THE QUANTUM DOUBLE AS A HOPF ALGEBRA

In this text we discuss the generalized quantum double construction. It is a detailed treatment of the results described without proofs in [2, Chpt. 3, §3].
We give several exercises in the text.
The **homework exercise** is 2.9.

1. Introduction

In the last lecture we have learned that the category of modules over a braided Hopf algebra H is a braided monoidal category. A braided Hopf algebra is a rather sophisticated algebraic object, it is not easy to give interesting nontrivial examples. In this text we develop a theory that will lead to a concrete recipe which produces a nontrivial *braided* Hopf algebra $\mathcal{D}(A)$ (called Drinfeld's quantum double) for any finite dimensional Hopf algebra A with invertible antipode! We will furthermore use this technique to produce an important example of a quantum group, namely the quantized universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2-matrices.
Conventions: In this text Hopf algebras will be assumed to have invertible antipodes. Although we will be working with several Hopf algebras at the same time, we use the same notations for the associated (co)algebra maps, units and antipodes: it will always be clear from context which maps we are dealing with.

2. Hopf algebra pairings

The linear dual $A^* = \mathrm{Hom}_k(A, k)$ of a finite dimensional Hopf algebra A canonically inherits a Hopf algebra structure from A. Indeed, by finite dimensionality we may identify $(A \otimes A)^* \simeq A^* \otimes A^*$ as vector spaces. The Hopf algebra maps of A^* are then given by

$$(fg)(a) = \sum_{(a)} f(a_{(1)})g(a_{(2)}), \qquad 1(a) = \epsilon(a),$$

$$\Delta(f)(a \otimes a') = f(aa'), \qquad \epsilon(f) = f(1), \qquad S(f)(a) = f(S(a))$$

for $a, a' \in A$ and $f, g \in A^*$, where we use Sweedler's notation $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$. In other words, the algebra (resp. coalgebra) maps of A induce the coalgebra (resp. algebra) maps of A^*. An important role will be played by Hopf algebras of the form $A^{*,cop}$, in which the comultiplication Δ and the antipode S of A^* are replaced by the opposite comultiplication Δ^{op} and S^{-1} respectively.

Exercise 2.1. (i) *Show that the above maps turn A^* into a Hopf algebra.*
(ii) *Let G be a finite group and $k[G]$ the group algebra of G, with its canonical Hopf algebra structure (see* [2, Chpt. 2 §1.4]*). Identify $k[G]^*$ as vector space with the space* $\mathrm{Fun}_k(G)$ *of*

k-valued functions on G (a function $f \in \mathrm{Fun}_k(G)$ uniquely extends to a k-linear functional on $k[G]$). Hence $\mathrm{Fun}_k(G)$ inherits a Hopf algebra structure from $k[G]^*$. Give the explicit formulas for the resulting Hopf algebra maps on $\mathrm{Fun}_k(G)$.

A second basic construction is the *product Hopf algebra* $A \otimes B$ of two Hopf algebras A and B. It is the Hopf algebra structure on the vector space $A \otimes B$ determined by

$$\begin{aligned}(a \otimes b)(a' \otimes b') &= aa' \otimes bb',\\ \Delta(a \otimes b) &= \sum_{(a),(b)} (a_{(1)} \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}),\\ \epsilon(a \otimes b) &= \epsilon(a)\epsilon(b),\\ S(a \otimes b) &= S(a) \otimes S(b)\end{aligned}$$

for $a, a' \in A$ and $b, b' \in B$.

As a Hopf algebra, the quantum double $\mathcal{D}(A)$ of a finite dimensional Hopf algebra A will be obtained from the product Hopf algebra $A \otimes A^{*,cop}$ by adjusting the algebra structure appropriately. The adjustment in the multiplication uses the fact that the two Hopf algebras A and $A^{*,cop}$ are dual to each other. It translates into the following elementary properties of the bilinear form $\varphi : A \times A^{*,cop} \to k$,

$$\varphi(a, b) = b(a), \qquad a \in A, b \in A^{*,cop}. \tag{2.1}$$

Lemma 2.2. *The pairing φ (see (2.1)) satisfies:*
(i) *Compatibility of the (co)units*

$$\begin{aligned}\varphi(a, 1) &= \epsilon(a),\\ \varphi(1, b) &= \epsilon(b).\end{aligned} \tag{2.2}$$

(ii) *Compatibility of the (co)multiplications*

$$\begin{aligned}\varphi(aa', b) &= \sum_{(b)} \varphi(a, b_{(2)})\varphi(a', b_{(1)}),\\ \varphi(a, bb') &= \sum_{(a)} \varphi(a_{(1)}, b)\varphi(a_{(2)}, b').\end{aligned} \tag{2.3}$$

