2(A)\},$$

which is a k-linear subspace of the quotient (A, A)-bimodule $\mathrm{rad}(A)/\mathrm{rad}^2(A)$. On the other hand, $e\,\mathrm{rad}(A)f$ is a k-vector space with subspace $e\,\mathrm{rad}^2(A)f$, hence we can form the quotient vector space $e\,\mathrm{rad}(A)f/e\,\mathrm{rad}^2(A)f$. As k-linear spaces we have

$$e\,\mathrm{rad}(A)f/e\,\mathrm{rad}^2(A)f \xrightarrow{\sim} e\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big)f \tag{8.3}$$

with isomorphism $x + e\,\mathrm{rad}^2(A)f \mapsto x + \mathrm{rad}^2(A)$ for $x \in e\,\mathrm{rad}(A)f$. We will freely use (8.3) to identify the two spaces in the remainder of the text.

Definition 8.22. *Let A be a basic finite dimensional associative k-algebra and $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents. Define the quiver Q_A of A as follows:*

- **a.** $(Q_A)_0 := \{1, \dots, n\}$.
- **b.** *For $i, j \in (Q_A)_0$,*

$$\dim_k\big(e_i(\mathrm{rad}(A)/\mathrm{rad}^2(A))e_j\big)$$

is the number of arrows $i \to j$ in Q_A.

Lemma 8.23. *Up to isomorphism, the quiver Q_A of the basic algebra A does not depend on the choice of compete set of primitive orthogonal idempotents.*

Proof. If $\{e'_1, \cdots, e'_r\}$ is another complete set of primitive orthogonal idempotents then $r = n$ and after renumber of the indices (which does not effect the isomorphism class of the quiver), we have $e_iA \simeq e'_iA$ as right A-modules for $i = 1, \dots, n$. Lemma 7.16 gives

$$e_i\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big)e_j \simeq \mathrm{Hom}_A(e_jA, e_i(\mathrm{rad}(A)/\mathrm{rad}^2(A)))$$

as vector spaces. Hence

$$\dim_k\big(e_i(\mathrm{rad}(A)/\mathrm{rad}^2(A))e_j\big) = \dim_k\big(e'_i(\mathrm{rad}(A)/\mathrm{rad}^2(A))e'_j\big)$$

if

$$e_i\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big) \simeq e'_i\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big)$$

as right A-modules. Recall that $e_i\mathrm{rad}(A) = \mathrm{rad}(e_iA)$, then we have as right A-modules,

$$\begin{aligned} e_i\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big) &\simeq e_i\mathrm{rad}(A)/e_i\mathrm{rad}^2(A) \\ &= \mathrm{rad}(e_iA)/\mathrm{rad}(e_iA)\mathrm{rad}(A) \\ &\simeq \mathrm{rad}(e'_iA)/\mathrm{rad}(e'_iA)\mathrm{rad}(A) \simeq e'_i\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big). \end{aligned} \tag{8.4}$$

This completes the proof. □

Remark 8.24. If $kQ/\mathcal{I}$ be a bounded quiver algebra then $Q_{kQ/\mathcal{I}} \simeq Q$ in view of Remark 8.17.

We fix for each $\alpha \in (Q_A)_1$ with $s(\alpha) = i$ and $t(\alpha) = j$ an element $x_\alpha \in e_i\mathrm{rad}(A)e_j$ such that

$$\{x_\alpha + e_i\mathrm{rad}^2(A)e_j \mid \alpha \in (Q_A)_1 \text{ with } s(\alpha) = i \ \&\ t(\alpha) = j\}$$

is a k-linear basis of $e_i\big(\mathrm{rad}(A)/\mathrm{rad}^2(A)\big)e_j$. Let $c = \alpha_1\alpha_2\cdots\alpha_\ell$ be a path in Q_A of length $\ell \geq 1$ with source i and target j. We sometimes denote such as path as

$$c := (i|\alpha_1\alpha_2\cdots\alpha_\ell|j).$$

Set

$$x_c := x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_\ell} \in e_i\mathrm{rad}^\ell(A)e_j.$$

Note that $x_cx_{c'} = 0$ if $t(c) \neq s(c')$ and $x_cx_{c'} = x_{cc'}$ if $t(c) = s(c')$.

Lemma 8.25. *Let A be a basic algebra and $i, j \in (Q_A)_0$. Then $e_i\mathrm{rad}(A)e_j$ is spanned by the x_c's with c running over the paths in Q_A from i to j of length ≥ 1.*

Proof. Since the x_α's with $\alpha \in (Q_A)_1$ having $s(\alpha) = i$ and $t(\alpha) = j$ form a basis of $e_i\mathrm{rad}(A)e_j$ modulo $e_i\mathrm{rad}^2(A)e_j$, it follows by induction to the length of the path and the fact that

$$e_i\mathrm{rad}^{\ell+1}(A)e_j = \sum_k (e_i\mathrm{rad}(A)e_k)(e_k\mathrm{rad}^\ell(A)e_j)$$

that the elements x_c with c a path in Q_A of length ℓ with source i and target j span $e_i\mathrm{rad}^\ell(A)e_j$ modulo $e_i\mathrm{rad}^{\ell+1}(A)e_j$. The lemma now follows from the fact that $\mathrm{rad}(A)$ is a nilpotent ideal. □

Exercise 8.26. *Let A be a basic algebra. Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents.*

- **a.** *Let $i \neq j$. Show that there does not exist an epimorphism $e_iA \twoheadrightarrow e_jA$ of right A-modules and conclude that* $\mathrm{Hom}_A(e_jA, e_iA) = \mathrm{Hom}_A(e_jA, e_i\mathrm{rad}(A))$.
- **b.** *Let $i \neq j$. Show that $e_iAe_j = e_i\mathrm{rad}(A)e_j$.*
- **c.** *Suppose that A is connected. Prove that Q_A is connected.*

Theorem 8.27. *Let A be a basic algebra. Suppose that k is algebraically closed. There exists an admissible ideal $\mathcal{I} \subseteq kQ_A$ such that*

$$A \simeq kQ_A/\mathcal{I}$$

as algebras (we call $kQ_A/\mathcal{I}$ a presentation of A as bounded quiver algebra).

Proof. Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. Write ϵ_i for the constant path at $i \in (Q_A)_0$. Let $\varphi : kQ_A \to A$ be the k-linear map satisfying

$$\varphi(\epsilon_i) := e_i, \qquad \varphi(c) := x_c$$

for $i \in (Q_A)_0$ and c a path in Q_A of length ≥ 1. By the multiplication rules

$$e_{s(c)}x_c = x_c = x_c e_{t(c)}, \qquad x_c x_{c'} = \delta_{t(c),s(c')} x_{cc'}$$

in A for paths c and c' in Q_A of length ≥ 1, it follows that φ is a morphism of algebras.

We first show that φ is surjective. Write $\pi : A \twoheadrightarrow A/\mathrm{rad}(A)$ for the canonical projection and denote $\overline{y} := \pi(y) = y + \mathrm{rad}(A)$ for $y \in A$. Let $x \in A$. By Corollary 7.13 and Proposition 8.9 we have

$$A/\mathrm{rad}(A) = \bigoplus_{i=1}^{n} k\overline{e}_i. \tag{8.5}$$

Hence there exists unique $\lambda_i \in k$ $(i = 1, \ldots, n)$ such that

$$x - \sum_{i=1}^{n} \lambda_i e_i \in \ker(\pi) = \mathrm{rad}(A).$$

Now $\mathrm{rad}(A)$ is spanned by the x_c for paths c in Q_A of length ≥ 1 in view of Lemma 8.25 and the fact that $\mathrm{rad}(A) = \bigoplus_{i,j}^{n} e_i \mathrm{rad}(A) e_j$. This shows that x lies in the image of φ.

Write $\mathcal{I} := \ker(\varphi) \subseteq kQ_A$. It remains to show that $\mathcal{I}$ is an admissible ideal.

Let $R_{Q_A} \subseteq kQ_A$ be the arrow ideal, generated by the arrows $(Q_A)_1$ of Q_A. Then $\varphi(R_{Q_A}) \subseteq A$ is the ideal generated by the x_α's ($\alpha \in (Q_A)_1$). Since $x_\alpha \in \mathrm{rad}(A)$ for $\alpha \in (Q_A)_1$, we conclude that $\varphi(R_{Q_A}) \subseteq \mathrm{rad}(A)$. Let $m \geq 1$ such that $\mathrm{rad}^m(A) = \{0\}$ (which exists since $\mathrm{rad}(A)$ is nilpotent), then $R_{Q_A}^m \subseteq \ker(\varphi) = \mathcal{I}$.

It remains to show that $\mathcal{I} \subseteq R_{Q_A}^2$. Choose $x \in \mathcal{I} \subseteq kQ_A$. Write

$$x = \sum_{i=1}^{n} \lambda_i \epsilon_i + \sum_{\alpha \in (Q_A)_1} \mu_\alpha \alpha + y$$

with $y \in R_{Q_A}^2$. Then in A,

$$0 = \varphi(x) = \sum_{i=1}^{n} \lambda_i e_i + \sum_{\alpha \in (Q_A)_1} \mu_\alpha x_\alpha + \varphi(y). \tag{8.6}$$

But $x_\alpha \in \mathrm{rad}(A)$ and $\varphi(y) \in \mathrm{rad}^2(A)$, hence

$$\overline{0} = \pi(\varphi(x)) = \sum_{i=1}^{n} \lambda_i \overline{e}_i$$

in $A/\mathrm{rad}(A)$. By (8.5) we conclude that $\lambda_i = 0$ for $i = 1, \ldots, n$, hence

$$x = \sum_{\alpha \in (Q_A)_1} \mu_\alpha \alpha + y$$

lies in R_{Q_A}. Returning to (8.6) we then have

$$\sum_{\alpha \in (Q_A)_1} \mu_\alpha x_\alpha \equiv 0$$

modulo $\mathrm{rad}^2(A)$, hence $\mu_\alpha = 0$ for all $\alpha \in (Q_A)_1$. This shows that $x = y \in R^2_{Q_A}$, hence $\mathcal{I} \subseteq R^2_{Q_A}$. □

Exercise 8.28. *Let k be an algebraically closed field and $\mathrm{Mat}_3(k)$ the algebra of 3×3-matrices over k. Consider the subalgebra*

$$A := \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \;\middle|\; a_{ij} \in k \right\}$$

of $\mathrm{Mat}_3(k)$.

a. *Show that A is a basic algebra.*

b. *Give the presentation of A as bounded quiver algebra.*

c. *Change in the presentation of A the orientation of one of the arrows. Which subalgebra of $\mathrm{Mat}_3(k)$ does this describe?*

9. Induction and restriction

In this section we associate to a finite dimensional associative k-algebra A a basic algebra A^b such that A and A^b are *Morita equivalent*, meaning that the associated categories of finite dimensional right modules are equivalent. In view of Theorem 8.27 this will imply that a finite dimensional k-algebra A is Morita equivalent to a bounded quiver algebra if the ground field k is algebraically closed.

In this section the conventions and notations of Section 8 are in place. Other sources for this section are Chapters 4 and 5 of Ralf Schiffler's book [4] and the book of Assem, Simson and Skowronski [1].

9.1. **The functors.** If $\mathcal{C}$ and $\mathcal{D}$ are pre-additive categories, then a covariant functor $F:\mathcal{C}\to\mathcal{D}$ is said to be additive if for all $M,N\in\mathrm{Ob}(\mathcal{C})$, the map

$$F:\mathrm{Hom}_{\mathcal{C}}(M,N)\to\mathrm{Hom}_{\mathcal{D}}(F(M),F(N)) \tag{9.1}$$

is a group homomorphism. If $\mathcal{C}$ and $\mathcal{D}$ are k-linear categories then a covariant functor $F:\mathcal{C}\to\mathcal{D}$ is called k-linear if the maps (9.1) are k-linear for all $M,N\in\mathrm{Ob}(\mathcal{C})$.

