QUANTUM GROUPS AND KNOT THEORY: WEEK 37

The basic text of this lecture is [7, Chapter 1 and 2].

1. Link diagrams and Reidemeister's Theorem

We work with combinatorial links in $\mathbb{R}^3$ in this section. Let $L \subset \mathbb{R}^3$ be a combinatorial link. We call L locally linear at $v \in L$ if there exists an open neighborhood $U \ni v$ of $v \in \mathbb{R}^3$, and an affine line $l \subset \mathbb{R}^3$ such that $L \cap U = l \cap U$. Let $V = V(L) \subset L$ be the finite subset of points of L where L is not locally linear. This finite subset of L is called the set of *vertices* of L.

Definition 1.1. *Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the standard projection $\pi(x, y, z) = (x, y)$. We say that L is in* general position *with respect to π if*

$$\forall p \in \pi(L) : |\pi^{-1}(p) \cap L| \leq 2 \tag{1.1}$$

while equality holds for only finitely many $p \in \pi(L)$, and if moreover

$$\forall p \in \pi(V) : |\pi^{-1}(p) \cap L| = 1. \tag{1.2}$$

A crossing *of $\pi(L)$ is a point $p \in \pi(L)$ such that the equality $|\pi^{-1}(p) \cap L| = 2$ holds. We call $\pi(L)$ a* regular projection *of L.*

Proposition 1.2. *Every equivalence class $[L]$ of combinatorial links contains a representative L which is in general position with respect to π.*

Proof. Let S be the triangulation of L whose set of 0-simplices is equal to $V = V(L)$. Let $v \in V$ and let σ and τ be the two 1-simplices of S containing v. There exists a small open ball $U \ni v$ such that for any $v' \in U$ we can modify L within its equivalence class to a link L' by a combination of a Δ and a Δ^{-1}-move replacing v by v' while adapting σ and τ accordingly. Let us call such a modification of L a *local deformation at* v of L. Observe the special case where we choose v' in the affine line l_τ defined by τ. This changes the direction of σ by changing the length (but not its direction) of τ by moving its end point at v inside the line determined by τ, while keeping the directions of all other 1-simplices of L fixed. A deformation of L is a sequence of local deformations.

A property of the embedding L is called *stable* if it is invariant for all *sufficiently small* local deformations of L, and *unstable* otherwise. For example, let $N = |V(L)|$. Consider the number $N(L, \pi) := |\pi(V(L))|$. Clearly $N(L, \pi) \leq N$, and the property $N(L, \pi) < N$ is unstable while the property $N(L, \pi) = N$ is stable. Hence for any L there exist arbitrarily small deformations L' of L such that $N(L', \pi) = N$. Without loss of generality we may therefore assume that $N(L, \pi) = N$. From now on we consider only sequences of local deformations of L which do not disturb this stable property. Then the images $\pi(\sigma)$ of the 1-simplices σ of L are all 1-simplices in $\mathbb{R}^2$. Let M denote the number of 1-simplices in S. Consider the number $M(L, \pi)$ of distinct *directions* of $\pi(\sigma)$ where σ runs over the set of 1-simplices of S. Clearly $M(L, \pi) \leq M$, and is stable only if equality holds. Hence we may assume from now on that all projections $\pi(\sigma)$ have distinct directions and that we are only considering sequences of local deformations from now on which do not disturb this property. In this situation the inverse image $\pi^{-1}(c)$ of any element $c \in \pi(L)$ is finite and there exist only finitely many points $c \in \pi(L)$ such that $|\pi^{-1}(c)| > 1$. For $v \in V(L)$, let $b(v)$ be the number of 1-simplices σ of L such that $\pi(v) \in \pi(\sigma)$. Thus $b(v) \geq 2$ for all $v \in V(L)$. Consider now the number

$$B(L,\pi) := \sum_{v \in V(L)} (b(v) - 2) \tag{1.3}$$

Clearly the only stable value of $B(L,\pi)$ is $B(L,\pi) = 0$. Let us assume this now. Observe next that the projection $\pi(L)$ can be deformed at any chosen vertex $\pi(v)$ by the projection of the deformations as discussed above: we can move $\pi(v)$ over a sufficiently small distance in the line determined by $\pi(\tau)$ while adapting $\pi(\tau)$ and $\pi(\sigma)$ accordingly and keeping the remaining part of $\pi(L)$ fixed (where σ and τ are the 1-simplices of L containing v, as above). We call the finite set C of points $c \in \pi(L)$ with $|\pi^{-1}(c)| > 1$ the set of crossing points of $\pi(L)$. Since $B(L,\pi) = 0$, no vertex $\pi(v)$ with $v \in V$ is a crossing point. Hence the order $d(c) := |\pi^{-1}(c)|$ of a crossing point can be defined as the number of 1-simplices $\pi(\sigma)$ such that $c \in \pi(\sigma)$. Let c_0 be a crossing point of order d_0, and apply the above type of local deformation of $\pi(L)$ to one of the projected 1-simplices $\pi(\sigma)$ such that $c_0 \in \pi(\sigma)$. This changes the direction of $\pi(\sigma)$ in a small but arbitrary way, while keeping the directions of the lines of the projections of the other 1-simplices in L fixed. Then c_0 is replaced after this deformation by $(d_0 - 1)$ new crossing points which have order 2, whereas the order of c_0 is reduced to $d_0 - 1$ (so if $d_0 = 2$, the original point c_0 is no longer a crossing point). Therefore, by such deformation of L the number

$$D(L,\pi) := \sum_{c \in C} (d(c) - 2) \in \mathbb{Z}_{\geq 0} \tag{1.4}$$

is reduced by 1 if $d(c_0) - 2$ is positive, and unchanged otherwise. Hence the only stable value of $D(L,\pi)$ is 0. So now we have finally obtained the situation $N(L,\pi) - N = M(L,\pi) - M = B(L,\pi) = D(L,\pi) = 0$, which amounts to saying that L is L regular for π. □

Definition 1.3. *A (PL-)link projection Π is the image of a PL-map $F : X \to \mathbb{R}^2$, where X is a finite disjoint union of PL-circles, such that F is injective except at finitely many double points $p \in \Pi$, and has the property of being locally linear at the pre-images of the double points.*

In particular, Π has the structure of a 4-valent planar graph (in which the 4-valent vertices are the double points of Π).