(iii) *Compatibility of the antipodes*

$$\varphi(S(a), b) = \varphi(a, S^{-1}(b)). \tag{2.4}$$

Proof. **(i)** The unit element of $A^{*,cop}$ is the counit ϵ of A, hence

$$\varphi(a, 1) = \varphi(a, \epsilon) = \epsilon(a).$$

The counit of $A^{*,cop}$ is $\epsilon(b) = b(1)$ ($b \in A^{*,cop}$), hence $\varphi(1, b) = \epsilon(b)$.
(ii) The comultiplication $\Delta(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}$ of $A^{*,cop}$ is characterized by the property

$$\sum_{(b)} b_{(2)}(a) b_{(1)}(a') = b(aa')$$

for $a, a' \in A$. Translated in terms of φ we get the first equality of (2.3). The multiplication in $A^{*,cop}$ is defined by

$$(bb')(a) = \sum_{(a)} b(a_{(1)})b'(a_{(2)})$$

for $b, b' \in B$ and $a \in A$. Translated in terms of φ we get the second equality of (2.3).
(iii) We have

$$\varphi(S(a), b) = b(S(a)) = (S^{-1}(b))(a) = \varphi(a, S^{-1}(b)),$$

where the second equality is by the definition of the antipode of $A^{*,cop}$. □

Definition 2.3. **(i)** *A bilinear pairing*

$$\varphi : A \times B \to k$$

of bialgebras A and B is called a bialgebra pairing if it satisfies (2.2) *and* (2.3) *for all $a, a' \in A$ and $b, b' \in B$.*
(ii) *If A and B are Hopf algebras then a bialgebra pairing $\varphi : A \times B \to k$ is called a Hopf pairing if it satisfies* (2.4) *for all $a \in A$ and $b \in B$.*

For a Hopf pairing $\varphi : A \times B \to k$ we define

$$I_A = \{a \in A \mid \varphi(a, b) = 0 \quad \forall\, b \in B\},$$
$$I_B = \{b \in B \mid \varphi(a, b) = 0 \quad \forall\, a \in A\}.$$

We say that the pairing φ is nondegenerate if $I_A = \{0\}$ and $I_B = \{0\}$.
Examples. **(i)** For two Hopf algebras A and B, the Hopf pairing

$$\varphi_0(a, b) = \epsilon(a)\epsilon(b)$$

is called the trivial Hopf pairing between A and B. It is highly degenerate, since $I_A = \mathrm{Ker}(\epsilon)$ and $I_B = \mathrm{Ker}(\epsilon)$ are the augmentation ideals of A and B, respectively.
(ii) The canonical Hopf pairing (2.1) between a finite dimensional Hopf algebra A and its dual $A^{*,cop}$ is nondegenerate.

Exercise 2.4. *Let $\varphi : A \times B \to k$ be a nondegenerate Hopf pairing between two finite dimensional Hopf algebras A and B. Show that $b \mapsto \varphi(\cdot, b)$ defines an isomorphism $\psi : B \xrightarrow{\sim} A^{*,cop}$ of Hopf algebras.*

Remark. Under the assumptions of the previous exercise, the identification of B with $A^{*,cop}$ as Hopf algebras by the map ψ turns the Hopf pairing φ into the canonical Hopf pairing (2.1): $(\psi(b))(a) = \varphi(a, b)$ for $a \in A$ and $b \in B$.

In examples it is often easy to check whether a given bialgebra pairing $A \times B \to k$ between Hopf algebras A and B is a Hopf pairing, in view of the following exercise.

Exercise 2.5. *Suppose that $\varphi : A \times B \to k$ is a bialgebra pairing of the Hopf algebras A and B. Suppose that A (resp. B) is generated as algebra by $\{a_1, \ldots, a_r\}$ (resp. $\{b_1, \ldots, b_s\}$) and that*

$$\Delta(a_i) \in V \otimes V, \qquad i = 1, \ldots, r$$

where $V = span_k\{a_1, \ldots, a_r, 1\} \subset A$. *Show that* φ *is a Hopf pairing if*

$$\varphi(S(a_i), b_j) = \varphi(a_i, S^{-1}(b_j))$$

for $i = 1, \ldots, r$ *and* $j = 1, \ldots, s$.

Lemma 2.6. *Let* $\varphi : A \times B \to k$ *be a Hopf pairing between Hopf algebras* A *and* B.
(i) $I_A \subset A$ *and* $I_B \subset B$ *are Hopf ideals.*
(ii) *The Hopf pairing* φ *descends canonically to a nondegenerate Hopf pairing* $\overline{\varphi}$ *between the quotient Hopf algebras* A/I_A *and* B/I_B.

Proof. **(i)** We consider $I_A \subset A$. First note that I_A is a two-sided ideal in A. Indeed, I_A is clearly closed under addition, and for $a, a' \in A$ and $x \in I_A$ we have for all $b \in B$,

$$\varphi\big(axa', b\big) = \sum_{(b)} \varphi\big(a, b_{(3)}\big)\varphi\big(x, b_{(2)}\big)\varphi\big(a', b_{(1)}\big) = 0,$$

hence $axa' \in I_A$.

Let $a \in I_A$ and $b \in B$. Then $\epsilon(a) = \varphi(a, 1) = 0$ and $\varphi(S(a), b) = \varphi(a, S^{-1}(b)) = 0$. It follows that $I_A \subset \mathrm{Ker}(\epsilon)$ and $S(I_A) \subseteq I_A$.

