The main goal of today’s lecture will be: A first introduction to Hopf algebras. The basic text of this course is [3], Chapter 2, and [2], Chapter 3. Unexplained notations and the numbering in this syllabus refer by default to [3].

1. The Yang Baxter equation and braid group representations

All the material in this chapter is covered unless stated otherwise.

1. The Yang Baxter equation.

Exercise (a). See [3, Exercise 1.5(a)].

Exercise (b). See [3, Exercise 1.5(b)]. However, an additional condition is necessary to make sure that the matrix is invertible.

2. Artin’s braid group. Most things here were already covered in week 37.

3. Alternative description of $B_n$. We have already seen two definitions of the braid group. The topological braid group (let us denote this by $B_n^{\text{top}}$) was defined as the group consisting of isotopy classes of braids with $n$ strands in $\mathbb{C} \times [0, 1]$, with the product defined by composition of braids. On the other hand the algebraic braid group $B_n^{\text{alg}}$ was defined in the preceding section by generators and relations. In week 37 it was shown that $B_n^{\text{top}} \approx B_n^{\text{alg}}$.

In the current section there is yet another incarnation of $B_n$: The fundamental group $B_n^{\text{fun}}$ of the configuration space $Y_n$ of $n$ distinct, unordered points in $\mathbb{C}$.

Definition 3.1. Let $p = (1, \ldots, n) \in X_n$ and let $\bar{p} = \pi(p) \in Y_n$. Let $B_n^{\text{fun}} = \Pi_1(Y_n, \bar{p})$, and let $P_n^{\text{fun}} = \Pi_1(X_n, p)$.

Proposition 3.2. The canonical quotient map $\pi : X_n \to Y_n$ is a regular covering map with deck transformation group $S_n$.

Proof. This is a standard result in basic algebraic topology: if a finite group $G$ acts freely on a Hausdorff space $X$ which is connected and locally pathwise connected then the quotient map $\pi : X \to G \backslash X$ is a regular covering with deck transformation group $G$, see [1, Proposition III 7.2, Exercise III 7.1].

Corollary 3.3. We have a fundamental exact sequence

$$1 \to P_n^{\text{fun}} \to B_n^{\text{fun}} \to S_n \to 1 \tag{3.1}$$

Proof. This is a standard fact about regular covering maps see e.g. [1, Corollary III 6.9].
It is useful to understand the exact sequence (3.1) in more detail. The *monodromy action* \( \mu : \Xi \times B_n^{\text{top}} \to \Xi \) of \( B_n^{\text{top}} \) on the fiber \( \Xi := \pi^{-1}(\bar{p}) = S_n \pi \) above \( \bar{p} \) is defined as follows. If \( b \in B_n^{\text{top}} \) and \( x \in \Xi \) and let \( l \) be a loop in \( Y_n \) representing \( b \). Then \( \mu(x, b) \) is the end point \( g(1) \in \Xi \) of the unique lifting \( g : [0, 1] \to X_n \) of \( l \) which starts at \( x \). The monodromy action yields a transitive right action of \( B_n^{\text{top}} \) on \( \Xi \). This right action commutes with the (simply transitive) left action of the deck transformation group \( S_n \) on \( \Xi \). This defines a surjective homomorphism \( \alpha : B_n^{\text{top}} \to S_n \) by requiring that \( \alpha(b)p = \mu(p, b) \). By definition of the monodromy action the kernel of \( \alpha \) is equal to \( \pi_* (P_n^{\text{top}}) \triangleleft B_n^{\text{top}} \). By the homotopy lifting theorem (see e.g. [1, Theorem III 3.4]) it is clear that \( \pi_* : P_n^{\text{top}} \to B_n^{\text{top}} \) is a monomorphism.

**Definition 3.4.** The group \( P_n^{\text{fun}} \) is identified with the normal subgroup \( \pi_* (P_n^{\text{fun}}) \triangleleft B_n^{\text{fun}} \) and is called the pure braid group.