Let $\Pi = F(X) \subset \mathbb{R}^2$ be a link projection. Clearly there exists a triangulation S of X such that F is simplicial on S and such that the pre-images of the double points of Π are interior points of 1-simplices of S. By a linear subdivision of S we may furthermore assume that no 1-simplex of S contains more than one pre-image of a double point of Π. We will call such a triangulation S of X an F-adapted triangulation. A *crossing datum* at a double point $p \in \Pi$ is the information saying which of the two 1-simplices of S containing the pre-images of p crosses over the other.

Definition 1.4. *A* (PL-)link diagram *D is a link projection Π dressed with* crossing data *at each double point $p \in \Pi$. The double points of D are called* crossings *of D.*

A link $L \subset \mathbb{R}^3$ which is regular with respect to π defines a link diagram D by the composition F of a PL-embedding $I : X \to \mathbb{R}^3$ where X is a finite disjoint union of PL-circles such that $L = I(X)$ with the projection π, using the z-coordinate to define the crossing data at the crossings. Conversely we have:

Proposition 1.5. *Each link diagram is the regular projection of a unique link class.*

Proof. Let $F : X \to D \subset \mathbb{R}^2$ be the PL-map defining the link diagram D. Equip X with an F-adapted triangulation S. We may apply a linear subdivision to each 1-simplex of S which crosses over another 1-simplex at a crossing of D in three 1-simplices such that the middle peace contains the crossing. Therefore we may and will assume that the an overcrossing 1-simplex of S is always accompanied on both sides with 1-simplices that do not contain a crossing and whose other neighboring 1-simplex is not an overcrossing 1-simplex. We now construct a link L by modifying the simplicial map F (while keeping S fixed) to an *injective* simplicial map $I : X = |S| \to \mathbb{R}^3$ by moving, at each crossing point, the upper 1-vertex vertically upwards over a small distance into the upper half space and leaving the lower one in the xy-plane, and adapting the map F on the neighboring 1-simplices (which do not contain crossings and whose other neighbor is not an overcrossing 1-simplex) accordingly. Then $L = I(X)$ is a regular PL-link with respect to π, and L projects onto D. This shows that there exists a PL-link projecting regularly onto D.

To show the uniqueness, suppose that L' is another PL-link which is regular with respect to π and which projects onto D. Recall from week 36 that we need to fix a triangulation S' of L' in order to define Δ-moves, although the combinatorial equivalence class does not depend on this choice. Using a sequence of Δ and Δ^{-1}-moves we can translate L' vertically over an arbitrary distance (the reader is invited visualize this procedure). Since such a translation changes neither the projection of L' nor its equivalence class, we may assume that L' lies entirely above L. Recall from week 36 that we can refine a given triangulation of L' arbitrarily by application of Δ and Δ^{-1} moves. We may thus replace the triangulations of L and L' within the same combinatorial equivalence class by refinements which coincide with each other after applying the projection π. Therefore it suffices to show that L' is combinatorially equivalent to L under the assumption that the image of the triangulation S' of L' under π is the same as the image of S under F. In this situation the 1-simplices of L and L' are in natural bijection such that corresponding pairs σ, σ' determine a trapezium T_σ with two vertical sides, with the side σ' at the top and σ at the bottom. If σ has no crossings or is "passing under" at a crossing then T_σ does not contain points of L'. Hence we can replace, by two Δ-moves, the 1-simplex σ' of L' by the three other sides of T_σ. We do this for all 1-simplices σ' of L' which are undercrossings or non-crossing. The intermediate link L'' thus obtained is in general not regular with respect to π because of the vertical 1-simplices it contains, but this is of no concern. Now observe that each of the over-crossing 1-simplices σ' of L' is also a 1-simplex of L'', and that T_σ does not contain any points of L''. We can now modify L'' by combinatorial moves applied to the overcrossing 1-simplices as we did before for the undercrossing and non-crossing 1-simplicies. After these modification we will finally obtain the link L, finishing the proof of the uniqueness. □

From the work of the first knot tabulator Peter G. Tait (around 1877) the following basic question emerges, marking the beginning of knot theory: "When do two link diagrams define the same equivalence class of links?" A fundamental insight into the nature of this question was provided by Kurt Reidemeister (1926):

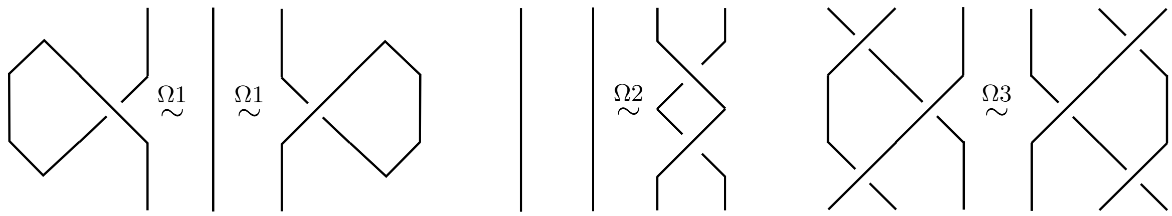

FIGURE 1. Reidemeister moves

Theorem 1.6. *Two link diagrams D, D' define the same link class iff D and D' are connected by a sequence of modifications consisting of* planar ambient isotopies *(i.e. PL-isotopies of $\mathbb{R}^2$ applied to the diagram) and of the three types of local (i.e. taking place within a small disk without changing the diagram outside this disk) moves called Reidemeister moves of type $\Omega 1$, $\Omega 2$ or $\Omega 3$ and their inverses. The three types of Reidemeister moves are those illustrated in Figure 1.*