Let $a \in I_A$ and write $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$. The linear functional

$$b \otimes b' \mapsto \sum_{(a)} \varphi(a_{(1)}, b)\varphi(a_{(2)}, b')$$

on $B \otimes B$ is identically zero, since the right hand side equals

$$\varphi(a, bb') = 0.$$

It is now a straightforward exercise in linear algebra to conclude that $\Delta(a) \in I_A \otimes A + A \otimes I_A$.
(ii) The nondegenerate Hopf pairing $\overline{\varphi}$ between A/I_A and B/I_B is defined by

$$\overline{\varphi}(a + I_A, b + I_B) = \varphi(a, b).$$

It satisfies all the desired properties. □

Exercise 2.7. *Fill in the remaining details of the proof of Lemma 2.6.*

We shortly recall the construction of the tensor algebra $T(V)$ of a k-vector space V. As a vector space, it equals

$$T(V) = \bigoplus_{m=0}^{\infty} V^{\otimes m}, \qquad V^{\otimes 0} := k.$$

The algebra structure is

$$(v_1 \otimes \cdots \otimes v_l)(v_1' \otimes \cdots \otimes v_m') := v_1 \otimes \cdots \otimes v_l \otimes v_1' \otimes \cdots \otimes v_m'$$

for $v_i, v_j' \in V$ with unit $1 \in k = V^{\otimes 0}$. Note that $v_1 v_2 \cdots v_m = v_1 \otimes \cdots \otimes v_m$ in the tensor algebra $T(V)$, hence $T(V)$ is algebraically generated by V. If V is finite dimensional with basis $a_1, \ldots, a_r$, then $T(V)$ is the free algebra with generators $a_1, \ldots, a_r$ (i.e. the set of words in $\{a_1, \ldots, a_r\}$ forms a basis of $T(V)$). This leads to the universal property of the tensor algebra,

$$\mathrm{Hom}_k(V, A) \simeq \mathrm{Hom}_{Alg}(T(V), A)$$

for any algebra A, where Hom_k stand for k-linear morphisms and Hom_{Alg} for algebra morphisms.

Remark. A tensor algebra $T(V)$ may have many different Hopf algebra structures, one of which is its canonical, cocommutative Hopf algebra structure determined by

$$\Delta(v) = v \otimes 1 + 1 \otimes v,$$
$$\epsilon(v) = 0, \qquad S(v) = -v$$

for $v \in V$.

Many examples of (Hopf) algebras are defined by specifying generators $\{a_1, \ldots, a_r\}$ and relations. It is constructed as the associated free algebra $T(V)$ divided out by a two-sided ideal incorporating the desired relations between the generators.

Example. If we impose the generators of a free algebra to be commutative we are led to symmetric algebras. The symmetric algebra $S(V)$ of a vector space V is $T(V)/I$ with I the two-sided ideal generated by $v \otimes w - w \otimes v$ for all $v, w \in V$. It satisfies the universal property

$$\mathrm{Hom}_k(V, A) \simeq \mathrm{Hom}_{Alg}(S(V), A)$$

for *commutative* algebras A.

Another important example is the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. It is a quotient of $T(\mathfrak{g})$ by a Hopf ideal $I(\mathfrak{g})$ (with respect to the canonical cocommutative Hopf algebra structure on $T(\mathfrak{g})$), hence $U(\mathfrak{g})$ is a cocommutative Hopf algebra. We come back to (deformations of) universal enveloping algebras in more detail at a later stage.

The following lemma shows that, under mild assumptions, nontrivial bialgebra pairings $\varphi : A \times B \to k$ can be easily constructed if A and B are free as algebras.

Lemma 2.8. *Suppose A (resp. B) is a free algebra with generators $a_1, \ldots, a_r$ (resp. $b_1, \ldots, b_s$). So $A = T(V)$ and $B = T(W)$ with V (resp. W) the vector space with linear basis $\{a_1, \ldots, a_r\}$ (resp. $\{b_1, \ldots, b_s\}$). Suppose that the free algebras A and B are equipped with bialgebra structures such that*

$$\Delta(V) \subset (V \oplus k)^{\otimes 2}.$$

Given a bilinear form $\psi : V \times W \to k$, there exists a unique bialgebra pairing $\varphi : A \times B \to k$ such that $\varphi|_{V \times W} = \psi$.

Remark. Under the assumptions of Lemma 2.8, $(V \otimes W)^*$ is in one-to-one correspondence with the set of bialgebra pairings $\varphi : A \times B \to k$. This identification turns the set of bialgebra pairings $\varphi : A \times B \to k$ into a vector space.

Proof. Suppose $\varphi : A \times B \to k$ is a bialgebra pairing extending the linear functional ψ on $V \times W$. Then by (2.3), for $v_1, \dots, v_k \in V$ and $b \in B$,

$$\varphi(v_1 v_2 \cdots v_k, b) = \sum_{(b)} \varphi(v_1, b_{(k)}) \varphi(v_2, b_{(k-1)}) \cdots \varphi(v_k, b_{(1)}). \tag{2.5}$$

Combined with (2.2) we conclude that φ is uniquely determined by its restriction to $\big(V \oplus k\big) \times B$. For $v \in V$ and $w_1, \dots, w_l \in W$ we have by (2.3),

$$\varphi(v, w_1 w_2 \cdots w_l) = \sum_{(v)} \varphi(v_{(1)}, w_1) \varphi(v_{(2)}, w_2) \cdots \varphi(v_{(l)}, w_l). \tag{2.6}$$

Since $v_{(i)} \in V \oplus k$ by the assumptions, (2.2) and (2.6) imply that φ is uniquely determined by $\varphi|_{V \times W}$.