We now construct a surjective anti-homomorphism \( \lambda : B_n^{\text{top}} \to B_n^{\text{fun}} \). Let \( b = [g] \in B_n^{\text{top}} \) be represented by a continuous map \( g = (g_1, g_2, \ldots, g_n) : [0, 1] \to X_n \). Thus the map \( g_i : [0, 1] \to \mathbb{C} \) represents the \( i \)-th strand of the topological braid \( b \). The condition that the strands are disjoint is equivalent to saying that the image of \( g \) is contained in \( X_n \). Observe that the path \( \pi \circ g \) is a closed path in \( Y_n \) which begins and ends at \( \bar{p} \), and hence determines an element \( \sigma_g \in B_n^{\text{fun}} \). It is clear that if \( g : [0, 1] \to X_n \) is isotopic to \( h \) in the sense of braids (i.e. we require at all times of the isotopy that the number of intersection points of the strands with a plane of the form \( \mathbb{C} \times \{t\} \) (with \( t \in (0, 1) \)) is equal to \( n \) then the loops \( g \circ \pi \) and \( h \circ \pi \) are homotopic, hence \( \sigma_g = \sigma_h \). (As an aside, it is known (Artin) that the equivalence relation defined on the set of braids by the above notion of braid isotopy is the same as the equivalence relation defined by allowing the more general isotopies of tangles). Therefore we have a well defined map \( \lambda : B_n^{\text{top}} \to B_n^{\text{fun}} \) by putting \( \lambda([g]) = \sigma_g \). This is clearly an anti-homomorphism.

**Exercise (c).** Check that \( \lambda \) is an anti-homomorphism.

It is also clear that \( \lambda \) is surjective. Any element of \( B_n^{\text{fun}} \) is of the form \([l] \in B_n^{\text{fun}} \) for a closed loop \( l \) in \( Y_n \) beginning at \( \bar{p} \). Then \( l \) has a unique lift \( g : [0, 1] \to X_n \) starting at \( p \in X_n \), which is clearly equivalent to saying that \( \lambda([g]) = [l] \).

It was shown by Artin that \( \lambda \) is also injective, in other words: if \( g \circ \pi \) and \( h \circ \pi \) are homotopic loops in \( Y_n \) (homotopy with fixed end points) then \([g], [h] \in B_n^{\text{top}} \) are isotopic braids. We will not show this result here. We summarize the above discussion in the main result of this section.

**Theorem 3.5.** (Artin) The map \( \lambda : B_n^{\text{top}} \to B_n^{\text{fun}} \) defined by \( \lambda([g]) = \sigma_g := [g \circ \pi] \) is an anti-isomorphism. The pure braid group \( P_n^{\text{top}} \) is mapped to \( P_n^{\text{fun}} \) by \( \lambda \).

**Exercise (d).** (The monodromy action) Recall the homomorphism \( f : B_n^{\text{top}} \to S_n \) discussed in week 37. Prove that \( \alpha(\lambda(b)) = f(b)^{-1} \), in other words prove that \( \mu(p, \lambda(b))_i = g_i(1) \), where \( g : [0, 1] \to X_n \) is a topological braid representing \( b \).

**Exercise (e).** We use the notation for the generators of the braid group as in [3, Definition 2.1]. The following element of the braid group is a central element: \( \beta_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n \in
Quantum Groups and Knot Theory Lecture: Week 38

\[ B_n^{\text{top}} \]. Let \( \gamma_n \in B_n^{\text{top}} \) be the isotopy class of braids represented by \( g = (g_1, \ldots, g_n) : [0, 1] \to \mathbb{C}^n \) given by \( g_k(s) = k \exp 2\pi is \).

(i) Show that \( \gamma_n \in B_n^{\text{top}} \) is central.

(ii) Draw \( \gamma_4 \) and prove that \( \gamma_4 = \beta_4 \).

(iii)* (bonus) Prove that (ii) generalizes to arbitrary \( n \).

2. Hopf Algebras and Monoidal Categories

A good additional reference for this chapter is [2].