Proof. We give a sketch of the proof. It is easy to see that two diagrams which differ by one of the 3 Reidemeister moves defined above, or by a planar ambient isotopy, define combinatorially equivalent links. The converse statement is less obvious and requires a serious analysis.

Suppose that we modify a given link L which is regular with respect to π by a Δ-move. We may first of all assume that the modified link L' is itself regular with respect to π. Indeed, from the proof of Proposition 1.2 it is clear that the set of directions such that both L and L' project regularly along this direction is the complement of finitely many lines and points of the real projective plane $\mathbb{P}^2(\mathbb{R})$. Hence there exists an open neighborhood U of the direction of the projection π (in the real projective plane of directions in $\mathbb{R}^3$) such that for all directions in U the projection of L along this new direction is still regular. Then the projection of L along a direction in U is planar ambient isotopic to D. But we have also seen that U contains directions with respect to which L' is regular. Therefore, if two diagrams define equivalent links then they can be connected by a sequence of planar ambient isotopies and steps obtained by the projection of a Δ or Δ^{-1}-move such that the link before and after the move is regular with respect to π.

Now every Δ-move is the composition of Δ and Δ^{-1}-moves associated with the six 2-simplices in the barycentric subdivision of the 2-simplex defining the Δ-move. Again, upon a small deformation of the direction of the projection π and application of planar ambient isotopies we may assume that the subdivided Δ-moves are all regular with respect to π. Since repeated barycentric subdivision produces a triangulation of the original 2-simplex with an arbitrarily small mesh (see week 36) we can make the participating Δ-moves small enough so as to ensure that the interior of $\pi(\Delta)$ of each subdivision 2-simplex Δ intersects nontrivially with at most one projected 1-simplex of L, or with two projected 1-simplices of L crossing each other in an interior point of of $\pi(\Delta)$. By the regularity assumptions we see that if the projection of a 1-simplex σ of L intersects the interior of $\pi(\Delta)$ nontrivially then either $\pi(\sigma)$ intersects the boundary of $\pi(\Delta)$ at two interior points of the faces of $\pi(\partial(\Delta))$, or σ is the neighboring 1-simplex in L of a face of $\partial(\Delta)$. Listing all possibilities for the projected Δ-moves of this kind, we finally need to establish that these modifications of D can all be realized by an appropriate combination of Reidemeister moves $\Omega 1$, $\Omega 2$ and $\Omega 3$ in combination

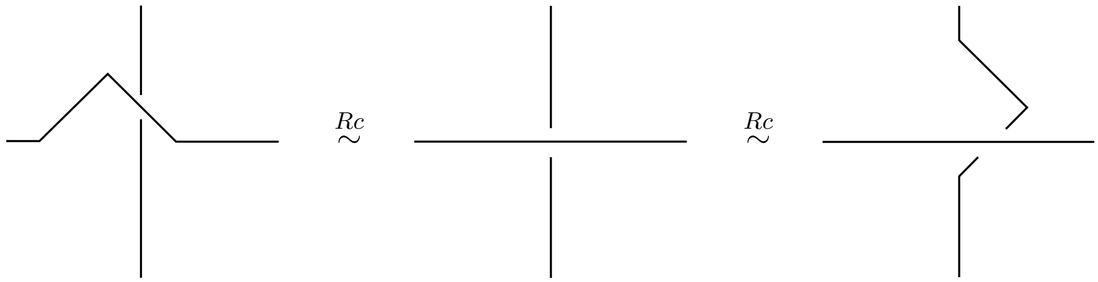

FIGURE 2. Diagram isotopies near a crossing

to planar ambient isotopies of the diagram. This is not difficult; we refer to [7] for further details. □

Exercise (a). *Let D be a link diagram with k components. Show that one can change the crossing data of D in such a way that the resulting new link diagram D' represents the k-component unlink.*

1.0.1. *Diagram isotopies.* Reidemeister's Theorem gives no description how to generate planar ambient PL-isotopies of link diagrams (called *diagram isotopies*) combinatorially (by "local moves", like the role of Δ-moves for ambient isotopies of PL-links in $\mathbb{R}^3$). For most applications this is not a problem because the invariance under diagram isotopies is otherwise easy to analyze, but for some of the later developments we will need to address this issue carefully. We will consider this problem now, while we are at it, before we return to some basic applications of Reidemeister's result.