Exercise 9.1. *Let $F:\mathcal{C}\to\mathcal{D}$ be an additive functor between additive categories $\mathcal{C}$ and $\mathcal{D}$. For $M,N\in\mathrm{Ob}(\mathcal{C})$ let $\xi_{M,N}:F(M)\oplus F(N)\to F(M\oplus N)$ be the morphism in $\mathcal{D}$ satisfying*

$$\xi_{M,N}\circ\iota_{F(M)}=F(\iota_M),\qquad \xi_{M,N}\circ\iota_{F(N)}=F(\iota_N).$$

Show that $\xi=\big(\xi_{M,N}\big)_{M,N\in\mathrm{Ob}(\mathcal{C})}:F_1\xrightarrow{\sim}F_2$ is a natural isomorphism with the functors $F_1,F_2:\mathcal{C}\times\mathcal{C}\to\mathcal{D}$ defined by $F_1(\cdot,-):=F(\cdot)\oplus F(-)$ and $F_2(\cdot,-):=F(\cdot\oplus -)$.

For a given finite dimensional k-algebra B with unit, we write Mod_B for the category of finite dimensional right B-modules.

Definition 9.2. *Let $e\in A$ be an idempotent and $eAe\subset A$ the corresponding idempotent subalgebra of A. We define two k-linear covariant functors*

$$\mathrm{Ind}_e:\mathrm{Mod}_{eAe}\to\mathrm{Mod}_A,$$
$$\mathrm{Res}_e:\mathrm{Mod}_A\to\mathrm{Mod}_{eAe}$$

as follows. On objects

$$\mathrm{Ind}_e(M):=M\otimes_{eAe}eA,\qquad M\in\mathrm{Mod}_{eAe},$$
$$\mathrm{Res}_e(X):=Xe,\qquad X\in\mathrm{Mod}_A$$

where $M\otimes_{eAe}eA$ is viewed as right A-module via the right action of A on eA (note that the subspace $Xe\subseteq X$ is invariant under the right action of eAe, giving Xe the structure of a right eAe-module). On morphisms we set

$$\mathrm{Ind}_e(\phi)(m\otimes_{eAe}a):=\phi(m)\otimes_{eAe}a,\qquad \phi\in\mathrm{Hom}_{eAe}(M,N),$$
$$\mathrm{Res}_e(\psi):=\psi|_{Xe},\qquad \psi\in\mathrm{Hom}_A(X,Y),$$

where $m \in M$ and $a \in eA$.

We call Ind_e and Res_e the induction and restriction functor associated to the idempotent e, respectively.

Exercise 9.3. *Show that* Ind_e *and* Res_e *are well defined k-linear functors.*

We now first establish a number of properties of the induction and restriction functor for an arbitrary idempotent subalgebra.

Lemma 9.4. *Let $e \in A$ be an idempotent.*
(i) Ind_e *is left adjoint to* Res_e*.*
(ii) $\mathrm{Res}_e\mathrm{Ind}_e \simeq 1_{\mathrm{Mod}_{eAe}}$*.*
(iii) Ind_e *is fully faithful.*

Proof. **(i)** Let $M \in \mathrm{Ob}\big(\mathrm{Mod}_{eAe}\big)$ and $X \in \mathrm{Ob}\big(\mathrm{Mod}_A\big)$. By [7, Example 7.4], Ind_e is left adjoint to $\mathrm{Hom}_A(eA, -)$, with the functorial isomorphism

$$\mathrm{Hom}_A(\mathrm{Ind}_e(M), X) \xrightarrow{\sim} \mathrm{Hom}_{eAe}(M, \mathrm{Hom}_A(eA, X))$$

mapping $\phi \in \mathrm{Hom}_A(\mathrm{Ind}_e(M), X)$ to the map

$$m \mapsto \{a \mapsto \phi(m \otimes_{eAe} a)\}$$

for $m \in M$ and $a \in eA$. The inverse sends $\psi \in \mathrm{Hom}_{eAe}(M, \mathrm{Hom}_A(eA, X))$ to the map $m \otimes_{eAe} a \mapsto \psi(m)(a)$ for $m \in M$ and $a \in eA$. Combined with the functorial isomorphism

$$\theta_X : \mathrm{Hom}_A(eA, X) \xrightarrow{\sim} \mathrm{Res}_e(X)$$

from Lemma 7.16 we get the result.
(ii) For $M \in \mathrm{Ob}\big(\mathrm{Mod}_{eAe}\big)$ we have

$$\mathrm{Res}_e\mathrm{Ind}_e(M) = \mathrm{Res}_e(M \otimes_{eAe} eA) \simeq M \otimes_{eAe} eAe \simeq M$$

as right eAe-modules.
(iii) For $M, N \in \mathrm{Ob}\big(\mathrm{Mod}_{eAe}\big)$ we have

$$\mathrm{Hom}_{eAe}(M, N) \simeq \mathrm{Hom}_{eAe}(M, \mathrm{Res}_e\mathrm{Ind}_e(N)) \simeq \mathrm{Hom}_A(\mathrm{Ind}_e(M), \mathrm{Ind}_e(N)).$$

Considering the explicit maps in this chain of isomorphisms, we see that the resulting isomorphism

$$\mathrm{Hom}_{eAe}(M, N) \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Ind}_e(M), \mathrm{Ind}_e(N))$$

is Ind_e (check this!). Hence Ind_e is fully faithful. □

Exercise 9.5. *Let $e \in A$ be an idempotent. Let $M \in \mathrm{Ob}(\mathrm{Mod}_{eAe})$ and $X \in \mathrm{Ob}(\mathrm{Mod}_A)$. Show that the natural isomorphism*

$$\eta_{M,X} : \mathrm{Hom}_A(\mathrm{Ind}_e(M), X) \xrightarrow{\sim} \mathrm{Hom}_{eAe}(M, \mathrm{Res}_e(X))$$

constructed in the proof of Lemma 9.4 is given explicitly by

$$\eta_{M,X}(\phi)(m) = \phi(m \otimes_{eAe} e)$$

for $\phi \in \mathrm{Hom}_A(\mathrm{Ind}_e(M), X)$ *and* $m \in M$*. Show that its inverse is given by*

$$\eta^{-1}_{M,X}(\psi)(m \otimes_{eAe} a) = \psi(m)a$$

for $\psi \in \mathrm{Hom}_{eAe}(M, \mathrm{Res}_e(X))$*,* $m \in M$ *and* $a \in eA$*.*

Exercise 9.6. *Let* $e \in A$ *be an idempotent. Let* $X \in \mathrm{Ob}(\mathrm{Mod}_A)$ *and suppose that there exists a module* $M \in \mathrm{Ob}(\mathrm{Mod}_{eAe})$ *such that* $\mathrm{Ind}_e(M) \simeq X$ *as right* A*-modules. Show that* $M \simeq \mathrm{Res}_e(X)$ *as right* eAe*-modules.*

9.2. **Morita equivalence.** Let $e \in A$ be an idempotent and decompose

$$eA = P_1 \oplus \cdots \oplus P_r$$

as direct sum of indecomposable right A-modules. Write $e = e_1 + \cdots + e_r$ with $e_i \in P_i$. Then $\{e_1, \ldots, e_r\}$ is a set of primitive orthogonal idempotents. Fix a subset $I_e \subseteq \{1, \ldots, r\}$ such that $P_j \simeq P_{i_j}$ for a unique $i_j \in I_e$ for each $j \in \{1, \ldots, r\}$.

Definition 9.7. *Let* P *be a finite dimensional right* A*-module. With the notations as above we say that* P *is supported by* e *if* $P \simeq \bigoplus_{i \in I_e} P_i^{\oplus n_i}$ *for certain* $n_i \in \mathbb{Z}_{\geq 0}$*.*

Note that if P is supported by e, then P is projective.

Lemma 9.8. *Let* $e \in A$ *be an idempotent and* P *a finite dimensional right* A*-module supported by* e*. Then* $\mathrm{Ind}_e(\mathrm{Res}_e(P)) \simeq P$*.*

Proof. We use the notations introduced above. By the k-linearity of the functors and Exercise 9.1 it suffices to prove the lemma for $P = e_iA$ with $i \in I_e$. Note that $i \in I_e$ implies that $e_ie = e_i = ee_i$.

For $P = e_iA$ we have $\mathrm{Ind}_e(\mathrm{Res}_e(e_iA)) = e_iAe \otimes_{eAe} eA$, hence it suffices to show that the morphism $\mu_i : e_iAe \otimes_{eAe} eA \to e_iA$ of right A-modules, defined by $\mu_i(a \otimes_{eAe} b) := ab$ for $a \in e_iAe$ and $b \in eA$, is an isomorphism.

Note that $e_i \in e_iAe$ and $\mu_i(e_i \otimes_{eAe} b) = b$ ($b \in eA$) since $e_ie = e_i$. Hence μ_i is an epimorphism.

Since $eAe = e_iAe \oplus (e - e_i)Ae$ as right eAe-modules, the additivity of Ind_e and Exercise 9.1 give

$$\begin{aligned} e_iAe \otimes_{eAe} eA \hookrightarrow &(e_iAe \otimes_{eAe} eA) \oplus ((e - e_i)Ae \otimes_{eAe} eA) \\ &\simeq (e_iAe \oplus (e - e_i)Ae) \otimes_{eAe} eA = eAe \otimes_{eAe} eA \simeq eA, \end{aligned}$$

where the last isomorphism is given by the multiplication map. The resulting map $e_iAe \otimes_{eAe} eA \hookrightarrow eA$ is μ_i, hence it is a monomorphism. □

Let $X \in \mathrm{Mod}_A$. Since X has a projective resolution there always exists an exact sequence

$$P' \xrightarrow{h} P \xrightarrow{\ell} X \longrightarrow 0 \tag{9.2}$$

with P, P' projective modules. In the following lemma we give conditions on P and P' that guarantee that X is isomorphic to $\mathrm{Ind}_e(M)$ for some $M \in \mathrm{Mod}_{eAe}$.

Lemma 9.9. *Let $e \in A$ be an idempotent. If $X \in \mathrm{Ob}(\mathrm{Mod}_A)$ admits an exact sequence* (9.2) *with the projective modules P and P' being supported by e, then $\mathrm{Ind}_e(M) \simeq X$ with*

$$M = \mathrm{coker}(\mathrm{Res}_e h).$$

In particular, $\mathrm{Ind}_e\big(\mathrm{Res}_e X\big) \simeq X$.

Proof. The last statement is due to Exercise 9.6. By (the proof of) Lemma 9.8 the multiplication maps

$$f_P : \mathrm{Ind}_e(\mathrm{Res}_e(P)) \xrightarrow{\sim} P, \qquad f_{P'} : \mathrm{Ind}_e(\mathrm{Res}_e(P')) \xrightarrow{\sim} P',$$

defined by $f_P(p \otimes_{eAe} a) := pa$ and $f_{P'}(p' \otimes_{eAe} a) := p'a$ for $p \in Pe$, $p' \in P'e$ and $a \in eA$, are isomorphisms of right A-modules. Applying the induction function Ind_e, which is right exact (see [7, Prop. 6.16]), to the exact sequence

$$\mathrm{Res}_e(P') \xrightarrow{\mathrm{Res}_e(h)} \mathrm{Res}_e(P) \xrightarrow{\pi} \mathrm{coker}(\mathrm{Res}_e h) \longrightarrow 0$$

with π the canonical map, we get an exact sequence

$$\mathrm{Ind}_e(\mathrm{Res}_e(P')) \longrightarrow \mathrm{Ind}_e(\mathrm{Res}_e(P)) \longrightarrow \mathrm{Ind}_e(\mathrm{coker}(\mathrm{Res}_e h)) \longrightarrow 0.$$

It extends to a diagram

$$\begin{array}{ccccccc}
\mathrm{Ind}_e(\mathrm{Res}_e(P')) & \xrightarrow{\ r\ } & \mathrm{Ind}_e(\mathrm{Res}_e(P)) & \xrightarrow{\mathrm{Ind}_e\pi} & \mathrm{Ind}_e(\mathrm{coker}(\mathrm{Res}_e h)) & \longrightarrow & 0 \\
{\scriptstyle f_{P'}}\downarrow & & {\scriptstyle f_P}\downarrow & & & & \\
P' & \xrightarrow{\ h\ } & P & \xrightarrow{\ \ell\ } & X & \longrightarrow & 0
\end{array}$$

with $r := \mathrm{Ind}_e(\mathrm{Res}_e h)$. The two rows are exact. We claim that the square is commutative. Indeed, for $p' \in P'e$ and $a \in eA$,

$$\begin{aligned}(f_P \circ r)(p' \otimes_{eAe} a) &= f_P((\mathrm{Res}_e h)(p') \otimes_{eAe} a) \\ &= f_P(h(p') \otimes_{eAe} a) = h(p')a = h(p'a) = (h \circ f_{P'})(p' \otimes_{eAe} a).\end{aligned}$$

We show that there exists a unique isomorphism

$$g : \mathrm{Ind}_e(\mathrm{coker}(\mathrm{Res}_e h)) \xrightarrow{\sim} X$$

such that the resulting second square in the diagram is commutative. Since $\mathrm{Ind}_e\pi$ is surjective, uniqueness follows immediately from existence.