1. Hopf Algebras. (See also [2, Chapter III, 1-3]). Excluded are Example 1.6 and Exercise 1.7(a).

1.8. Sweedler’s sigma notation. The Sweedler notation is also denoted e.g. as follows:

\[ (1.1) \quad (\text{id} \otimes \Delta) \Delta(x) = \sum_{(x)} x' \otimes x'' \otimes x''' \]

Exercise (f). Suppose that \( A \) and \( B \) are bialgebras.

(i) Show that \( (a \otimes b)(a' \otimes b') = aa' \otimes bb' \) determines a unique algebra structure on \( A \otimes B \).

(ii) Show that the map \( \Delta : A \otimes B \to (A \otimes B) \otimes (A \otimes B) \) defined by \( \Delta = (\text{id}_A \otimes \tau_{A,B} \otimes \text{id}_B) \circ \Delta_A \otimes \Delta_B \) together with the map \( \epsilon : A \otimes B \to k \) defined by \( \epsilon(a \otimes b) := \epsilon_A(a) \epsilon_B(b) \) defines a bialgebra structure on \( A \otimes B \) with comultiplication \( \Delta \) and counit \( \epsilon \).

Exercise (g). Show that \( A \otimes B \) is a Hopf algebra if \( A \) and \( B \) are Hopf algebras.

Exercise (h). (i) Given two bialgebras \( A, B \) define a convolution product \( * \) on \( \text{Hom}(A, B) \) generalizing the convolution product on \( \text{End}(A) \) for a bialgebra \( A \).

(ii) Prove that the convolution product defines an algebra structure on \( \text{Hom}(A, B) \) with unit \( \eta_B \circ \epsilon_A \).

Exercise (i). Let \( A = (A, \mu, \eta, \Delta, \epsilon) \) be a bi-algebra. Show that \( A^{\text{op}} = (A, \mu^{\text{op}}, \eta, \Delta, \epsilon) \) and \( A^{\text{cop}} = (A, \mu, \eta, \Delta^{\text{op}}, \epsilon) \) are also bi-algebras.

Exercise (j). Let \( H \) be an Hopf algebra. A two-sided ideal \( I \subset H \) is called a Hopf ideal if

\[ I \subset \ker(\epsilon), \]

\[ \Delta(I) \subset I \otimes H + H \otimes I, \]

\[ S(I) \subset I. \]

Show that the quotient algebra \( H/I \) has a unique Hopf algebra structure such that the canonical map \( \pi : H \to H/I, \pi(h) = h + I \), becomes a morphism of Hopf algebras.

Exercise (k). Let \( H \) be a Hopf algebra with antipode \( S \).

(i) We equip \( A := \text{Hom}_k(H \otimes H, H) \) with convolution algebra structure as defined above. Let \( \mu \in \text{Hom}(H \otimes H, H) \) denote the multiplication and define \( \nu = \mu^{\text{op}} \circ (S \otimes S) \in A \) and \( \rho = S \circ \mu \in A \). Prove that \( \rho * \mu = \mu * \nu = \eta \circ \epsilon \circ \mu \).

(ii) Prove that \( S : H \to H^{\text{op}} \) is an algebra homomorphism.
(iii) Prove dually that $S : H \to H^{\text{cop}}$ is a morphism of co-algebras.
(iv) Show that $H^{\text{op, cop}}$ is also a Hopf algebra with antipode $S$, and that $S : H \to H^{\text{op, cop}}$ is a morphism of Hopf algebras.

Exercise (I). Let $H$ be a Hopf algebra with antipode $S$.
(i) Show that the convolution products on $\text{End}_k(H^{\text{op}})$ and on $\text{End}_k(H^{\text{cop}})$ are each others opposite.
(ii) Show that $S^2 \in \text{End}_k(H^{\text{op}})$ is is a left and right inverse for $S$ with respect to convolution.
(iii) Show that $S^2 = \text{id}_H$ iff $H^{\text{op}}$ (or equivalently $H^{\text{cop}}$) is a Hopf algebra with antipode $S$.
(iv) Suppose that $S$ is invertible. Show that $H^{\text{op}}$ and $H^{\text{cop}}$ are Hopf algebras with antipode $S^{-1}$.

References