Lemma 1.7. *Diagram isotopies are generated by planar Δ- and Δ^{-1}-moves applied to non-crossing 1-simplices (these are sometimes called R0-moves), together with the planar Δ- and Δ^{-1}-moves applied to a 1-simplex involved in a crossing (and leaving the other 1-simplex involved in the same crossing fixed), provided that the 2-simplex defining the move does not have any other intersection with the diagram outside the two 1-simplices involved in the crossing (an example of such a move is indicated in Figure 2). We call moves of the latter kind Rc-moves.*

Proof. The R0- and Rc-moves define diagram isotopies, as one easily sees. To see that these indeed generate all diagram isotopies we mimic the proof of Proposition [4, Proposition 1.10] (also see week 36) of the corresponding result for PL-links in $\mathbb{R}^3$. This proof shows that it suffices to prove that these local combinatorial moves generate the translations of diagrams in $\mathbb{R}^2$. This is easy to see (again following [4, Proposition 1.10]). □

Lemma 1.8. *Let D and D' be link diagrams, and let L, L' be PL-links projecting regularly onto D and D' respectively via the projection π. The following assertions are equivalent:*

(i) *D, D' are connected by a planar ambient isotopy.*
(ii) *L, L' are connected by a sequence of Δ and Δ^{-1}-moves each of which has the property of representing an ambient isotopy $t \to L_t$ such that L_t is regular with respect to π for all $t \in [0,1]$).*

Proof. (i) implies (ii) by the above Lemma, since the R0- and Rc-moves in the Lemma are indeed projections of Δ- and Δ^{-1}-moves with the required property.

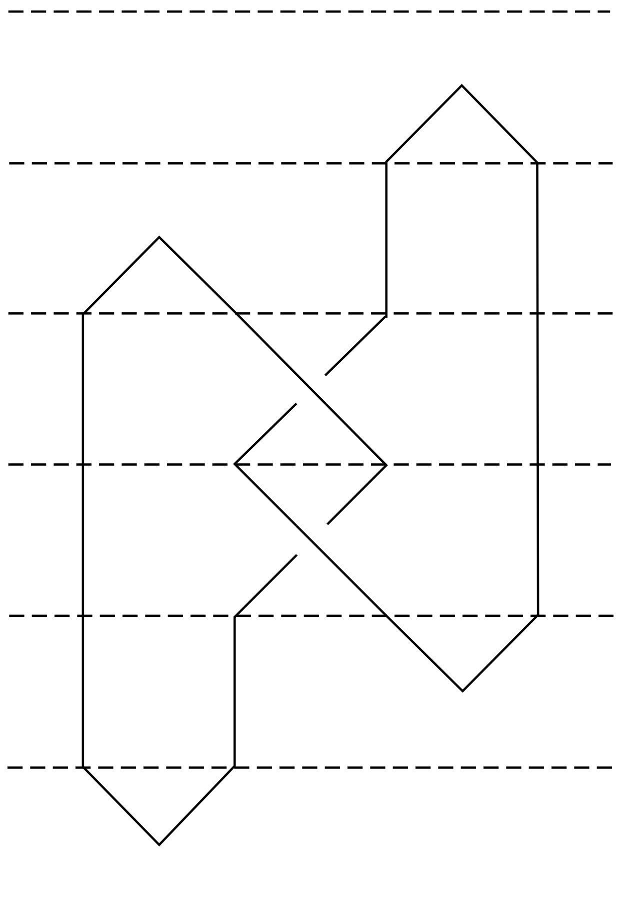

FIGURE 3.

For the converse implication we proceed as in the proof of Reidemeister's Theorem, by projecting a sufficiently small Δ-move via π. By assumption the Δ-move is the end result of a family of such moves with the property that all intermediate steps project regularly with respect to π. It is an easy verification (as in the proof of Reidemeister's Theorem in [7]) that such projected Δ-move (after a suitable linear subdivision if necessary) are of the kind $R0$ or Rc. $\square$

1.0.2. *Diagram isotopies and generic diagrams.* It is an extremely useful idea to think of a link diagram as being composed of horizontal strips such that each strip contains at most one crossing or local maximum or minimum, see Figure 3. To formalize this, fix a nonzero linear functional $h : \mathbb{R}^2 \to \mathbb{R}$. We call h the height function. Unless specified otherwise we will choose $h = y$.

Definition 1.9. *A link diagram D is called* generic *(with respect to h) if the restriction of h to Π_D (the link projection underlying D) has finitely many local extrema, and if the local extrema and the crossings all have distinct heights. We call the union of the finite set of local*