It remains to show that a given $\psi \in (V \otimes W)^*$ extends to a bialgebra pairing $\varphi : A \times B \to k$. We first *define* $\varphi : A \times B \to k$ as the unique bilinear form satisfying (2.2), $\varphi|_{V \times W} = \psi$, and satisfying (2.6) and (2.5). The pairing φ is well defined since $\Delta(V) \subset (V \oplus k)^{\otimes 2}$. It remains to show that φ is a bialgebra pairing.

The (co)unit axioms (2.2) are by assumption. We discuss now the proof of the first equality

$$\varphi(aa', b) = \sum_{(b)} \varphi(a, b_{(2)}) \varphi(a', b_{(1)}), \qquad a, a' \in A,\ b \in B \tag{2.7}$$

of the (co)multiplication axiom (2.3).

For $a = 1$ the equality (2.7) is valid by (2.2) and by the counit axiom,

$$\sum_{(b)} \varphi(1, b_{(2)}) \varphi(a', b_{(1)}) = \varphi\big(a', \sum_{(b)} \epsilon(b_{(2)}) b_{(1)}\big) = \varphi(a', b).$$

In the same way one shows that (2.7) is valid for $a' = 1$. By linearity it remains to prove (2.7) in case that $a = v_1 \cdots v_m$ and $a' = v_{m+1} \cdots v_l$ for $v_i \in V$ and $1 \leq m < l$. In this case we expand $\varphi(aa', b)$ by (2.5) and contract the first m terms (respectively the last $l - m$ terms) by the same formula applied in reversed order,

$$\begin{aligned}
\varphi(v_1 \cdots v_l, b) &= \sum_{(b)} \varphi(v_1, b_{(l)}) \cdots \varphi(v_m, b_{(l-m+1)}) \varphi(v_{m+1}, b_{(l-m)}) \cdots \varphi(v_l, b_{(1)}) \\
&= \sum_{(b)} \varphi(v_1 \cdots v_m, b_{(l-m+1)}) \varphi(v_{m+1}, b_{(l-m)}) \cdots \varphi(v_l, b_{(1)}) \\
&= \sum_{(b)} \varphi(v_1 \cdots v_m, b_{(2)}) \varphi(v_{m+1} \cdots v_l, b_{(1)}).
\end{aligned}$$

The proof of the second equality of the (co)multiplication axiom (2.3) can be verified in an analogous manner. □

Remark. If the free algebras A and B in the above lemma are Hopf algebras, then Exercise 2.5 gives a handy criterion to verify whether the constructed bialgebra pairing φ is a Hopf pairing.

Exercise 2.9. *Let A and B be Hopf algebras with Hopf pairing $\varphi : A \times B \to k$.*
(i) *Show that the bilinear map $B \times A \to B$, $(b, a) \mapsto b^a$ with*

$$b^a = \sum_{(b)} \varphi\big(a, b_{(2)}\big) b_{(1)},$$

is a right A-action on B (i.e. $b^1 = b$ and $(b^a)^{a'} = b^{aa'}$).
(ii) *Prove that the bilinear map $B \times A \to A$, $(b, a) \mapsto b \cdot a$ with*

$$b \cdot a = \sum_{(a)} \varphi\big(S^{-1}(a_{(1)}), b\big) a_{(2)},$$

is a left B-action on A (i.e. $1 \cdot a = a$ and $b \cdot (b' \cdot a) = (bb') \cdot a$). Here you may use that $\epsilon(S(a)) = \epsilon(a)$ for $a \in A$ (which you will be asked to prove in Exercise 3.2(i)).

3. The quantum double as a Hopf algebra

In this section we discuss the following theorem in detail.

Theorem 3.1. *Let $\varphi : A \times B \to k$ be a Hopf pairing between Hopf algebras A and B. There exists a unique Hopf algebra structure on $A \otimes B$ satisfying the following conditions:*

- **(i)** *The canonical linear embedding $a \mapsto a \otimes 1$ (resp. $b \mapsto 1 \otimes b$) of A (resp. B) into $A \otimes B$ is a morphism of Hopf algebras.*
- **(ii)** *For $a \in A$ and $b \in B$ the multiplication rules are determined by*

$$\begin{aligned} &(a \otimes 1)(1 \otimes b) = a \otimes b, \\ &(1 \otimes b)(a \otimes 1) = \sum_{(a),(b)} \varphi\big(S^{-1}(a_{(1)}), b_{(1)}\big)\varphi\big(a_{(3)}, b_{(3)}\big) a_{(2)} \otimes b_{(2)}. \end{aligned} \tag{3.1}$$

The resulting Hopf algebra is denoted by $\mathcal{D}_\varphi(A, B)$. It is called the generalized double of A and B with respect to the pairing φ.

Note that the second equation in (3.1) tells you how to rewrite a product of the form $(1 \otimes b)(a \otimes 1)$ into a sum of products of the form $(a' \otimes 1)(1 \otimes b') = a' \otimes b'$. You can thus think of it as a "straightening rule" which allows you, in combination with the first property **(i)** of the new algebra structure, to rewrite any complicated product in $\mathcal{D}_\varphi(A, B)$ as a finite sum of terms of the form $a' \otimes b'$.