For the existence of such a morphism g it suffices to show that

$$\mathrm{Ker}(\mathrm{Ind}_e\pi) \subseteq \mathrm{Ker}(\ell \circ f_P).$$

Let $p \in \mathrm{Ker}(\mathrm{Ind}_e\pi)$. There exists a $p' \in \mathrm{Ind}_e(\mathrm{Res}_e(P'))$ such that $r(p') = p$ since $\mathrm{Ker}(\mathrm{Ind}_e\pi) = \mathrm{Im}(r)$. Then

$$(\ell \circ f_P)(p) = (\ell \circ f_P \circ r)(p') = (\ell \circ h \circ f_{P'})(p') = 0,$$

where the first equality follows from the commutativity of the square in the diagram, and the second equality from the exactness of the second row in the diagram.

In the same way one proves the existence of a unique morphism $g' : X \to \mathrm{Ind}_e(\mathrm{coker}(\mathrm{Res}_e h))$ such that the resulting second square in the diagram is commutative. Now

$$(g \circ g') \circ (\ell \circ f_P) = g \circ \mathrm{Ind}_e \pi = \ell \circ f_P$$

and $\ell \circ f_P$ is an epimorphism, hence $g \circ g' = \mathrm{Id}_X$. Similarly, $g' \circ g = \mathrm{Id}_{\mathrm{Ind}_e(\mathrm{coker}(\mathrm{Res}_e h))}$. This completes the proof of the lemma. □

Lemma 9.10. *Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents. Let $\{j_1, \ldots, j_s\} \subseteq \{1, \ldots, n\}$ such that $e_{j_i}A \simeq e_{j_\ell}A$ as right A-module iff $i = \ell$, and such that each e_rA $(1 \leq r \leq n)$ is isomorphic to some $e_{j_i}A$ as right A-module. Consider the idempotent $e_A := \sum_{i=1}^{s} e_{j_i} \in A$ and consider the idempotent subalgebra*

$$A^b := e_A A e_A$$

of A. Then

- **a.** *$\{e_{j_i}\}_{i=1}^{s}$ is a complete set of primitive orthogonal idempotents of A^b.*
- **b.** *A^b is a basic algebra.*
- **c.** *Up to algebra isomorphism, A^b is independent of the choices.*

Proof. **a.** Note that $e_A e_{j_i} e_A = e_{j_i}$, hence $e_{j_i} \in A^b$ for all $i = 1, \ldots, s$. The result now immediately follows since e_A is the unit of A^b and $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A.

b. Note that the indecomposable projective right A-module $e_{j_k}A$ is supported by e_A. Hence by Lemma 9.8,

$$e_{j_k}A \simeq \mathrm{Ind}_{e_A}\big(\mathrm{Res}_{e_A}(e_{j_k}A)\big).$$

Note that $\mathrm{Res}_{e_A}(e_{j_k}A) = e_{j_k}Ae_A = e_{j_k}e_A A e_A = e_{j_k}A^b$ with the natural right A^b-module structure, hence

$$e_{j_k}A \simeq \mathrm{Ind}_{e_A}\big(e_{j_k}A^b\big), \qquad 1 \leq k \leq r$$

as right A-modules. Consequently, if $e_{j_i}A^b \simeq e_{j_k}A^b$ then $e_{j_i}A \simeq e_{j_k}A$, hence $i = k$. This shows that A^b is a basic algebra.

c. Recall that by Lemma 7.16 we have $A^b \simeq \mathrm{End}_A(e_A A)$ as algebras. But as right A-modules we have, by part **a** and **b** of the lemma,

$$e_A A = \bigoplus_{i=1}^{s} e_{j_i}A \simeq \bigoplus_{i=1}^{s} P_i$$

with $\{P_i\}_{i=1}^{s}$ a complete set of representatives of the equivalence classes of the indecomposable projective right A-modules. Hence

$$A^b \simeq \mathrm{End}_A(P_1 \oplus \cdots \oplus P_s).$$

This shows that A^b is independent of the choices up to algebra isomorphism. □

Theorem 9.11. *The induction functor* $\mathrm{Ind}_{e_A} : \mathrm{Mod}_{A^b} \to \mathrm{Mod}_A$ *is an equivalence of categories with quasi-inverse* Res_{e_A}.

Proof. We use the notations from Lemma 9.10. By [7, Thm. 5.24] the induction functor $\mathrm{Ind}_{e_A} : \mathrm{Mod}_{A^b} \to \mathrm{Mod}_A$ is an equivalence of categories if Ind_{e_A} is fully faithful and essentially surjective. We have already proven that Ind_{e_A} is fully faithful. Let $X \in \mathrm{Ob}\big(\mathrm{Mod}_A\big)$. Since

$$e_A A \simeq P_1 \oplus \cdots \oplus P_s$$

with $\{P_1, \ldots, P_s\}$ a complete set of isoclasses of indecomposable projective right A-modules, Lemma 9.9 shows that $X \simeq \mathrm{Ind}_e(M)$ for some $M \in \mathrm{Ob}\big(\mathrm{Mod}_{A^b}\big)$ (all projective right A-modules are supported by e_A). Hence Ind_{e_A} is essentially surjective. The last statement is immediate. □

We say that two finite dimensional unital k-algebras B and C are Morita equivalent if there exists a k-linear equivalence $\mathrm{Mod}_B \xrightarrow{\sim} \mathrm{Mod}_C$ of categories. Clearly isomorphic algebras are Morita equivalent. The above theorem shows that A is Morita equivalent to the basic algebra A^b. Combined with Theorem 8.27 we conclude the following.

Corollary 9.12. *Suppose that k is algebraically closed. Then A is Morita equivalent to a bounded quiver algebra.*

Exercise 9.13. *Let $n \geq 1$. Show that there exists a k-linear equivalence of categories*

$$\mathrm{Mod}_A \simeq \mathrm{Mod}_{\mathrm{Mat}_n(A)},$$

with $\mathrm{Mat}_n(A)$ the k-algebra of $n \times n$-matrices with coefficients in A.
Hint: *Consider the matrix units $E_{i,j} \in \mathrm{Mat}_n(A)$ with 1 at the (i,j)th entry and zeroes everywhere else. Then consider the idempotents $e_i := E_{i,i}$ $(1 \leq i \leq n)$ and the induction functor $\mathrm{Ind}_{e_1} : \mathrm{Mod}_{e_1\mathrm{Mat}_n(A)e_1} \to \mathrm{Mod}_{\mathrm{Mat}_n(A)}$.*

10. Dynkin quivers

The last goal of the course is to describe the connected quivers without loops whose representation category has finitely many isoclasses of indecomposable quiver representations (Gabriel's Theorem). For this we need various facts about Ext groups and Tits forms (Tits forms are certain integral quadratic forms naturally associated to quivers), which will be the topic of this section. We classify the quivers whose Tits forms are positive definite (the so-called Dynkin quivers).

Other sources for this section are for instance [4, §2.4] (for Ext-groups) and [4, §8.2] (for Tits forms). We will use without proofs some basic results on Ext-groups, as discussed for instance in [7, Chpt. 11].

In this section we write $n(Q)$ for the number of vertices of a quiver Q. We identify the set of vertices Q_0 of Q with $\{1, \ldots, n(Q)\}$ by choosing an enumeration of the vertices.

10.1. **Ext-groups.**

In this subsection Q will be a connected quiver without cycles.

Let $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ and choose a projective resolution of M,

$$\cdots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0. \tag{10.1}$$

For a second quiver representation $N \in \mathrm{Ob}(\mathrm{Rep}_Q)$ apply the left-exact contravariant Hom-functor $\mathrm{Hom}(-, N)$ to the projection resolution. We obtain the cochain complex

$$0 \longrightarrow \mathrm{Hom}(M, N) \xrightarrow{f_0^*} \mathrm{Hom}(P_0, N) \xrightarrow{f_1^*} \mathrm{Hom}(P_1, N) \xrightarrow{f_2^*} \cdots \tag{10.2}$$

in the category Vect_k of finite dimensional vector spaces over k, where

$$f_j^* : \mathrm{Hom}(P_{j-1}, N) \to \mathrm{Hom}(P_j, N), \qquad \psi \mapsto \psi \circ f_j$$

is the pull-back of f_j (here $P_{-1} := M$). Note that

$$0 \longrightarrow \mathrm{Hom}(M, N) \xrightarrow{f_0^*} \mathrm{Hom}(P_0, N) \xrightarrow{f_1^*} \mathrm{Hom}(P_1, N) \tag{10.3}$$

is exact by the left exactness of $\mathrm{Hom}(-, N)$, and hence $\mathrm{im}(f_0^*) = \ker(f_1^*)$. So the first cohomology group of (10.2) is always trivial. For cohomological purposes it therefore suffices to look at the truncated cochain complex

$$0 \longrightarrow \mathrm{Hom}(P_0, N) \xrightarrow{f_1^*} \mathrm{Hom}(P_1, N) \xrightarrow{f_2^*} \mathrm{Hom}(P_2, N) \xrightarrow{f_3^*} \cdots \tag{10.4}$$

obtained from (10.2) by removing $\mathrm{Hom}(M, N)$ from the cochain. The i^{th} cohomology group of (10.4) is denoted by $\mathrm{Ext}^i(M, N)$. It is called the i^{th} Ext group. Concretely we have

$$\mathrm{Ext}^0(M, N) = \ker(f_1^*), \qquad \mathrm{Ext}^i(M, N) = \ker(f_{i+1}^*)/\mathrm{im}(f_i^*)$$

for $i \geq 1$. Note that the k-linearity of Rep_Q implies that the abelian Ext groups are vector spaces over k. The Ext groups are independent of the projective resolution up to isomorphism, by a well-known theorem of Cartan and Eilenberg.

Lemma 10.1. $\mathrm{Ext}^0(M,N) \simeq \mathrm{Hom}(M,N)$ *and* $\mathrm{Ext}^i(M,N) = 0$ *for* $i > 1$.

Proof. By the exactness of (10.3) we have

$$\mathrm{Ext}^0(M,N) = \ker(f_1^*) = \mathrm{im}(f_0^*) \simeq \mathrm{Hom}(M,N)$$

(the last isomorphism is due to the fact that f_0^* is mono).

Recall that the projective dimension of M is ≤ 1, i.e., there exists a projective resolution

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0 \tag{10.5}$$

of length one of M. In the notation of (10.1), we have for $j \geq 2$ that $P_j = 0$, and $f_j : P_j \to P_{j-1}$ is the zero map. Its truncated cochain complex (10.4) becomes

$$0 \longrightarrow \mathrm{Hom}(P_0,N) \xrightarrow{f_1^*} \mathrm{Hom}(P_1,N) \xrightarrow{0} 0 \longrightarrow \cdots,$$

i.e., $\mathrm{Hom}(P_j,N) = 0$ and $f_j^* : \mathrm{Hom}(P_{j-1},N) \to \mathrm{Hom}(P_j,N)$ is the zero map when $j \geq 2$. Hence for $i > 1$,

$$\mathrm{Ext}^i(M,N) = \ker(f_{i+1}^*)/\mathrm{im}(f_i^*) = \mathrm{Hom}(P_i,N) = 0.$$

□

Remark 10.2. Computing $\mathrm{Ext}^1(M,N)$ with respect to a projective resolution (10.5) of M of length one, we have

$$\mathrm{Ext}^1(M,N) = \mathrm{Hom}(P_1,N)/\mathrm{im}(f_1^*) = \mathrm{coker}(f_1^*).$$

Hence in this case (10.3) extends to an exact sequence

$$0 \longrightarrow \mathrm{Hom}(M,N) \xrightarrow{f_0^*} \mathrm{Hom}(P_0,N) \xrightarrow{f_1^*} \mathrm{Hom}(P_1,N) \xrightarrow{\pi} \mathrm{Ext}^1(M,N) \longrightarrow 0,$$

with π the canonical map. It is important to realise that this only holds true for projective resolutions of length one!