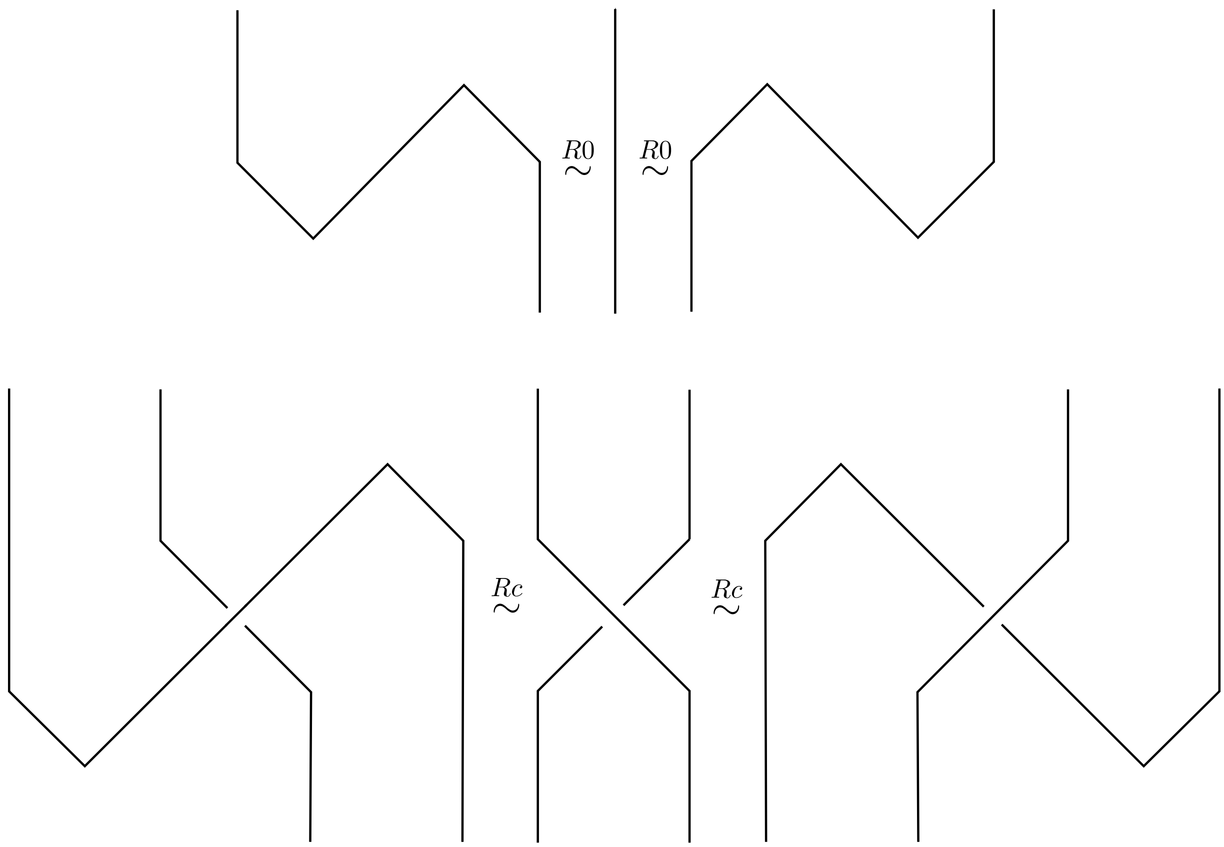

FIGURE 4. The Reidemeister moves R0 and Rc (+refl. in vert. line).

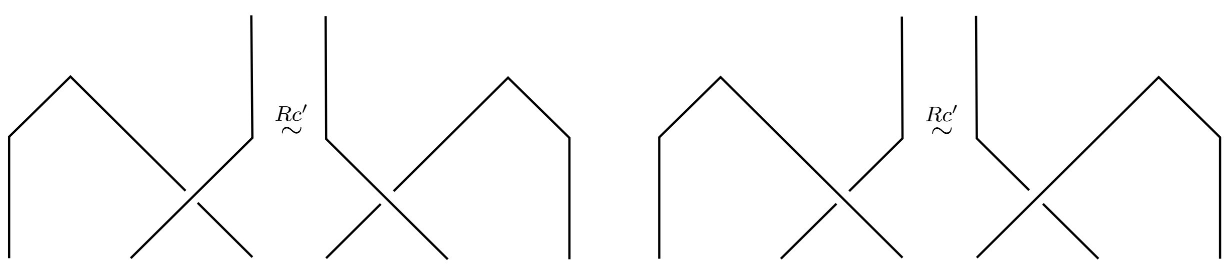

FIGURE 5. Nongeneric diagram isotopies (+refl. in hor. line)

extrema of h on a generic diagram D with the finite set of crossings of D the set of singular points *of D (with respect to h).*

Every link diagram is ambient isotopic to a generic link diagram. For example, for any given link diagram D, its rotations D_ϕ will be generic for all but finitely many angles $\phi \in [0, 2\pi]$. Every generic link diagram can be cut into horizontal strips each of which contains at most one singular point, besides a number of disjoint polygonal segments without local extrema with one boundary point on the bottom of the strip and its other boundary point on the top of the strip. Such strips are called *elementary strips.*

Definition 1.10. *Let D_0 and D_1 be generic link diagrams. A diagram isotopy* $[0,1] \ni t \to D_t$ *from D_0 to D_1 is called* generic *if D_t is generic for all* $t \in [0,1]$.

A general diagram isotopy between two generic link diagrams can be broken up as a composition of a sequence of diagram isotopies consisting of a generic diagram isotopies in an elementary strip, or non-generic diagram isotopies of the type R0 or Rc. A move of type R0 or Rc can be non-generic: the simplest way in which this may happen involves the creation or annihilation of two consecutive elementary strips containing a local maximum and a local minimum (the R0-move) or two such elementary strips bordering an elementary strip with a crossing as shown in Figure 4 (together with the reflection of the Rc-move in a vertical line (or the plane of the diagram)).

Another type of non-generic move induced by R0 and Rc-moves amounts to interchanging the relative heights of two singular points. Consider a (non-elementary) strip containing two singular points which have the same height. By diagram isotopies it can be deformed and decomposed as a composition of two elementary strips in two different ways (with a different order of the heights of the two singular points). The resulting diagram isotopy between these two compositions of elementary strips is referred to as "changing the relative heights of non-interacting singular points". Such non-generic diagram isotopies are consequence of R0- and Rc-moves (in the sense of Figure 2) as well.

Exercise (b). *Consider the composition of an elementary strip with a local maximum and a strip without singular points. Show how to change the order of the two strips using R0-moves (in the sense of 2). Do the same for an elementary strip containing a crossing, using Rc-moves. Prove that this implies that "changing the relative heights of non-interacting singular points" is a consequence of R0- and Rc-moves.*

Conversely we have:

Theorem 1.11. *Any diagram isotopy between generic link diagrams is a composition of elementary moves of the following kind:*

(i) *Generic diagram isotopies in elementary strips.*
(ii) *Inserting or deleting a strip without singular points.*
(iii) *Changing the relative heights of non-interacting singular points.*
(iv) *Nongeneric R0-moves (as in Figure 4).*
(v) *Nongeneric Rc-moves (as in Figure 4, and its reflection in a vertical line).*

Proof. It is easy to see that every R0- or Rc-moves is a composition of these five types of moves; conversely (as we argued in the text above) these moves are consequences of the R0- and Rc-moves. By Lemma 1.7 the result follows. □

Exercise (c). *Prove that in Theorem 1.11 the nongeneric Rc-moves can be replaced by the nongeneric moves of type Rc' of Figure 5.*

As a corollary we obtain the following refined version of Reidemeister's Theorem:

Theorem 1.12. *Two generic link diagrams represent isotopic links if and only if they can be connected by a sequence of moves of the type listed in Theorem 1.11 together with the three Reidemeister moves* $\Omega 1$, $\Omega 2$ *and* $\Omega 3$ *(as listed in Figure 1).*

1.1. **Variations**

Two additional structures play a crucial role in various aspects of the theory.