Observe that the straightening rule in $\mathcal{D}_\varphi(A, B)$ can be expressed in terms of the two actions of exercise 2.9 as

$$(1 \otimes b)(a \otimes 1) = \sum_{(a),(b)} b_{(1)} \cdot a_{(1)} \otimes b_{(2)}^{a_{(2)}}$$

for $a \in A$ and $b \in B$. This links the present constructions to Kassel's [1, Chpt. IX] treatment of quantum doubles using matched pairs of Hopf algebras.

Under the additional assumptions that $\varphi : A \times B \to k$ is a nondegenerate Hopf pairing between finite dimensional Hopf algebras A and B, we will show next week that the associated generalized quantum double $\mathcal{D}_\varphi(A, B)$ is a braided Hopf algebra.

Example. We consider examples with $A = k[G]$ and $B = \mathrm{Fun}_k(G)^{cop}$ for a finite group G.
(i) Let $\varphi_0 : k[G] \times \mathrm{Fun}_k(G)^{cop} \to k$ be the trivial Hopf pairing. It is given by $\varphi_0(g, f) = f(e)$ for $g \in G$ and $f \in \mathrm{Fun}_k(G)$, with $e \in G$ the unit element. In this case the straightening rule gives

$$\begin{aligned}(e \otimes f)(g \otimes 1) &= \sum_{(f)} \varphi_0(g^{-1}, f_{(1)})\varphi_0(g, f_{(3)}) g \otimes f_{(2)} \\ &= g \otimes \sum_{(f)} f_{(1)}(e) f_{(3)}(e) f_{(2)} \\ &= g \otimes f(e \cdot e) = g \otimes f,\end{aligned}$$

hence $\mathcal{D}_\varphi(A, B)$ is the usual product Hopf algebra $A \otimes B$.
(ii) Let $\varphi : k[G] \times \mathrm{Fun}_k(G)^{cop} \to k$ be the canonical Hopf pairing. It is given by $\varphi(g, f) = f(g)$ for $g \in G$ and $f \in \mathrm{Fun}_k(G)$. In this case the straightening rule gives

$$\begin{aligned}(e \otimes f)(g \otimes 1) &= \sum_{(f)} \varphi(g^{-1}, f_{(1)})\varphi(g, f_{(3)}) g \otimes f_{(2)} \\ &= g \otimes \sum_{(f)} f_{(1)}(g^{-1}) f_{(3)}(g) f_{(2)} \\ &= g \otimes (f \circ \mathrm{Ad}(g)),\end{aligned}$$

where $\mathrm{Ad}(g) : G \to G$ is given by $\mathrm{Ad}(g)(g') = gg'g^{-1}$ for $g, g' \in G$. The resulting Hopf algebra $\mathcal{D}(G) := \mathcal{D}_\varphi(A, B)$ is called the quantum double of the finite group G.

Exercise 3.2. *Let A be a Hopf algebra.*
(i) *Show that $\epsilon(S(a)) = \epsilon(a)$ for all $a \in A$.*
Fix a second Hopf algebra B and let $\varphi_0 : A \times B \to k$ be the trivial Hopf pairing.
(ii) *Prove that $\mathcal{D}_{\varphi_0}(A, B)$ is the standard product Hopf algebra $A \otimes B$.*

For the remainder of the section we fix two Hopf algebras A and B and a Hopf pairing $\varphi : A \times B \to k$. We divide the proof of Theorem 3.1 in two parts. In the first step we deal with the algebra structure of $\mathcal{D}_\varphi(A, B)$.

Lemma 3.3. *There exists a unique algebra structure on $A \otimes B$ satisfying*

- **(i)** *The canonical linear embeddings of A and B into $A \otimes B$ are algebra homomorphisms.*
- **(ii)** *The multiplication rules* (3.1) *hold.*

We will denote the resulting algebra by $\mathcal{D}_\varphi(A, B)$ (in accordance with Theorem 3.1).

Proof. If $A \otimes B$ has a multiplication turning it into an associative algebra and satisfying the conditions of the lemma, then it is given on elements in the spanning set $\{a \otimes b\}_{a \in A, b \in B}$ of $A \otimes B$ by

$$(a \otimes b)(a' \otimes b') = \sum_{(a'),(b)} \varphi\big(S^{-1}(a'_{(1)}), b_{(1)}\big)\varphi\big(a'_{(3)}, b_{(3)}\big) a a'_{(2)} \otimes b_{(2)} b' \tag{3.2}$$

for $a, a' \in A$ and $b, b' \in B$. Indeed, this follows from writing $(a \otimes b)(a' \otimes b')$ as the product $(a \otimes 1)(1 \otimes b)(a' \otimes 1)(1 \otimes b')$ and using the straightening rule for the product of the two middle terms. This implies the uniqueness of the algebra structure.

For the existence we use (3.2) as the definition of a linear map $\mu : (A \otimes B)^{\otimes 2} \to A \otimes B$. Write $(a \otimes b)(a' \otimes b') := \mu((a \otimes b) \otimes (a' \otimes b'))$. We have to show that

1. $A \otimes B$ is an associative algebra with respect to μ, with unit $1 \otimes 1$.
2. With respect to the algebra structure on $A \otimes B$ as defined in **1**, the canonical linear embeddings of A and B into $A \otimes B$ are algebra homomorphisms.
3. With respect to the algebra structure on $A \otimes B$ as defined in **1**, the multiplication rules (3.1) are valid in $A \otimes B$.