Definition 10.3. (1) *A short exact sequence*

$$0 \longrightarrow N \hookrightarrow E \twoheadrightarrow M \longrightarrow 0 \tag{10.6}$$

in Rep_Q *is called an extension of* M *by* N.

(2) *Two extensions*

$$0 \longrightarrow N \hookrightarrow E \twoheadrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow N \hookrightarrow E' \twoheadrightarrow M \longrightarrow 0$$

of M *by* N *are said to be equivalent if there exists a diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \hookrightarrow & E & \twoheadrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_N} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle 1_M} & & \\ 0 & \longrightarrow & N & \hookrightarrow & E' & \twoheadrightarrow & M & \longrightarrow & 0 \end{array}$$

in Rep_Q *with the squares commuting.*

Exercise 10.4. *Show that the isomorphism ξ in Definition 10.3(2) is an isomorphism.*

We write $\mathcal{E}(M,N)$ for the extensions of M by N up to equivalence. One should think of $\mathcal{E}(M,N)$ as the possible ways to "glue" N onto M in order to form a new representation E of the quiver Q. The trivial extension of M by N is the equivalence class of

$$0 \hookrightarrow N \overset{\iota_N}{\hookrightarrow} M \oplus N \overset{\pi_M}{\twoheadrightarrow} M \longrightarrow 0 \tag{10.7}$$

in $\mathcal{E}(M,N)$ (see Section 3 for the notations).

The extension (10.6) of M by N induces a long exact sequence

$$0 \longrightarrow \mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(E,N) \longrightarrow \mathrm{End}(N) \xrightarrow{f} \mathrm{Ext}^1(M,N) \longrightarrow \mathrm{Ext}^1(E,N) \longrightarrow \cdots$$

(see, e.g., [7, Thm. 11.5]). Set $\theta(e) := f(1_N) \in \mathrm{Ext}^1(M,N)$, with $e \in \mathcal{E}(M,N)$ the equivalence class of the extension. This is well defined, and we have

Theorem 10.5. *The map $\theta : \mathcal{E}(M,N) \to \mathrm{Ext}^1(M,N)$ is a bijection.*

Proof. See, e.g., [7, §11.3]. □

The abelian group structure on $\mathrm{Ext}^1(M,N)$ can be transported to $\mathcal{E}(M,N)$ through the bijection θ. The resulting group operation on $\mathcal{E}(M,N)$ is called the Baer sum. It can be described intrinsically, see [4, §2.4]. The trivial extension (10.7) of M by N is the neutral element.

10.2. **The Tits form of a quiver.** In the remainder of this section all quivers will be without loops (i.e., there are no edges α satisfying $s(\alpha) = t(\alpha)$). The quivers are allowed to have cycles, unless explicitly stated otherwise. We will use the notations

Q': an arbitrary quiver without loops,
$\overline{Q}$: an Euclidean quiver,
Q: a Dynkin quiver,

where Euclidean and Dynkin quivers are special types of quivers without loops introduced below.

Definition 10.6. *A map $q : \mathbb{Z}^m \to \mathbb{Z}$ is called an integral quadratic form if there exist $b_{ij} \in \mathbb{Z}$ $(1 \leq i,j \leq m)$ such that*

$$q(x) = \sum_{i,j=1}^{m} b_{ij} x_i x_j \qquad \forall\, x = (x_1, \ldots, x_m) \in \mathbb{Z}^m.$$

An integral quadratic form q gives rise to a symmetric $\mathbb{Z}$-bilinear form $(\cdot,\cdot) : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ by the formula

$$(x,y) := q(x+y) - q(x) - q(y) \qquad (x,y \in \mathbb{Z}^m).$$

Concretely we have $(x,y) = \sum_{i,j=1}^m a_{ij}x_iy_j$ with $a_{ij} := b_{ij} + b_{ji}$, and hence $(x,x) = 2q(x)$. We call the symmetric matrix $A = (a_{ij})_{i,j=1}^m$ the matrix associated to q.

Definition 10.7. *The Tits form $q_{Q'}$ is the integral quadratic form $q_{Q'} : \mathbb{Z}^{n(Q')} \to \mathbb{Z}$ defined by*

$$q_{Q'}(x) := \sum_{i=1}^{n(Q')} x_i^2 - \sum_{\alpha \in Q'_1} x_{s(\alpha)}x_{t(\alpha)}.$$

Its associated symmetric $\mathbb{Z}$-bilinear form is denoted by $(\cdot,\cdot)_{Q'}$. It is called the Euler form of Q'.

For $i,j \in Q'_0$ let n'_{ij} be the number of edges in Q' connecting i and j. Here we count both the edges $i \to j$ and the edges $j \to i$ in Q' (in particular, $n'_{ij} = n'_{ji}$). Note that $n'_{ii} = 0$ by the assumption that Q' has no loops. The matrix $A = (a_{ij})_{i,j=1}^{n(Q')}$ associated to $q_{Q'}$ is thus given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -n'_{ij} & \text{if } i \neq j. \end{cases}$$

We furthermore have

$$\begin{aligned} q_{Q'}(x) &= \sum_{i=1}^{n(Q')} x_i^2 - \sum_{1 \le i < j \le n(Q')} n'_{ij}x_ix_j, \\ (x,y)_{Q'} &= 2\sum_{i=1}^{n(Q')} x_iy_i - \sum_{1 \le i \neq j \le n(Q')} n'_{ij}x_iy_j. \end{aligned} \tag{10.8}$$

Note that the Tits form and the Euler form only depend on the unoriented graph underlying Q'.

The explicit form of the standard projective resolution of M of length one (see Section 5) now gives the following result.

Proposition 10.8. *Let $M \in \mathrm{Ob}(\mathrm{Rep}_{Q'})$ and write*

$$d_M := (\dim_k(M_1), \ldots, \dim_k(M_{n(Q')})) \in \mathbb{Z}_{\geq 0}^{n(Q')}$$

for its dimension vector. If M has no cycles, then

$$q_{Q'}(d_M) = \dim_k(\mathrm{End}(M)) - \dim_k(\mathrm{Ext}^1(M,M)).$$

Proof. Take the standard projective resolution of M of length one. The corresponding two projective representations P_0 and P_1 in (10.5) are then given by

$$P_0 = \bigoplus_{i \in Q_0'} P(i)^{M_i}, \qquad P_1 = \bigoplus_{\alpha \in Q_1'} P(t(\alpha))^{M_{s(\alpha)}} \tag{10.9}$$

(recall that $P(i)^V$ is the direct sum of $\dim_k(V)$ copies of $P(i)$). By Remark 10.2, we obtain an exact sequence

$$0 \longrightarrow \mathrm{End}(M) \longrightarrow \mathrm{Hom}(P_0, M) \longrightarrow \mathrm{Hom}(P_1, M) \longrightarrow \mathrm{Ext}^1(M, M) \longrightarrow 0 \tag{10.10}$$

of vector spaces. Then

$$\dim_k(\mathrm{End}(M)) - \dim_k(\mathrm{Ext}^1(M,M)) = \dim_k(\mathrm{Hom}(P_0, M)) - \dim_k(\mathrm{Hom}(P_1, M))$$

(here we used the well known fact that

$$\dim_k(V) = \dim_k(\ker(T)) + \dim_k(\mathrm{im}(T))$$

for a linear operator $T : V \to W$ between finite dimensional vector spaces V and W). So it suffices to show that the right hand side of this identity equals $q_{Q'}(d_M)$.

For this recall that

$$\dim_k(\mathrm{Hom}(P(i), M)) = \dim_k(M_i) \qquad (i \in Q_0')$$

(see Theorem 4.3), so that

$$\dim_k(\mathrm{Hom}(P_0, M)) = \sum_{i=1}^{n(Q')} \dim_k(M_i) \dim_k(\mathrm{Hom}(P(i), M)) = \sum_{i=1}^{n(Q')} \dim_k(M_i)^2,$$
$$\dim_k(\mathrm{Hom}(P_1, M)) = \sum_{\alpha \in Q_1'} \dim_k(M_{s(\alpha)}) \dim_k(M_{t(\alpha)}).$$

Hence the desired result follows from the definition of the Tits form. □

Exercise 10.9. *Prove the following generalisation of Proposition 10.8:*

$$\begin{aligned}\big(d_{M_1}, d_{M_2}\big)_{Q'} &= \dim_k(\mathrm{Hom}(M_1, M_2)) + \dim_k(\mathrm{Hom}(M_2, M_1)) \\ &\quad - \dim_k(\mathrm{Ext}^1(M_1, M_2)) - \dim_k(\mathrm{Ext}^1(M_2, M_1))\end{aligned}$$

for $M_1, M_2 \in \mathrm{Ob}(\mathrm{Rep}_{Q'})$.

10.3. **Euclidean quivers.**

Definition 10.10. *An integral quadratic form $q : \mathbb{Z}^m \to \mathbb{Z}$ is called*

(1) positive definite if $q(x) > 0$ for all $x \in \mathbb{Z}^m \setminus \{0\}$.
(2) positive semi-definite if $q(x) \geq 0$ for all $x \in \mathbb{Z}^m$.

Note that the previous proposition bounds the dimension of $\mathrm{Ext}^1(M, M)$ by $\dim_k(\mathrm{End}(M))$ when $q_{Q'}$ is positive (semi)definite. This will play an important role in the proof of Gabriel's Theorem (see Section 11).

Example 10.11. (1) *Consider the Kronecker quiver $\overline{Q}$, given by*

$$\bullet \rightrightarrows \bullet$$

Then $q_{\overline{Q}}(x) = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 - x_2)^2$, so $q_{\overline{Q}}$ is positive semi-definite but not positive definite.
(2) *Consider the quiver Q, given by*

$$\bullet \rightarrow \bullet$$

In this case

$$q_Q(x) = x_1^2 + x_2^2 - x_1x_2 = \frac{1}{2}\big(x_1 + (-\frac{1}{2} + \frac{1}{2}\sqrt{3})x_2\big)^2 + \frac{1}{2}\big(x_1 + (-\frac{1}{2} - \frac{1}{2}\sqrt{3})x_2\big)^2,$$

and hence q_Q is positive definite.

The following proposition gives an easy criterium for positive semi-definiteness of a Tits form.

Proposition 10.12. *Let $\overline{Q}$ be a connected quiver without loops. Suppose there exists a nonzero vector $d = (d_1, \ldots, d_{n(\overline{Q})}) \in \mathbb{Z}_{\geq 0}^{n(\overline{Q})}$ such that*

$$(d, x)_{\overline{Q}} = 0 \qquad \forall\, x \in \mathbb{Z}^{n(\overline{Q})}.$$

Then

(1) *$q_{\overline{Q}}$ is positive semi-definite.*
(2) *$d_i > 0$ for $i = 1, \ldots, n(\overline{Q})$.*
(3) *$q_{\overline{Q}}(x) = 0 \;\Leftrightarrow\; x \in \mathbb{Z}^{n(\overline{Q})} \cap \mathbb{Q}d$.*

Proof. Write $\epsilon_i \in \mathbb{Z}^{n(\overline{Q})}$ for the i^{th} standard basis element of $\mathbb{Z}^{n(\overline{Q})}$ (having a one at the i^{th} entry and zeroes everywhere else). Then $(d, \epsilon_i)_{\overline{Q}} = 0$ for $1 \leq i \leq n(\overline{Q})$ implies that

$$d_i = \frac{1}{2}\sum_{j \neq i} \overline{n}_{ij} d_j \qquad (1 \leq i \leq n(\overline{Q})) \tag{10.11}$$

by (10.8), with $\overline{n}_{ij}$ the number of edges in $\overline{Q}$ connecting i and j.