1.1.1. *Oriented links.* An *oriented link* is a link in which all components are dressed with an orientation. Two oriented links are called equivalent if they are ambient isotopic and the isotopy preserves the orientation.

It is clear that there is a corresponding notion of equivalence of oriented link diagrams. Generally this complicates the application of Reidemeister's Theorem a little bit because one needs to consider the Reidemeister moves with all the possible orientations of the strands involved.

1.1.2. *Framed links and regular isotopy.* A *framed link* is a *smooth* link L dressed with a non-vanishing normal vector field f. The framing of L determined by f is the homotopy class of f within the space of non-vanishing normal vector fields. An isotopy of framed links is a smooth ambient isotopy of the underlying unframed links such that the framings correspond under the isotopy.

Although this notion is most naturally defined in the smooth context, the notion can be expressed in terms of combinatorial equivalence of link diagrams as well. This leads to the notion of *regular isotopy* and this will be discussed later, in relation to *ribbon links*.

2. **First applications of Reidemeister's Theorem**

One of the basic goals of knot theory is to describe the set of $\mathcal{L}$ of equivalence classes of tame links. Reidemeister's Theorem provides an answer by describing the set $\mathcal{L}$ as the quotient of the set $\mathcal{D}$ of planar equivalence classes of link diagrams in $\mathbb{R}^2$ by the equivalence relation generated by the 3 Reidemeister moves $\Omega 1$, $\Omega 2$ and $\Omega 3$. The problem with this description lies in the long and complicated chains of Reidemeister moves and their inverses that are in general necessary to establish the equivalence of two given diagrams. Understanding the complexity of algorithms deciding on the equivalence of given diagrams (e.g. the unknotting problem) is a field of current research.

The following theorem is an equivalent "dual formulation" of Reidemeister's Theorem which is often useful in practice:

Definition 2.1. *A* link invariant *is a map $f : \mathcal{L} \to V$ to a set V.*

Theorem 2.2. *A map $f : \mathcal{D} \to V$ defines a link invariant iff f is invariant for the* 3 *Reidemeister moves.*

Proof. This is an obvious consequence of Theorem 1.6. □

Knowing all link invariants describes the set $\mathcal{L}$. However, the above theorem is not suitable to *find* link invariants, but rather to verify if a given function defined on $\mathcal{D}$ is a link invariant. Later in the course we will discuss a sophisticated modern construction of link invariants based on the representation theory of quantum groups, and (a version of) the above theorem will indeed be used to prove the result. As such Reidemeister's result is of great theoretical use.

In this section we will discuss some classical link invariants, using Reidemeister's Theorem when necessary.

2.0.3. *The number of components.* One of the first interesting link invariants is the number of components of a link. It is trivial that this is an invariant of tame links (in the topological category). It is also easy to see directly that this defines an invariant of PL-links for combinatorial equivalence.

2.0.4. *The linking number.* The linking number is not an ordinary link invariant, since it is only defined naturally on oriented two component links.

Suppose that K and L are disjoint, oriented links and suppose that $K \cup L$ is regular with respect to the standard projection π on the xy-plane. Let D_K and D_L be the link diagrams obtained from the regular projection of K and L respectively onto the xy-plane via π, and let $V(K,L)$ be the set of crossings of $D_{K\cup L}$ which involve both a strand from K and a strand from L. We define

$$lk(D_K, D_L) := 1/2 \sum_{c \in V(K,L)} \epsilon(c) \tag{2.1}$$

where $\epsilon(c) = 1$ if the upper strand points to the right when we pass the crossing following the lower strand, and $\epsilon(c) = -1$ otherwise.

Theorem 2.3. *The number $lk(D_K, D_L)$ only depends on the ambient isotopy class of the oriented link $K \cup L$ and its decomposition as a disjoint union of K and L. Therefore we may denote this number as $Lk(K, L)$ (the* linking number *of K and L). Moreover, we have $Lk(K, L) \in \mathbb{Z}$.*

Proof. First of all observe that the number $lk(K, L)$ depends only on the diagram isotopy class of the diagram $D_{K\cup L}$. Hence to establish the result it suffices to show that $lk(D_K, D_L)$ is invariant under all the (oriented versions of) the Reidemeister moves $\Omega 1$, $\Omega 2$ and $\Omega 3$. The invariance under moves of type $\Omega 1$ is obvious since the crossings involved in such moves are never in $V(K, L)$. We leave it to the reader to perform the easy task of showing invariance under $\Omega 2$ and $\Omega 3$. Finally we will prove that $Lk(K, L) \in \mathbb{Z}$, using that this number is invariant for ambient isotopy. By Exercise **a** we may change the crossing data of $D_{K\cup L}$ in such a way that the resulting link diagram D' represents an unlink $K' \cup L'$ projecting regularly onto D'. But the unlink can also be represented by a link diagram without crossings. Since $Lk(K', L') = lk(D_{K'}, D_{L'})$ depends only on the ambient isotopy class, this implies that $Lk(K', L') = 0$. But on the other hand, a change of a crossing datum shifts the number lk by 1. The conclusion of these two facts is that $Lk(K, L) \in \mathbb{Z}$. $\square$

2.0.5. *The space of colorings modulo p.* Let D_L be a link diagram, and X an abelian group. We will define an abelian group $C(L, X)$ which will only depend on the link class L defined by D_L up to isomorphism. In the special case $X = \mathbb{F}_p$, the additive group of the finite field with p elements (where p is a prime) we will give an explicit presentation for $C(L, p) := C(L, \mathbb{F}_p)$, the "space of colorings of L modulo p."