Proof of **1.** The element $1 \otimes 1 \in A \otimes B$ serves as right unit, since

$$\begin{aligned}(a \otimes b)(1 \otimes 1) &= \sum_{(b)} \varphi(1, b_{(1)})\varphi(1, b_{(3)}) a \otimes b_{(2)} \\ &= a \otimes \Big(\sum_{(b)} \epsilon(b_{(1)})\epsilon(b_{(3)}) b_{(2)}\Big) \\ &= a \otimes b\end{aligned}$$

where the last equality is by the counit axiom applied twice. Similarly one shows that $1 \otimes 1$ serves as left unit.

Proving the associativity of the product (3.2) is more elaborate. It consists of showing that $((a \otimes b)(a' \otimes b'))(a'' \otimes b'')$ and $(a \otimes b)((a' \otimes b')(a'' \otimes b''))$ are equal to

$$\sum_{(b),(a'),(b'),(a'')} \big\{\varphi\big(S^{-1}(a'_{(1)}), b_{(1)}\big)\varphi\big(a'_{(3)}, b_{(5)}\big)\varphi\big(S^{-1}(a''_{(1)}), b'_{(1)}\big)\varphi\big(S^{-1}(a''_{(2)}), b_{(2)}\big) \times \varphi\big(a''_{(4)}, b_{(4)}\big)\varphi\big(a''_{(5)}, b'_{(3)}\big)\big\} a a'_{(2)} a''_{(3)} \otimes b_{(3)} b'_{(2)} b''.$$

This is a typical exercise in advanced Hopf-algebra computations!
Proof of **2.** If $a = a' = 1$ in (3.2), then it gives

$$\begin{aligned}(1 \otimes b)(1 \otimes b') &= \sum_{(b)} \varphi\big(1, b_{(1)}\big)\varphi\big(1, b_{(3)}\big) 1 \otimes b_{(2)} b' \\ &= 1 \otimes \sum_{(b)} \epsilon(b_{(1)})\epsilon(b_{(3)}) b_{(2)} b' \\ &= 1 \otimes b b'\end{aligned}$$

for $b, b' \in B$. Similarly, $(a \otimes 1)(a' \otimes 1) = aa' \otimes 1$ for $a, a' \in A$.
Proof of **3.** For $b = 1$ and $a' = 1$, (3.2) gives

$$(a \otimes 1)(1 \otimes b') = \varphi(1,1)\varphi(1,1)a1 \otimes 1b' = a \otimes b'.$$

For $a = 1$ and $b' = 1$ (3.2) immediately reduces to the straightening rule of (3.1). □

Exercise 3.4. *Fill in the details of part* **1** *of the proof of Lemma 3.3.*

The proof of Theorem 3.1 is now completed by the following lemma.

Lemma 3.5. *The algebra $\mathcal{D}_\varphi(A, B)$ of Lemma 3.3 has a unique Hopf algebra structure such that the canonical algebra embeddings of A and B in $\mathcal{D}_\varphi(A, B)$ are Hopf algebra homomorphisms.*

Proof. To avoid confusion, we denote during the proof of the lemma Δ_A, ϵ_A and S_A (resp. Δ_B, ϵ_B and S_B) for the comultiplication, counit and antipode of A (resp. B).

If the algebra $\mathcal{D}_\varphi(A, B)$ has a comultiplication Δ, counit ϵ and antipode S turning $\mathcal{D}_\varphi(A, B)$ into a Hopf algebra and satisfying the additional requirements of the lemma, then Δ, ϵ and S are unique. To show this fact, we express Δ, ϵ and S in terms of the Hopf algebra maps of A and B.

For the counit, $\epsilon : \mathcal{D}_\varphi(A, B) \to k$ is an algebra homomorphism, so

$$\epsilon(a \otimes b) = \epsilon(a \otimes 1)\epsilon(1 \otimes b) = \epsilon_A(a)\epsilon_B(b). \tag{3.3}$$

Similarly, the comultiplication $\Delta : \mathcal{D}_\varphi(A, B) \to \mathcal{D}_\varphi(A, B)^{\otimes 2}$ is an algebra homomorphism, so

$$\Delta(a \otimes b) = \Delta(a \otimes 1)\Delta(1 \otimes b) = \sum_{(a),(b)} (a_{(1)} \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)}), \tag{3.4}$$

which should be viewed as identities in the product algebra $\mathcal{D}_\varphi(A, B)^{\otimes 2}$. Finally, $S : \mathcal{D}_\varphi(A, B) \to \mathcal{D}_\varphi(A, B)^{op}$ is an algebra homomorphism, so

$$S(a \otimes b) = S(1 \otimes b)S(a \otimes 1) = (1 \otimes S_B(b))(S_A(a) \otimes 1), \tag{3.5}$$

viewed as identities in the algebra $\mathcal{D}_\varphi(A, B)$.