We now first show (2). Suppose that $d_i = 0$ for some $i \in \{1, \ldots, n(\overline{Q})\}$. Then (10.11) and $d_k \geq 0$ ($1 \leq k \leq n(\overline{Q})$) implies that $d_j = 0$ if j is connected to i by an edge. Repeating this argument and using that $\overline{Q}$ is connected, we conclude that $d_j = 0$ for all $j \in \{1, \ldots, n(\overline{Q})\}$. This is a contradiction.

We now return to the proof of (1) and (3). It follows from (10.11) and the previous paragraph that

$$1 = \frac{1}{2d_i}\sum_{j \neq i} \overline{n}_{ij} d_j \qquad (1 \leq i \leq n(\overline{Q})).$$

Hence

$$\sum_{i=1}^{n(\overline{Q})} x_i^2 = \sum_{i=1}^{n(\overline{Q})} \frac{x_i^2}{2d_i} \sum_{j\neq i} \overline{n}_{ij} d_j \tag{10.12}$$
$$= \sum_{1\leq i\neq j\leq n(\overline{Q})} \frac{\overline{n}_{ij} d_j}{2} \frac{x_i^2}{d_i}$$
$$= \sum_{1\leq i<j\leq n(\overline{Q})} \Big(\frac{\overline{n}_{ij} d_j}{2} \frac{x_i^2}{d_i} + \frac{\overline{n}_{ij} d_i}{2} \frac{x_j^2}{d_j} \Big) = \sum_{1\leq i<j\leq n(\overline{Q})} \frac{\overline{n}_{ij} d_i d_j}{2} \Big(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} \Big)$$

(here we used $\overline{n}_{ij} = \overline{n}_{ji}$ in the third equality). As a consequence

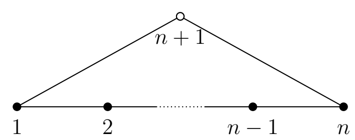
$$q_{\overline{Q}}(x) = \sum_{i=1}^{n(\overline{Q})} x_i^2 - \sum_{1\leq i<j\leq n(\overline{Q})} \overline{n}_{ij} x_i x_j = \sum_{1\leq i<j\leq n(\overline{Q})} \frac{\overline{n}_{ij} d_i d_j}{2} \Big(\frac{x_i}{d_i} - \frac{x_j}{d_j} \Big)^2,$$

which immediately imply (1) and (3) (for (3) use once more that $\overline{Q}$ is connected). □

Example 10.13. *For the quiver $\overline{Q}$ from example 10.11*(1) *the vector $d = (1,1)$ satisfies $(d,x)_{\overline{Q}} = 0$ for all $x \in \mathbb{Z}^2$.*

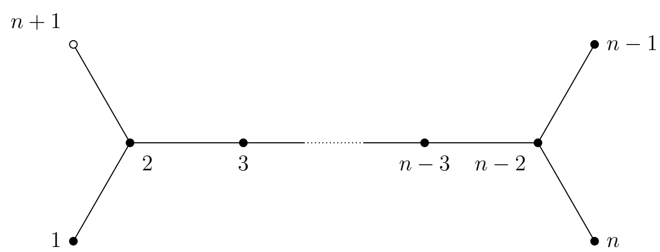
Consider the following two infinite families of connected graphs with enumerated vertices,

$\widetilde{A}_n$:



$(n \geq 1)$

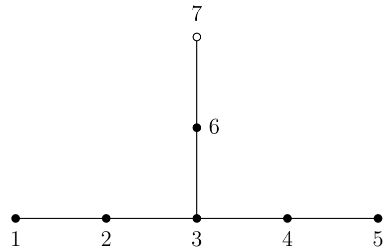
$\widetilde{D}_n$:



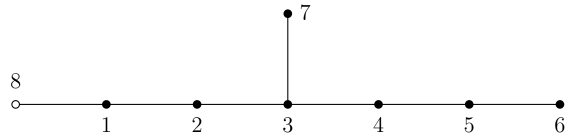
$(n \geq 4)$

as well as the following three exceptional ones,

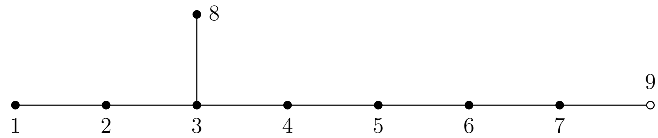
$\widetilde{E}_6$:



$\widetilde{E}_7$:



$\widetilde{E}_8$:



Here $\widetilde{X}_n$ $(X = A, D, E)$ is called the type of the graph. Note that a graph of type $\widetilde{X}_n$ has $n+1$ vertices.

We have denoted one of the vertices by a circle instead of a solid dot. This vertex will play a special role later on. We call it the *affine vertex*.

Definition 10.14. *A quiver $\overline{Q}$ is said to be Euclidean if the underlying unoriented graph is of type $\widetilde{A}_n$ $(n \geq 1)$, $\widetilde{D}_n$ $(n \geq 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$.*

Note that an Euclidean quiver does not have loops. We will say that an Euclidean quiver $\overline{Q}$ is of type $\widetilde{X}_n$ if the underlying unoriented graph is of type $\widetilde{X}_n$.

We will see in a moment that Euclidean quivers are the connected quivers without loops with positive semi-definite Tits forms. The following lemma is the first step.

Lemma 10.15. *Suppose $\overline{Q}$ is an Euclidean quiver.*

There exists a nonzero vector $d \in \mathbb{Z}_{\geq 0}^{n(\overline{Q})}$ such that $(d, x)_{\overline{Q}} = 0$ for all $x \in \mathbb{Z}^{n(\overline{Q})}$. In particular, $q_{\overline{Q}}$ is positive semi-definite, but not positive definite.

Proof. The second part of the lemma follows from Proposition 10.12. For each of the types we need to construct a vector $d \in \mathbb{Z}_{>0}^{n(\overline{Q})}$ such that (10.11) holds true for $i = 1, \ldots, n(\overline{Q})$.

If the unoriented graph underlying $\overline{Q}$ is of type $\widetilde{A}_1$ then $d = (1,1)$ does the job (see Example 10.11(1) and Example 10.13). If the type of the unoriented graph underlying $\overline{Q}$ is not $\widetilde{A}_1$ then $\overline{n}_{ij} = 1$ for all $i \neq j$, and (10.11) is the rule that for each vertex i,

$$2d_i = \sum_{j:j\sim i} d_j, \tag{10.13}$$

with $j \sim i$ meaning that the vertices i and j are connected by an edge. One then easily determines for each type $\widetilde{X}_n$ a vector $d \in \mathbb{Z}_{>0}^{n+1}$ satisfying (10.13), using the explicit shape of the graph. For instance, for $\overline{Q}$ of type $\widetilde{E}_8$ the 9-vector

$$d = (2, 4, 6, 5, 4, 3, 2, 3, 1)$$

does the job (here we have used the enumeration of the vertices as indicated in the graph of type $\widetilde{E}_8$ above). □

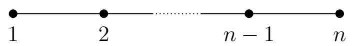
Exercise 10.16. *Write down a vector* $d \in \mathbb{Z}_{>0}^{n(\overline{Q})}$ *such that* $q_{\overline{Q}}(d) = 0$ *for all Euclidean quivers* $\overline{Q}$.

10.4. ADE classification of the Dynkin quivers.

Definition 10.17. *The quivers obtained from the Euclidean quivers by removing the affine vertex and the edges connected to it, are called Dynkin quivers.*

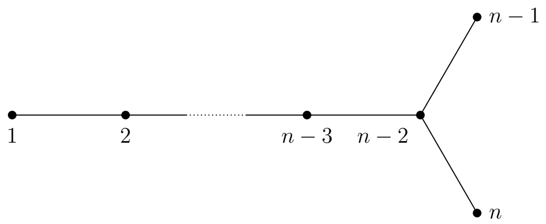
The unoriented graphs underlying the Dynkin quivers are called simply laced Dynkin diagrams. There are two infinite families of simply laced Dynkin diagrams,

A_n :



$(n \geq 1)$

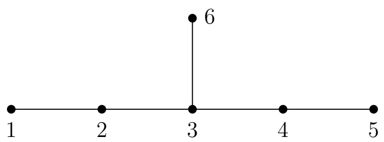
D_n :



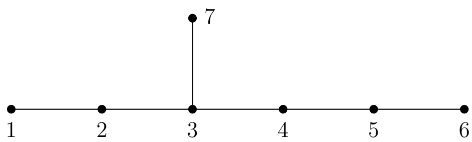
$(n \geq 4)$

and three exceptional simply laced Dynkin diagrams

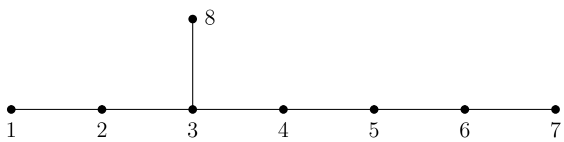
E_6 :



E_7 :



E_8 :



We call X_n ($X = A, D, E$) the type of the Dynkin graph. We will furthermore say that a Dynkin quiver Q is of type X_n if the underlying unoriented graph is of type X_n. Note that $n(Q) = n$ if Q is of type X_n. Furthermore, a Dynkin quiver Q is connected, and it does not have cycles.

The matrix $(a_{ij})_{i,j=1}^n \in \mathrm{Mat}_{n,n}(\mathbb{Z})$ associated to the Tits form q_Q of a Dynkin quiver Q of type X_n is called the Cartan matrix of type X_n. Note that it satisfies $a_{ii} = 2$ and $a_{ij} = a_{ji} \leq 0$ for $i \neq j$.

Exercise 10.18. *Write down the Cartan matrices of the Dynkin quivers. Are they invertible?*

Remark 10.19. Dynkin diagrams also naturally appear as part of the Killing-Cartan classification of the simple Lie algebras over an algebraically closed field of characteristic zero. In fact, the simply laced Dynkin diagrams have the habit of showing up unexpectedly in many apparently different contexts (representation theory, Lie theory, singularity theory, mathematical physics, string theory....). This was already noticed by Arnold in the seventies (see, e.g., [2] for a discussion).

Lemma 10.20. *The Tits form q_Q of a Dynkin quiver Q is positive definite.*

Proof. Suppose that Q is obtained from the Euclidean quiver $\overline{Q}$ of type $\widetilde{X}_n$ by deleting the affine vertex (which is labeled in the diagrams above by $n+1$) and deleting the edges connected to it. Comparing the Tits forms $q_{\overline{Q}}$ and q_Q we have

$$q_{\overline{Q}}(x) = q_Q(x) + x_{n+1}^2 - \sum_{i=1}^{n} \overline{n}_{i,n+1} x_i x_{n+1} \qquad (x \in \mathbb{Z}^{n+1}), \tag{10.14}$$

with $\overline{n}_{ij}$ the number of edges in $\overline{Q}$ connecting i to j.

Let $d \in \mathbb{Z}_{>0}^{n+1}$ such that $(d,x)_{\overline{Q}} = 0$ for all $x \in \mathbb{Z}^{n+1}$ (see Exercise 10.16). Suppose that $y \in \mathbb{Z}^n$ with $q_Q(y) \leq 0$. By (10.14) we then have $q_{\overline{Q}}((y,0)) \leq 0$, and hence $(y,0) \in \mathbb{Q}d \cap \mathbb{Z}^{n+1}$ by Proposition 10.12(3). By Proposition 10.12(2) the zero vector is the only vector in $\mathbb{Q}d \cap \mathbb{Z}^{n+1}$ with last entry equal to zero. Hence $y = 0$, and consequently q_Q is positive definite. □

Lemma 10.21. *Suppose that Q' has at least one vertex. Suppose furthermore that Q' is neither a Dynkin quiver nor an Euclidean quiver. Then Q' contains a proper Euclidean subquiver* (*proper in the sense that it contains less vertices and/or less edges*).

Proof. We may assume without loss of generality that Q' is connected. If the unoriented graph underlying Q' contains a cycle, then it contains an Euclidean subquiver of type $\widetilde{A}_m$ for some $m \geq 1$. This subquiver will be proper, since Q' is not Euclidean.