An *arc* of D_L is defined to be a connected component of D_L after cutting the lower strands at each crossing of D_L. Let A be the set of arcs, and let C be the set of crossings of D_L.

A *coloring* of D_L is a function $f : A \to X$ such that at each crossing $c \in C$ the relation $2f(u) = f(l_-) + f(l_+)$ holds, where u denotes the arc of the upper strand at the crossing c, and l_-, l_+ denote the arcs of the two pieces of the cutted lower strand at the crossing.

Consider the free abelian groups $\mathbb{Z}[C]$ and $\mathbb{Z}[A]$ on the finite sets C and A respectively. We define a homomorphism of abelian groups $M_L : \mathbb{Z}[C] \to \mathbb{Z}[A]$ by

$$M_L(c) = 2u - l_- - l_+, \tag{2.2}$$

(where the meaning of u, l_- and l_+ is as above) and extending this by linearly to $\mathbb{Z}[C]$. Then the above description of the abelian group $C(L, X)$ of X-valued colorings of D_L amounts to

$$C(L, X) := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}(M_L), X) \tag{2.3}$$

Theorem 2.4. *Up to isomorphisms, the group $C(L, X)$ only depends on the ambient isotopy class of links represented by D_L.*

Proof. It is clearly sufficient to show that the abelian group $\mathrm{Coker}(M_L)$ only depends on the ambient isotopy class $[L]$ of links represented by D_L. The presentation

$$\mathbb{Z}[C] \xrightarrow{M_L} \mathbb{Z}[A] \to \mathrm{Coker}(M_L) \to 0 \tag{2.4}$$

is clearly independent for diagram isotopies. Hence it remains to check the invariance (up to isomorphisms) for the Reidemeister moves $\Omega 1$, $\Omega 2$ and $\Omega 3$. The reader is invited to verify the details. □

In the special case $X = \mathbb{F}_p$, the finite field of order p (with p a prime number), it follows that

$$C(L, p) := C(L, \mathbb{F}_p) = \mathrm{Ker}({}^T M_{L,p}) \tag{2.5}$$

where ${}^T M_{L,p}$ denotes the transpose of the reduction modulo p of the matrix M_L, reducing the problem of computing $C(L, p)$ to a linear algebra problem over the field $\mathbb{F}_p$.

Exercise (d). *Show that $C(L, 2) \xrightarrow{\sim} \mathbb{F}_2^k$ where k denotes the number of components of L.*

Exercise (e). *Using the link invariants $C(L, p)$ for various p, prove that*

(i) *The 3-component borromean rings link is not equivalent to the 3-component unlink.*
(ii) *The trefoil knot and the eight knot are nontrivial and distinct.*

3. The braid group

3.0.6. *The algebraic braid group.*

Definition 3.1. *The algebraic braid group B_n^{alg} on n strands is the group generated by $n-1$ generators σ_i, $i = 1, \ldots, n-1$, and relations*

(i) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *if* $|i - j| > 1$, *and*
(ii) $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ *for all* $i = 1, \ldots, n-2$.

Corollary 3.2. *The symmetric group $\mathfrak{S}_n$ is a quotient of B_n^{alg} by the map π^{alg} sending σ_i to the transposition $(i, i+1)$.*

3.0.7. *The topological braid group.*

Definition 3.3. *A topological braid on n-strands is a disjoint union β of n segments (i.e. embeddings of $[0,1]$) in $\mathbb{R}^2 \times [0,1] \subset \mathbb{R}^3$ such that:*

(i) $|\beta \cap \{z = t\}| = n$ *for all* $t \in (0,1)$.
(ii) $\beta \cap \{z = t\} = \{1, \ldots, n\} \times \{0\} \times \{t\}$ *for all* $t \in \{0,1\}$.

Definition 3.4. *Similarly we define PL-braids and smooth braids, where for smooth braids we require that there exists an $\epsilon > 0$ such that the strands are all vertical for $t \in [0, \epsilon)$ and for $t \in (1-\epsilon, 1]$.*

Definition 3.5. *We call two braids β_1 and β_2* strongly isotopic, *notation $\beta_1 \sim_s \beta_2$, if there exists an ambient isotopy transforming β_1 in β_2 which respects the planes $z = h$ for all $h \in (0,1)$ and all $t \in [0,1]$, and which is the identity for all $t \in [0,1]$ in the planes $z = 0$ and $z = 1$.*

The proof of the following result is standard and reminiscent of the proof that the fundamental group of a topological space (with base point) is a group:

Theorem 3.6. *The set of strong isotopy classes of topological braids on n-strands form a group B_n^{top} such that the composition $[\beta_1][\beta_2]$ is defined by the strong isotopy class $[\beta]$ of the braid β obtained by putting β_1 on top of β_2, and reducing the heights (the z-coordinate) by a factor 2. The unit element is the braid consisting of n vertical straight segments of length one.*

Remark 3.7. *Observe that the composition of smooth braids is well defined in this way, since the strands of the smooth braids are required to be vertical near their end points.*

Definition 3.8. *A braid diagram is a regular projection of a PL-braid onto the xz-plane, where regularity refers to the projection π_y alon the y-axes.*

The following crucial result is nontrivial.