To prove the existence of the Hopf algebra structure on $\mathcal{D}_\varphi(A, B)$, we use (3.3), (3.4) and (3.5) as the definition of $\epsilon : \mathcal{D}_\varphi(A, B) \to k$, $\Delta : \mathcal{D}_\varphi(A, B) \to \mathcal{D}_\varphi(A, B)^{\otimes 2}$ and $S : \mathcal{D}_\varphi(A, B) \to \mathcal{D}_\varphi(A, B)^{op}$ as *linear* maps. We need to verify the following properties.

1. The three maps ϵ, Δ and S are algebra homomorphisms.
2. The algebra $\mathcal{D}_\varphi(A, B)$ becomes a Hopf algebra with comultiplication Δ, counit ϵ and antipode S.
3. The canonical algebra embeddings of A and B into $\mathcal{D}_\varphi(A, B)$ are Hopf-algebra homomorphisms.

Proof of **1.** For the map ϵ we have to show that $\epsilon(1 \otimes 1) = 1$, which is trivial, and that

$$\epsilon((a \otimes b)(a' \otimes b')) = \epsilon_A(a)\epsilon_B(b)\epsilon_A(a')\epsilon_B(b'). \tag{3.6}$$

We start with the left hand side, which by (3.2) can be expressed as

$$\sum_{(a'),(b)} \varphi\big(S_A^{-1}(a'_{(1)}), b_{(1)}\big)\varphi\big(a'_{(3)}, b_{(3)}\big)\epsilon\big(aa'_{(2)} \otimes b_{(2)}b'\big).$$

By the definition of ϵ and by the bilinearity of φ, this equals

$$\sum_{(a'),(b)} \varphi\big(S_A^{-1}(a'_{(1)}), b_{(1)}\epsilon_B(b_{(2)})\big)\varphi\big(\epsilon_A(a'_{(2)})a'_{(3)}, b_{(3)}\big)\epsilon_A(a)\epsilon_B(b').$$

Applying the counit axiom for A and B, we get

$$\sum_{(a'),(b)} \varphi\big(S_A^{-1}(a'_{(1)}), b_{(1)}\big)\varphi\big(a'_{(2)}, b_{(2)}\big)\epsilon_A(a)\epsilon_B(b')$$

Now we apply the (co)multiplication axiom (2.3) for the pairing φ to see that the last expression equals

$$\sum_{(a')} \varphi\big(a'_{(2)}S_A^{-1}(a'_{(1)}), b\big)\epsilon_A(a)\epsilon_B(b').$$

The antipode axiom for A and the fact that $S_A : A \to A^{op}$ is an algebra homomorphism now reduces this expression to

$$\epsilon_A(a')\varphi(1, b)\epsilon_A(a)\epsilon_B(b').$$

Applying the (co)unit axiom (2.2) for φ we finally obtain the right hand side of (3.6).

The verifications that Δ and S are algebra homomorphisms are even more elaborate! For the multiplicativity of Δ we have, in view of (3.4) and (3.2), that

$$\begin{aligned}
\Delta(a \otimes b)\Delta(a' \otimes b') &= \sum_{(a),(b),(a'),(b')} \big((a_{(1)} \otimes b_{(1)})(a'_{(1)} \otimes b'_{(1)})\big) \otimes \big((a_{(2)} \otimes b_{(2)})(a'_{(2)} \otimes b'_{(2)})\big) \\
&= \sum_{(a),(b),(a'),(b')} \varphi\big(S^{-1}(a'_{(1)}), b_{(1)}\big)\varphi\big(a'_{(3)}, b_{(3)}\big)\varphi\big(S^{-1}(a'_{(4)}), b_{(4)}\big)\varphi\big(a'_{(6)}, b_{(6)}\big) \\
&\qquad\qquad \times (a_{(1)}a'_{(2)} \otimes b_{(2)}b'_{(1)}) \otimes (a_{(2)}a'_{(5)} \otimes b_{(5)}b'_{(2)}) \\
&= \sum_{(a),(b),(a'),(b')} \varphi\big(S^{-1}(a'_{(1)}), b_{(1)}\big)\varphi\big(a'_{(4)}, b_{(4)}\big) \\
&\qquad \times \big((a_{(1)} \otimes 1)\Psi(a'_{(2)}, b_{(2)})(1 \otimes b'_{(1)})\big) \otimes (a_{(2)}a'_{(3)} \otimes b_{(3)}b'_{(2)})
\end{aligned}$$

with

$$\Psi(a, b) = \sum_{(a),(b)} \varphi\big(a_{(2)}, b_{(2)}\big)\varphi\big(S^{-1}(a_{(3)}), b_{(3)}\big)a_{(1)} \otimes b_{(1)} \in \mathcal{D}_\varphi(A, B).$$

We singled out the term $\Psi(a,b)$ because it can be drastically simplified by (2.3), the antipode axiom and the counit axiom,

$$\begin{aligned}\Psi(a,b) &= \sum_{(a),(b)} \varphi\big(a_{(2)},b_{(2)}\big)\varphi\big(S^{-1}(a_{(3)}),b_{(3)}\big)a_{(1)}\otimes b_{(1)}\\ &= \sum_{(a),(b)} \varphi\big(S^{-1}(a_{(3)})a_{(2)},b_{(2)}\big)a_{(1)}\otimes b_{(1)}\\ &= \sum_{(a),(b)} \epsilon_A(a_{(2)})\epsilon_B(b_{(2)})a_{(1)}\otimes b_{(1)}\\ &= a\otimes b.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\Delta(a\otimes b)\Delta(a'\otimes b') = \sum_{(a),(b),(a'),(b')} &\varphi\big(S^{-1}(a'_{(1)}),b_{(1)}\big)\varphi\big(a'_{(4)},b_{(4)}\big)\\ &\times (a_{(1)}a'_{(2)}\otimes b_{(2)}b'_{(1)})\otimes(a_{(2)}a'_{(3)}\otimes b_{(3)}b'_{(2)}).\end{aligned}$$