So we only need to consider the case that the unoriented graph underlying Q' has no cycles, i.e., Q' is an oriented tree. Q' is not a Dynkin diagram of type A_m ($m \geq 1$), hence Q' must have at least one branch point. If Q' has more than one branch point, then Q' contains a subquiver of type $\widetilde{D}_m$ for some $m \geq 5$, which will be proper since Q' is not Euclidean. If Q' has one branch point with valency ≥ 4, then Q' properly contains a subquiver of type $\widetilde{D}_4$.

So it remains to consider the case that Q' has a single branch point of valency 3. Denote by (k, ℓ, m) the number of edges of its three branches with $1 \leq k \leq \ell \leq m$. Since Q' is not a Dynkin quiver of type D_m, we have $(k, \ell) \neq (1, 1)$. If $k \geq 2$ then Q' properly contains a subquiver of type $\widetilde{E}_6$. If $k = 1$ and $\ell \geq 2$ then

$$(\ell, m) \notin \{(2,2), (2,3), (2,4), (2,5), (3,3)\}$$

since Q' is not of type E_6, E_7, E_8, $\widetilde{E}_8$ and $\widetilde{E}_7$, respectively. If $(k, \ell) = (1, 2)$ and $m \geq 6$ then Q' properly contains an Euclidean subquiver of type $\widetilde{E}_8$. If $(k, \ell) = (1, 3)$ and $m \geq 4$ then Q' properly contains an Euclidean subquiver of type $\widetilde{E}_7$. Finally, if $k = 1$ and $4 \leq \ell \leq m$ then Q' properly contains an Euclidean subquiver of type $\widetilde{E}_7$. This completes the proof of the lemma. □

Theorem 10.22. *If Q' is a nonempty connected quiver without loops, then*

(1) *Q' is a Dynkin quiver $\Leftrightarrow$ $q_{Q'}$ is positive definite.*
(2) *Q' is an Euclidean quiver $\Leftrightarrow$ $q_{Q'}$ is positive semi-definite, but not positive definite.*

Proof. Suppose that $q_{Q'}$ is positive semi-definite. By Lemma 10.20 and Lemma 10.15 it suffices to show that Q' is a Dynkin quiver or an Euclidean quiver.

Suppose that Q' is neither Dynkin nor Euclidean. Let $\overline{Q}$ be a proper Euclidean subquiver, which exists due to Lemma 10.21. Write $\overline{n}_{ij}$ (resp. n'_{ij}) for the number of edges connecting the vertices i and j in $\overline{Q}$ (resp. in Q'). Then $n(\overline{Q}) \leq n(Q')$ and $\overline{n}_{ij} \leq n'_{ij}$ for $1 \leq i \neq j \leq n(\overline{Q})$. Let $d \in \mathbb{Z}_{>0}^{n(\overline{Q})}$ such that $q_{\overline{Q}}(d) = 0$ (see Proposition 10.12).

Suppose that $n(\overline{Q}) = n(Q')$. Then there exist indices $1 \leq k < \ell \leq n(\overline{Q})$ such that $\overline{n}_{k\ell} < n'_{k\ell}$, and hence

$$\begin{aligned} q_{Q'}(d) &= \sum_{i=1}^{n(\overline{Q})} d_i^2 - \sum_{1 \leq i < j \leq n(\overline{Q})} n'_{ij} d_i d_j \\ &< \sum_{i=1}^{n(\overline{Q})} d_i^2 - \sum_{1 \leq i < j \leq n(\overline{Q})} \overline{n}_{ij} d_i d_j = q_{\overline{Q}}(d) = 0, \end{aligned}$$

contradicting the assumption that $q_{Q'}$ is positive semi-definite.

Suppose that $n(\overline{Q}) < n(Q')$. Choose vertices $k \in \overline{Q}_0$ and $\ell \in Q'_0 \setminus \overline{Q}_0$ such that $n'_{k\ell} \geq 1$ (they exist since Q' is connected). Define $\delta \in \mathbb{Z}_{\geq 0}^{n(Q')}$ by

$$\delta_i = \begin{cases} 2d_i & \text{if } i \in \overline{Q}_0, \\ 1 & \text{if } i = \ell, \\ 0 & \text{if } i \in Q'_0 \setminus (\overline{Q}_0 \cup \{\ell\}). \end{cases}$$

Then

$$q_{Q'}(\delta) \leq 4q_{\overline{Q}}(d) + 1 - 2\sum_{i=1}^{n(\overline{Q})} n'_{i\ell} d_i \leq 1 - 2n'_{k\ell} d_k < 0,$$

which again contradicts the assumption that $q_{Q'}$ is positive semi-definite. □

11. Gabriel's Theorem

In this last section of the lecture notes we discuss Gabriel's Theorem, which states that the category of representations of a connected quiver without loops has finitely many isoclasses of indecomposable quiver representations if and only if the quiver is a Dynkin quiver. This is also discussed in, e.g., [4, §8.3 & §8.4].

In this part of the syllabus Q' will be a connected quiver without cycles and at least one vertex unless stated explicitly otherwise. We write $n(Q')$ for the number of vertices of Q'. We identify the set of vertices Q'_0 of Q' with $\{1, \ldots, n(Q')\}$ by choosing some enumeration of the vertices.

11.1. **Root systems.** Note that the simple representations $S(i)$ of Q' ($i \in Q'_0$) satisfy

$$\mathrm{End}(S(i)) = k1_{S(i)}.$$

The following notion of a brick thus generalises simple representations.

Definition 11.1. *$M \in \mathrm{Ob}(\mathrm{Rep}_{Q'})$ is called a brick if $M \neq 0$ and $\mathrm{End}(M) = k1_M$.*

Remark 11.2. A brick M is indecomposable since $\mathrm{End}(M) = k1_M$ is a local ring.

For a representation $M = (M_i, \phi_\alpha)_{i \in Q'_0, \alpha \in Q'_1}$ of Q' we write

$$d_M := (\dim_k(M_1), \ldots, \dim_k(M_{n(Q')})) \in \mathbb{Z}_{\geq 0}^{n(Q')}$$

for its dimension vector. Recall that $q_{Q'}$ is the Tits form of Q'.

Lemma 11.3. *Suppose that Q is a Dynkin quiver and $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ is a brick. Then*

(1) $\mathrm{Ext}^1(M, M) = 0$,
(2) $q_Q(d_M) = 1$.

Proof. Note that q_Q is positive definite since Q is Dynkin, and $d_M \neq 0$ since M is a brick. Hence $q_Q(d_M) \in \mathbb{Z}_{>0}$. On the other hand, by Proposition 10.8,

$$q_Q(d_M) = \dim_k(\mathrm{End}(M)) - \dim_k(\mathrm{Ext}^1(M, M)) = 1 - \dim_k(\mathrm{Ext}^1(M, M)),$$

where we use that M is a brick for the second equality. These two conditions force that $q_Q(d_M) = 1$ and $\dim_k(\mathrm{Ext}^1(M, M)) = 0$. □

Definition 11.4. *Let Q' be an Euclidean quiver or a Dynkin quiver.*

(a) *We call $x \in \mathbb{Z}^{n(Q')}$ a root if $x \neq 0$ and $q_{Q'}(x) \in \{0, 1\}$.*

We write $\Phi_{Q'}$ for the set of roots in Q'.

(b) *A root $x \in \Phi_{Q'}$ is called a real root if $q_{Q'}(x) = 1$, and an imaginary root if $q_{Q'}(x) = 0$.*

Remark 11.5. (1) For a Dynkin quiver Q all roots are real.
(2) Let $\overline{Q}$ be an Euclidean quiver. Let $d \in \mathbb{Z}_{>0}^{n(\overline{Q})}$ be the unique vector with strictly positive coordinates such that $(d, x)_{\overline{Q}} = 0$ for all $x \in \mathbb{Z}^{n(\overline{Q})}$ and $\mathrm{ggd}(\{d_i\}_{i=1}^{n(\overline{Q})}) = 1$

(see Proposition 10.12 and Lemma 10.15). Then $\mathbb{Z}^\times d$ is the subset of imaginary roots in $\Phi_{\overline{Q}}$. In particular, all the coordinates of an imaginary root are nonzero.

Write $\epsilon_i \in \mathbb{Z}^{n(Q')}$ for the i^{th} standard basis vector in $\mathbb{Z}^{n(Q)}$, with its i^{th} coordinate equal to one and the other coordinates zero.

Note that ϵ_i is the dimension vector of the simple representation $S(i)$.

Definition 11.6. *Let Q' be an Euclidean quiver or a Dynkin quiver.*

1. *The real roots $\epsilon_i \in \Phi_{Q'}$ $(1 \le i \le n(Q'))$ are said to be simple. We write $\Delta_{Q'} = \{\epsilon_1, \dots, \epsilon_{n(Q')}\}$ for the set of simple real roots.*
2. *A root $\alpha \in \Phi_{Q'}$ is said to be positive if $\alpha \in \sum_{i=1}^{n(Q')} \mathbb{Z}_{\ge 0}\epsilon_i$, and negative if $\alpha \in \sum_{i=1}^{n(Q')} \mathbb{Z}_{\le 0}\epsilon_i$. We write $\Phi^+_{Q'}$ and $\Phi^-_{Q'}$ for the subset of positive and negative roots, respectively.*

Note that $\Delta_{Q'} \subseteq \Phi^+_{Q'}$ and $\Phi^+_{Q'} \cap \Phi^-_{Q'} = \emptyset$. The Tits form $q_{Q'}$ is a quadratic form, so $q_{Q'}(-x) = q_{Q'}(x)$. Hence $-\alpha \in \Phi_{Q'}$ if $\alpha \in \Phi_{Q'}$. In particular, $\Phi^-_{Q'} = -\Phi^+_{Q'}$.

Lemma 11.7. *Let $\overline{Q}$ be an Euclidean quiver and $d \in \mathbb{Z}^{n(\overline{Q})}_{>0}$ as in Remark 11.5(2). Then $\alpha - d \in \Phi_{\overline{Q}}$ for all $\alpha \in \Phi_{\overline{Q}} \setminus \{d\}$.*

Proof. For $\alpha \in \Phi_{\overline{Q}} \setminus \{d\}$ we have $\alpha - d \neq 0$ and

$$\begin{aligned} q_{\overline{Q}}(\alpha - d) &= -(\alpha, d)_{\overline{Q}} + q_{\overline{Q}}(\alpha) + q_{\overline{Q}}(-d) \\ &= 0 + q_{\overline{Q}}(\alpha) + 0 \in \{0, 1\}, \end{aligned}$$

hence $\alpha - d \in \Phi_{\overline{Q}}$. □

Proposition 11.8. *Let Q' be an Euclidean quiver or a Dynkin quiver. Then*

$$\Phi_{Q'} = \Phi^+_{Q'} \cup \Phi^-_{Q'}.$$

Proof. Let $\alpha \in \Phi_{Q'}$ and write $\alpha = \sum_{i=1}^{n(Q')} \alpha_i \epsilon_i$ with $\alpha_i \in \mathbb{Z}$. Set $I = \{i \mid \alpha_i > 0\}$ and $J = \{j \mid \alpha_j < 0\}$, and write

$$\alpha = \beta + \gamma$$

with $\beta := \sum_{i \in I} \alpha_i \epsilon_i$ and $\gamma := \sum_{j \in J} \alpha_j \epsilon_j$. By (10.8) we conclude that

$$(\beta, \gamma)_{Q'} = -\sum_{(i,j) \in I \times J} n'_{ij} \alpha_i \alpha_j \ge 0.$$

Hence

$$\begin{aligned} \{0, 1\} \ni q_{Q'}(\alpha) = q_{Q'}(\beta + \gamma) &= (\beta, \gamma)_{Q'} + q_{Q'}(\beta) + q_{Q'}(\gamma) \\ &\ge q_{Q'}(\beta) + q_{Q'}(\gamma). \end{aligned}$$

On the other hand, $q_{Q'}(\beta) \ge 0$ and $q_{Q'}(\gamma) \ge 0$ since $q_{Q'}$ is positive semi-definite. It follows that $q_{Q'}(\beta) = 0$ or $q_{Q'}(\gamma) = 0$ (or both). Without loss of generality we may assume that $q_{Q'}(\beta) = 0$ (otherwise replace α by $-\alpha$). If $\beta = 0$ then