Theorem 3.9. *(See* [1]*) Two braid diagrams D_1 and D_2 are strongly isotopic iff D_1 and D_2 are connected by a finite sequence of moves consisting of braid diagram isotopies (i.e. ambient diagram isotopies respecting the height function z) and the Reidemeister moves $\Omega 2$ and $\Omega 3$.*

Corollary 3.10. *The map $t : \sigma_i \to X_i$ (for $i = 1, \ldots, n-1$) extends to an isomorphism $t : B_n^{alg} \to B_n^{top}$. Here X_i is the braid on n-strands where all but two strands are vertical segments, except for the strands i and $i+1$ which cross each other (and the strand from $(i,0,0)$ to $(i+1,0,1)$ crosses over the other strand).*

Proof. The X_i are generators of the group B_n^{top}. Indeed, we can represent any braid β on n-strands such that all crossings have distinct heights, and cutting such a braid in horizontal strips such that every strip contains at most one crossing gives a representation of β as a product of elements of the form $(X_i)^{\pm 1}$. The generators X_i satisfy the Reidemeister equivalence Ω_3, and thus we obtain a surjective homomorphism $t : B_n^{alg} \to B_n^{top}$. The injectivity of the map follows from Artin's Theorem 3.9. □

We define a map $\pi : B_n^{top} \to \mathfrak{S}_n$ by requiring that $\pi(\beta)(i) = j$ if the strand of β connected to the point $(i,0,0)$ ends in $(j,0,1)$. It is easy to see that π^{top} is a surjective homomorphism. It follows easily that $\pi^{alg} = \pi^{top} \circ t$ by checking it on the generators σ_i.

Definition 3.11. *The kernel* $P_n^{top} \subset B_n^{top}$ *of* π^{top} *is called the* pure *braid group.*

Corollary 3.12. *There exists a canonical short exact sequence*

$$1 \to P_n^{top} \to B_n^{top} \xrightarrow{\pi^{top}} \mathfrak{S}_n \to 1 \tag{3.1}$$

3.0.8. *The braid group and configuration spaces of points in* $\mathbb{C}$. Let $Y_n = \mathbb{C}^n \setminus \cup_{i<j} \{z_i = z_j\}$ be the configuration space of n distinct ordered points in $\mathbb{C}$, and let $X_n = \mathfrak{S}_n \backslash X_n$ be the configuration space of n distinct unordered complex numbers. We define $P_n^{fun} := \Pi_1(Y_n, (1, 2, \ldots, n))$ and $B_n^{fun} := \Pi_1(X_n, \overline{(1, 2, \ldots, n)})$. Since the action of the finite group $\mathfrak{S}_n$ on Y_n is free, the canonical quotient map $\pi : Y_n \to X_n$ is a regular covering map [3, Proposition 7.2] with group of deck transformations group $\mathfrak{S}_n$. By standard theory of regular covering maps [3, Proposition 6.9, Proposition 7.2] we get the exact sequence of monodromy:

$$1 \to P_n^{fun} \to B_n^{fun} \xrightarrow{\pi^{fun}} \mathfrak{S}_n \to 1 \tag{3.2}$$

Given a topological braid β we define a path $f(\beta)$ in X_n by requiring for all $t \in [0, 1]$:

$$f(\beta)(t) := \beta \cap \{(x, y, z) \in \mathbb{R}^3 \mid z = t\} \tag{3.3}$$

Then $f(\beta)$ is closed (as a path in X_n) with begin and end point $\overline{(1, 2, \ldots, n)}$. It is not hard to see that a strong isotopy of topological braids $\beta_1 \sim_s \beta_2$ defines a homotopy $f(\beta_1) \sim f(\beta_2)$ of closed paths keeping the end point fixed. Hence we obtain a map (also denoted by f):

$$f : B_n^{top} \to B_n^{fun} \tag{3.4}$$

In view of the definitions of the group structures in B_n^{top} and B_n^{fun} it is easy to see that $f(\beta_1\beta_2) = f(\beta_2)f(\beta_1)$ or in other words, f is an anti-homomorphism of groups.

Theorem 3.13. *The homomorphism* $f : B_n^{top,op} \to B_n^{fun}$ *is an isomorphism.*

Proof. We need still to verify that f is bijective. The surjectivity is easy to see, by associating a topological braid β to a closed path λ in X_n which begins and ends in $\overline{(1, 2, \ldots, n)}$ a topological braid β such that $f(\beta) = \lambda$. The injectivity is a quite nontrivial matter, for which we refer to [1]. □

The exact sequence (3.1) is related to the exact sequence of monodromy (3.2) via f, but we need to accommodate for changing to the opposite groups. It follows that $\pi^{fun}(f([\beta])) = \pi^{top}([\beta])^{-1}$ for any topological braid β.

References

[1] Artin, E., *The Theory of Braids*, Annals of Mathematics, Vol. **48**, No. 1 (1947), pp. 101–126.
[2] P. Alexandroff, H. Hopf, *Topologie*, Springer Verlag (1935).
[3] G.E. Bredon, *Topology and Geometry*, Springer GTM 139 (1993).
[4] G. Burde, H. Zieschang, *Knots*, De Gruyter (1985).
[5] C. Kassel, *Quantum groups*, Springer GTM 155 (1995)
[6] C. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et syntheses **5**, Soc. Math. de France (1997)
[7] C. Livingston, *Knot Theory*, The Carus Mathematical Monographs 24 (1993).
[8] C.P. Rourke, B.J. Sanderson, *Introduction to Piecewise linear Topology*, Ergebnisse Math., Band 69, Springer Verlag (1972).