On the other hand, by (3.4) and (3.2) we have

$$\begin{aligned}\Delta((a\otimes b)(a'\otimes b')) &= \sum_{(a'),(b)} \varphi\big(S^{-1}(a'_{(1)}),b_{(1)}\big)\varphi\big(a'_{(3)},b_{(3)}\big)\Delta(aa'_{(2)}\otimes b_{(2)}b')\\ &= \sum_{(a),(b),(a'),(b')} \varphi\big(S^{-1}(a'_{(1)}),b_{(1)}\big)\varphi\big(a'_{(4)},b_{(4)}\big)\\ &\qquad\times (a_{(1)}a'_{(2)}\otimes b_{(2)}b'_{(1)})\otimes(a_{(2)}a'_{(3)}\otimes b_{(3)}b'_{(2)})\end{aligned}$$

in $\mathcal{D}_\varphi(A,B)^{\otimes 2}$. Comparing the expressions we conclude that

$$\Delta(a\otimes b)\Delta(a'\otimes b') = \Delta((a\otimes b)(a'\otimes b'))$$

in $\mathcal{D}_\varphi(A,B)^{\otimes 2}$. The verification of

$$S((a\otimes b)(a'\otimes b')) = S(a'\otimes b')S(a\otimes b) \tag{3.7}$$

in $\mathcal{D}_\varphi(A,B)$ is left as an exercise.
Proof of **2** and **3.** By the definitions of the maps Δ, ϵ and S we have

$$\Delta(a\otimes 1) = \sum_{(a)}(a_{(1)}\otimes 1)\otimes(a_{(2)}\otimes 1), \qquad \epsilon(a\otimes 1) = \epsilon_A(a), \qquad S(a\otimes 1) = S_A(a)\otimes 1 \tag{3.8}$$

for $a\in A$, and analogous formulas hold when Δ, ϵ and S are applied to elements of the form $1\otimes b$ for $b\in B$. Hence the Hopf algebra axioms for Δ, ϵ and S are trivially satisfied on the subalgebras $A\otimes 1$ and $1\otimes B$ of $\mathcal{D}_\varphi(A,B)$. Since these subalgebras generate $\mathcal{D}_\varphi(A,B)$, and Δ, ϵ and S are algebra morphisms by **1**, we conclude that the Hopf algebra axioms hold in general. Returning to the formulas (3.8) and the analogous formulas for B, we conclude that the canonical algebra embeddings of A and B into $\mathcal{D}_\varphi(A,B)$ are Hopf algebra morphisms. □

Exercise 3.6. *Prove that*

$$S(a \otimes b) = \sum_{(a),(b)} \varphi\big(a_{(1)}, b_{(1)}\big)\varphi\big(a_{(3)}, S(b_{(3)})\big)\big(S(a_{(2)}) \otimes S(b_{(2)})\big)$$

in $\mathcal{D}_\varphi(A, B)$.

Exercise 3.7. *This is a continuation of exercise 2.1 part* **(ii)**. *We thus identify* $\mathrm{Fun}_k(G)$ *with* $k[G]^*$ *for the given finite group* G. *The canonical nondegenerate Hopf pairing for* $k[G]$ *is*

$$\varphi(g, f) = f(g), \qquad g \in G, \ f \in \mathrm{Fun}_k(G)$$

when viewed as bilinear map $\varphi : k[G] \times \mathrm{Fun}_k(G)^{cop} \to k$.

(i) *For* $g \in G$ *define* $e_g \in \mathrm{Fun}_k(G)$ *by* $e_g(h) = \delta_{g,h}$ *for* $h \in G$, *where* $\delta_{g,h}$ *is the Kronecker delta function (it equals one if* $g = h$, *and zero otherwise). Show that* $\{e_g\}_{g\in G}$ *is a linear basis of* $\mathrm{Fun}_k(G)$ *satisfying*

$$e_g e_h = \delta_{g,h} e_g, \qquad 1 = \sum_{g\in G} e_g,$$

$$\Delta(e_g) = \sum_{\substack{u,v\in G:\\ uv=g}} e_u \otimes e_v, \qquad \epsilon(e_g) = \delta_{g,e},$$

$$S(e_g) = e_{g^{-1}}$$

for $g, h \in G$.

(ii) *The Hopf algebra* $\mathcal{D}(G) := \mathcal{D}_\varphi(k[G], \mathrm{Fun}_k(G)^{cop})$ *is called the quantum double of the finite group* G. *Describe its Hopf algebra structure in terms of the linear basis* $B = \{g \otimes e_h\}_{g,h\in G}$ *of* $\mathcal{D}(G)$.

References

[1] C. Kassel, *Quantum groups*, Springer GTM 155 (1995)

[2] C. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et syntheses **5**, Soc. Math. de France (1997)