$\alpha = \gamma \in \Phi_{Q'}^-$ and we are done. If $\beta \neq 0$ then β is an imaginary root of $\Phi_{Q'}$. This forces $I = \{1, \ldots, n(Q')\}$ by Remark 11.5(2), hence $\gamma = 0$ and $\alpha = \beta \in \Phi_{Q'}^+$. □

Corollary 11.9. *If Q is a Dynkin quiver then $\#\Phi_Q < \infty$.*

Proof. Let $\overline{Q}$ be an Euclidean quiver such that Q is its subquiver obtained by removing the affine vertex and the edges connected to it. We take the enumeration of the vertices as in Section 10, such that the affine vertex of $\overline{Q}$ comes last. Let $d \in \mathbb{Z}_{>0}^{n(\overline{Q})}$ be as in Remark 11.5(2). For $\alpha \in \Phi_Q^+$ we have

$$q_{\overline{Q}}((\alpha, 0)) = q_Q(\alpha) = 1.$$

In particular $(\alpha, 0) \in \Phi_{\overline{Q}}^+$ is a real root, which is not equal to d. By Lemma 11.7 we have $\beta := (\alpha, 0) - d \in \Phi_{\overline{Q}}$, which is a negative root by Proposition 11.8 and the fact that $\beta_{n(\overline{Q})} = -d_{n(\overline{Q})} < 0$. Hence $\beta_j \leq 0$ for all $1 \leq j \leq n(Q)$, i.e.,

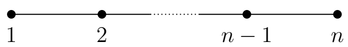
$$0 \leq \alpha_i \leq d_i \qquad \forall\, i \in \{1, \ldots, n(Q)\}.$$

We conclude that

$$\Phi_Q^+ \subseteq \Big\{ \sum_{i=1}^{n(Q)} \alpha_i \epsilon_i \;\mid\; 0 \leq \alpha_i \leq d_i \quad (i = 1, \ldots, n(Q)) \Big\}, \tag{11.1}$$

hence Φ_Q^+ is a finite set. Proposition 11.8 implies that Φ_Q is also a finite set. □

Exercise 11.10. *Let Q be the Dynkin quiver of type A_n, whose underlying graph is given by*



$(n \geq 1)$.

Show that

$$\Phi_Q^+ = \{ \sum_{r=s}^{t} \epsilon_r \;\mid\; 1 \leq s \leq t \leq n \}.$$

11.2. **Formulation of the theorem.** In this subsection we assume that k is algebraically closed. We write $\mathcal{I}_{Q'}$ for the isoclasses of indecomposable representations of Q'. We write $[M]$ for the isoclass of the indecomposable representation M of Q'. In this subsection we classify the following class of quivers.

Definition 11.11. *Q' is said to be of finite representation type if $\#\mathcal{I}_{Q'} < \infty$.*

The main result is as follows.

Theorem 11.12 (Gabriel)**.** *We have*

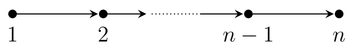
(1) *Q' is of finite representation type iff Q' is a Dynkin quiver.*

Suppose that Q is a Dynkin quiver.

(2) *A nonzero representation M of Q is indecomposable iff M is a brick.*
(3) *The assignment $[M] \mapsto d_M$ defines a bijection $\mathcal{I}_Q \xrightarrow{\sim} \Phi_Q^+$.*

Part (2) and (3) of the theorem are very helpful in explicitly describing the indecomposable representations of a Dynkin quiver Q. First of all, the set of roots can be determined using Exercise 10.16, (11.1) and the explicit expression of the Tits form (see Exercise 11.10 for type A_n). Next one constructs a $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ with dimension vector $d_M \in \Phi_Q^+$ such that $\mathrm{End}(M) = k1_M$.

Example 11.13. *Consider the Dynkin quiver Q of type A_n, given by*



Number the edges from left to right. For $1 \leq s \leq t \leq n$ let $M = (M_i, \lambda_j)_{1\leq i\leq n; 1\leq j<n}$ be the representation of Q defined by

$$M_i = \begin{cases} 0 & \text{if } 1 \leq i < s \text{ or } t < i \leq n, \\ k & \text{if } s \leq i \leq t \end{cases}$$

with connecting maps $\lambda_j \in k$ (it is automatically the zero map for $1 \leq j < s$ and for $t \leq j < n$). Then $d_M \in \Phi_Q^+$, and it is an easy check that $\mathrm{End}(M) = k1_M$ iff $\lambda_s\lambda_{s+1}\cdots\lambda_{t-1} \neq 0$ (this should be read as no condition if $s = t$). Writing M^{st} for the representation with $\lambda_s = \lambda_{s+1} = \cdots = \lambda_{t-1} = 1$, we conclude that $\{M^{st} \mid 1 \leq s \leq t \leq n\}$ is a complete set of representatives of the isoclasses of the indecomposable representations of Q.

11.3. **The proof.** We prove Gabriel's Theorem in a number of steps.

Lemma 11.14. *Theorem 11.12(2) is correct. In other words, if Q is a Dynkin quiver then a nonzero $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ is indecomposable iff M is a brick.*

Proof. We already noted $\Leftarrow$ (see Remark 11.2). It thus suffices to show $\Rightarrow$. We prove it by induction to the total dimension $\dim(M) := \sum_{i=1}^{n(Q)} \dim_k(M_i)$ of M. If M is indecomposable and $\dim(M) = 1$ then $M \simeq S(i)$ for some i, hence M is a brick. Let M be indecomposable with $\dim(M) > 1$, and suppose that $L \in \mathrm{Ob}(\mathrm{Rep}_Q)$ is a brick if L is nonzero, indecomposable and $\dim(L) < \dim(M)$.

Suppose M is not a brick. Write $\mathcal{S}_M$ for the set of nonzero endomorphisms $f \in \mathrm{End}(M)$ such that $f^2 = 0$. We first show that $\mathcal{S}_M \neq \emptyset$.

Let $g \in \mathrm{End}(M) \setminus k1_M$. Let $\lambda \in k$ be an eigenvalue of g. Then

$$0 \neq h := g - \lambda 1_M \in \mathrm{End}(M)$$

has a nontrivial kernel, hence h is a nilpotent endomorphism by Fitting's lemma (Lemma 3.12). Then $h^n \in \mathcal{S}_M$ with $n \in \mathbb{Z}_{>0}$ the largest positive integer such that $h^n \neq 0$.

We now choose a nonzero $f \in \mathcal{S}_M$ with $\dim(\mathrm{im}(f))$ as small as possible. Consider the corresponding inclusions of subrepresentations

$$0 \neq \mathrm{im}(f) \subseteq \ker(f) \subsetneq M.$$

By the Krull-Schmidt Theorem (Theorem 3.18) we can decompose

$$\ker(f) = L_1 \oplus \cdots \oplus L_r$$

with L_j nonzero indecomposable representations of Q. By the induction hypothesis all the direct summands L_j are bricks. Let $\pi_j : \ker(f) \twoheadrightarrow L_j$ be the canonical projection. Then $\pi_j(\mathrm{im}(f)) \neq 0$ for some j, and we write $0 \neq p_j \in \mathrm{Hom}(\mathrm{im}(f), L_j)$ for the restriction of π_j to $\mathrm{im}(f)$. Let $r \in \mathrm{End}(M)$ be the composition of the following sequence of maps,

$$M \overset{f}{\twoheadrightarrow} \mathrm{im}(f) \xrightarrow{p_j} L_j \hookrightarrow \ker(f) \hookrightarrow M, \tag{11.2}$$

with the two unspecified morphisms the canonical embeddings. Then $\dim(\mathrm{im}(r)) \leq \dim(\mathrm{im}(f))$. Furthermore, $r \neq 0$ since $p_j \neq 0$ and $r^2 = 0$ since $\mathrm{im}(r) \subseteq \ker(f)$, hence $r \in \mathcal{S}_M$. The choice of f thus forces $\dim(\mathrm{im}(r)) = \dim(\mathrm{im}(f))$, and hence p_j is a monomorphism.

Consider now the short exact sequence

$$0 \longrightarrow \mathrm{im}(f) \overset{p_j}{\hookrightarrow} L_j \twoheadrightarrow \mathrm{coker}(p_j) \longrightarrow 0.$$

Applying the contravariant hom-functor $\mathrm{Hom}(-, L_j)$ ([7, Thm. 11.5]) we get the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathrm{coker}(p_j), L_j) \longrightarrow \mathrm{End}(L_j) \longrightarrow \mathrm{Hom}(\mathrm{im}(f), L_j) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1(\mathrm{coker}(p_j), L_j) \longrightarrow \mathrm{Ext}^1(L_j, L_j) \longrightarrow \mathrm{Ext}^1(\mathrm{im}(f), L_j) \longrightarrow 0,$$

where the zero at the far right hand side is due to the fact that $\mathrm{Ext}^2(\mathrm{coker}(p_j), L_j) = 0$ (see Lemma 10.1). Since L_j is a brick we have $\mathrm{Ext}^1(L_j, L_j) = 0$ by Lemma 11.3, and hence we conclude that

$$\mathrm{Ext}^1(\mathrm{im}(f), L_j) = 0. \tag{11.3}$$

The short exact sequence

$$0 \longrightarrow \ker(f) \overset{\iota}{\hookrightarrow} M \overset{f}{\twoheadrightarrow} \mathrm{im}(f) \longrightarrow 0$$

extends to a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(f) & \overset{\iota}{\hookrightarrow} & M & \overset{f}{\twoheadrightarrow} & \mathrm{im}(f) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle 1_{\mathrm{im}(f)}} & & \\ 0 & \longrightarrow & L_j & \overset{u_2}{\hookrightarrow} & X & \overset{f'}{\twoheadrightarrow} & \mathrm{im}(f) & \longrightarrow & 0 \end{array} \tag{11.4}$$

with exact rows and commuting squares (see [4, Exerc. 1.9] for the construction of X, which is called the push-out of the morphisms $\iota \in \mathrm{Hom}(\ker(f), M)$ and $\pi_j \in \mathrm{Hom}(\ker(f), L_j)$, as well as for the definitions of the maps u_1, u_2 and f').

By (11.3) the lower short exact sequence splits, hence u_2 is a section. We write $j_2 \in \mathrm{Hom}(X, L_j)$ for a morphism such that $j_2 \circ u_2 = 1_{L_j}$.

Let $\phi \in \mathrm{Hom}(L_j, M)$ for the composition of the two canonical monomorphisms

$$L_j \overset{\iota_j}{\hookrightarrow} \ker(f) \overset{\iota}{\hookrightarrow} M.$$

By the commutativity of the left square in (11.4) we have $u_2 = (u_2 \circ \pi_j) \circ \iota_j = u_1 \circ \phi$, and hence

$$(j_2 \circ u_1) \circ \phi = j_2 \circ u_2 = \mathrm{id}_{L_j}.$$

Hence $\phi \in \mathrm{Hom}(L_j, M)$ is a section. Applying the splitting lemma to the short exact sequence

$$0 \longrightarrow L_j \overset{\phi}{\hookrightarrow} M \twoheadrightarrow \mathrm{coker}(\phi) \longrightarrow 0$$

we conclude that L_j is a direct summand of M. But M is indecomposable, so this implies that $M = L_j$ and hence M is a brick. This is the desired contradiction. $\square$

We can now make a first step in proving Theorem 11.12(3).

Corollary 11.15. *If Q is a Dynkin quiver and $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ is indecomposable then $d_M \in \Phi_Q^+$.*

Proof. Combine Lemma 11.3 and Lemma 11.14. $\square$

As a next step we show that the map $\mathcal{I}_Q \to \Phi_Q^+$, $[M] \mapsto d_M$, is injective for Dynkin quivers Q (which is part of Theorem 11.12(3)). For this we need to use some basic definitions and results on algebraic varieties and algebraic groups, which we will not recall here in detail – see, e.g., the classic textbook [5] for a thorough treatment of these topics. The main idea of the proof will be clear without prior algebraic geometric knowledge since we are dealing with a context where the algebraic geometric concepts have a natural intuitive meaning.

Fix a vector $\delta \in \mathbb{Z}_{\geq 0}^{n(Q')}$ and write E_δ for the set of representations of Q of the form $(k^{\delta_i}, \phi_\alpha)_{i \in Q_0', \alpha \in Q_1'}$ with connecting maps $\phi_\alpha \in \mathrm{Hom}_k(k^{\delta_{s(\alpha)}}, k^{\delta_{t(\alpha)}})$. Through the canonical bijection

$$\prod_{\alpha \in Q_1'} \mathrm{Hom}_k\big(k^{\delta_{s(\alpha)}} k^{\delta_{t(\alpha)}}\big) \xrightarrow{\sim} E_\delta, \qquad (\phi_\alpha)_\alpha \mapsto (k^{\delta_i}, \phi_\alpha)_{i,\alpha}, \tag{11.5}$$

E_δ inherits the structure of a vector space over k of dimension

$$\dim_k(E_\delta) = \sum_{\alpha \in Q_1'} \delta_{s(\alpha)} \delta_{t(\alpha)}. \tag{11.6}$$

The dimension $\dim(E_\delta)$ of E_δ as irreducible algebraic variety is equal to its dimension as a vector space. Consider the algebraic group

$$G_\delta := \prod_{i=1}^{n(Q')} \mathrm{GL}_{\delta_i}(k),$$

which is of dimension

$$\dim(G_\delta) = \sum_{i=1}^{n(Q')} \delta_i^2. \tag{11.7}$$

From the definition of the Tits form we conclude that

$$q_{Q'}(\delta) = \dim(G_\delta) - \dim(E_\delta). \tag{11.8}$$

It acts linearly and algebraically on E_δ by

$$\mathbf{g} \cdot \underline{\phi} := \left(g_{t(\alpha)}\phi_\alpha g_{s(\alpha)}^{-1}\right)_{\alpha\in Q'_1}$$

for $\mathbf{g} = (g_1, \ldots, g_{n(Q')}) \in G_\delta$ and $\underline{\phi} = (\phi_\alpha)_{\alpha\in Q'_1} \in E_\delta$ (here and below we use the isomorphism (11.5) to identify an element $M = (k^{\delta_i}, \phi_\alpha)_{i,\alpha}$ in E_δ with the associated tuple $(\phi_\alpha)_\alpha$ of connecting maps).

Definition 11.16. *Let $M = (M_i, \psi_\alpha)_{i\in Q'_0,\alpha\in Q'_1} \in \mathrm{Ob}(\mathrm{Rep}_{Q'})$ with dimension vector d_M equal to $\delta \in \mathbb{Z}_{\geq 0}^{n(Q')}$. We write*

$$\mathcal{O}_M \subseteq E_\delta$$

for the set of representations $(k^{\delta_i}, \phi_\alpha)_{i\in Q'_0,\alpha\in Q'_1}$ of Q' isomorphic to M.

With the notations as in the definition, fix k-linear isomorphisms $h_i : M_i \xrightarrow{\sim} k^{\delta_i}$ for all $i \in Q'_0$ and set $M' := (k^{\delta_i}, \psi'_\alpha)_{i\in Q'_0,\alpha\in Q'_1} \in E_\delta$ with connecting maps $\psi'_\alpha := h_{t(\alpha)}\psi_\alpha h_{s(\alpha)}^{-1}$ for $\alpha \in Q'_1$. Then $M \simeq M'$ as representations of Q, with isomorphism $\mathbf{h} = (h_i)_{i\in Q'_0} : M \xrightarrow{\sim} M'$.

Lemma 11.17. *With the above notations, we have*

(1) *$\mathcal{O}_M$ is the G_δ-orbit of $\underline{\psi}'$ within E_δ,*

$$\mathcal{O}_M = G_\delta \cdot \underline{\psi}'. \tag{11.9}$$

(2) *For $M_j \in \mathrm{Ob}(\mathrm{Rep}_{Q'})$ $(j = 1, 2)$ with dimension vector δ,*

$$\mathcal{O}_{M_1} = \mathcal{O}_{M_2} \quad \Leftrightarrow \quad M_1 \simeq M_2. \tag{11.10}$$

Proof. This is straightforward. □

Exercise 11.18. *Give the proof of Lemma 11.17.*

Write $\mathrm{Iso}_{Q'}(\delta)$ for the isoclasses of representations of Q' with dimension vector $\delta \in \mathbb{Z}_{\geq 0}^{n(Q')}$, and $\mathcal{I}_{Q'}(\delta) := \mathrm{Iso}_{Q'}(\delta) \cap \mathcal{I}_{Q'}$ for the subset of isoclasses of indecomposable representations. We then have the disjoint union

$$E_\delta = \bigcup_{[M]\in \mathrm{Iso}_{Q'}(\delta)} \mathcal{O}_M \tag{11.11}$$

of E_δ in non-singular subvarieties $\mathcal{O}_M$. Fix, for $M = (M_i, \psi_\alpha)_{i,\alpha} \in \mathrm{Ob}(\mathrm{Rep}_{Q'})$ with dimension vector δ, an element $\underline{\psi}' \in \mathcal{O}_M$. In view of (11.9) we have

$$\mathcal{O}_M \simeq G_\delta/\mathrm{Stab}(\underline{\psi}') \tag{11.12}$$

as G_δ-set and as varieties, with $\mathrm{Stab}(\underline{\psi}') \subseteq G_\delta$ the fixpoint subgroup of $\underline{\psi}'$.

Lemma 11.19. *With the above notations we have*

$$\dim(\mathcal{O}_M) = \dim(E_\delta) - \dim_k(\mathrm{Ext}^1(M, M)),$$

with $\dim(\mathcal{O}_M)$ *the dimension of* $\mathcal{O}_M$ *as subvariety of* E_δ*.*

Proof. By (11.12) we have

$$\dim(\mathcal{O}_M) = \dim(G_\delta) - \dim(\mathrm{Stab}(\underline{\psi}')). \tag{11.13}$$

Now note that $\mathrm{Stab}(\underline{\psi}') = \mathrm{GL}(M') \simeq \mathrm{GL}(M)$, with $\mathrm{GL}(M) := \mathrm{End}(M)^\times$ the automorphism group of M. Hence

$$\dim(\mathrm{Stab}(\underline{\psi}')) = \dim(\mathrm{GL}(M)) = \dim_k(\mathrm{End}(M)).$$

We conclude that

$$\begin{aligned} \dim(\mathcal{O}_M) &= \dim(G_\delta) - \dim_k(\mathrm{End}(M)) \\ &= \dim(E_\delta) + q_{Q'}(\delta) - \dim_k(\mathrm{End}(M)) \\ &= \dim(E_\delta) - \dim_k(\mathrm{Ext}^1(M, M)), \end{aligned} \tag{11.14}$$

where the second equality is by (11.8) and the third equality is by Proposition 10.8. □

Corollary 11.20. *Let Q be a Dynkin quiver.*

The assignment $\mathcal{I}_Q \to \Phi_Q^+$, $[M] \mapsto d_M$ is injective. In particular, Q is of finite representation type.

Proof. The second part of the lemma follows from Corollary 11.9. Suppose that $M_1, M_2 \in \mathrm{Ob}(\mathrm{Rep}_Q)$ are indecomposable representations of Q with the same dimension vector $\delta \in \Phi_Q^+$. We have to show that $M_1 \simeq M_2$.

Note that M_1 and M_2 are bricks by Lemma 11.14, and hence $\mathrm{Ext}^1(M_1, M_1) = 0 = \mathrm{Ext}^1(M_2, M_2)$ by Lemma 11.3. It follows from Lemma 11.19 that the subvarieties $\mathcal{O}_{M_1}, \mathcal{O}_{M_2}$ of E_δ are of maximal dimension $\dim(E_\delta)$. Hence $\mathcal{O}_{M_1}$ and $\mathcal{O}_{M_2}$ are open and dense in E_δ. Since E_δ is irreducible, we conclude that $\mathcal{O}_{M_1} \cap \mathcal{O}_{M_2} \neq \emptyset$. Then (11.9) implies that $\mathcal{O}_{M_1} = \mathcal{O}_{M_2}$, hence $M_1 \simeq M_2$ by (11.10). □

Lemma 11.21. *Theorem 11.12(1) is correct. In other words, Q' is of finite representation type iff Q' is a Dynkin quiver.*

Proof. In view of the previous corollary it suffices to show the following: if Q' is not a Dynkin quiver, then Q' is not of finite representation type.

Suppose Q' is not a Dynkin quiver. Let $\overline{Q}$ be an Euclidean subquiver of Q', which exists by Lemma 10.21 (if Q' is Euclidean, then take $\overline{Q} = Q'$). By Lemma

10.15 there exists a vector $\overline{\delta} \in \mathbb{Z}_{>0}^{n(\overline{Q})}$ such that $q_{\overline{Q}}(\overline{\delta}) = 0$. Extend $\overline{\delta}$ by zeroes to a vector $\delta \in \mathbb{Z}_{\geq 0}^{n(Q')}$. Then $q_{Q'}(\delta) = 0$. For all $[M] \in \mathrm{Iso}_{Q'}(\delta)$ we then have by (11.14),

$$\dim(\mathcal{O}_M) = \dim(E_\delta) - \dim_k(\mathrm{End}(M)) < \dim(E_\delta).$$

Hence all representations M with dimension vector δ, including the indecomposable ones, produce orbits $\mathcal{O}_M$ of dimension strictly smaller than $\dim(E_\delta)$. Then (11.11) implies that $\#\mathrm{Iso}_{Q'}(\delta) = \infty$ (note here that $\#k = \infty$ since k is algebraically closed). The result now follows from the following exercise. □

Exercise 11.22. *Suppose that $\#\mathrm{Iso}_{Q'}(\delta) = \infty$ for some $\delta \in \mathbb{Z}_{\geq 0}^{n(Q')}$. Show that Q' is not of finite representation type.*

The following lemma completes the proof of Gabriel's Theorem.

Lemma 11.23. *Suppose that Q is a Dynkin quiver. Then $\mathcal{I}_Q \hookrightarrow \Phi_Q^+$, $[M] \mapsto d_M$ is surjective.*

Proof. Fix a positive root $d \in \Phi_Q^+$. Choose $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ with dimension vector d such that $\dim(\mathcal{O}_M)$ is as large as possible. Since

$$\dim(\mathcal{O}_M) = \dim(G_d) - \dim_k(\mathrm{End}(M))$$

by the first equality in (11.14), this corresponds to a choice of a representation $M \in \mathrm{Ob}(\mathrm{Rep}_Q)$ with dimension vector d such that $\dim_k(\mathrm{End}(M))$ is as small as possible. We show that M is indecomposable.

Suppose that $M = M_1 \oplus M_2$ is a direct sum decomposition of M. It provides the trivial extension

$$0 \longrightarrow M_1 \hookrightarrow M \twoheadrightarrow M_2 \longrightarrow 0$$

of M_2 by M_1. Consider now an arbitrary extension

$$0 \longrightarrow M_1 \hookrightarrow N \twoheadrightarrow M_2 \longrightarrow 0$$

of M_2 by M_1. Then N has dimension vector d. Furthermore, $\dim(\mathcal{O}_N) \geq \dim(\mathcal{O}_M)$ with equality iff $N \simeq M$, see [4, Lem. 8.3]. By our choice of M this forces all extensions of M_2 by M_1 to be trivial, and hence $\mathrm{Ext}^1(M_1, M_2) = 0$. Interchanging the role of M_1 and M_2 we also have $\mathrm{Ext}^1(M_2, M_1) = 0$. By Exercise 10.9 we conclude that

$$(d_{M_1}, d_{M_2})_Q = \dim_k(\mathrm{Hom}(M_1, M_2)) + \dim_k(\mathrm{Hom}(M_2, M_1)) \geq 0.$$

Since d is a root of the Dynkin quiver Q we get

$$\begin{aligned} 1 = q_Q(d) &= q_Q(d_{M_1} + d_{M_2}) \\ &= q_Q(d_{M_1}) + q_Q(d_{M_2}) + (d_{M_1}, d_{M_2})_Q \\ &\geq q_Q(d_{M_1}) + q_Q(d_{M_2}). \end{aligned}$$

But q_Q is positive definite, so the inequality forces $d_{M_1} = 0$ or $d_{M_2} = 0$. Hence M is indecomposable. □

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