Symmetric Spherical and Planar Patterns

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Abstract
In this article we enumerate the discrete symmetry groups of patterns on the sphere and on the Euclidean plane. In our derivation, based on ideas of Coxeter (1969) and Fejes Tóth (1965), we only use elementary results from Euclidean geometry and group theory. We employ an adapted version of Conway’s signature notation. As a by-product we obtain Conway’s ‘Magic Theorems’ for spherical and planar patterns.

1 Spherical Patterns
1.1 Introduction
Many objects in daily life exhibit various forms of symmetry. Balls, bowls, cylinders or cones have infinitely many reflections and rotations as symmetries, at least if there are no special patterns on their surface. But for other symmetric objects the number of symmetries is finite. Take, for instance, a rectangular table, a chair, a cupboards or, in a more mathematical language, a cube, a regular prism or a regular pyramid. Or take balls with characteristic symmetric patterns on their surface, such as a soccer ball, a volleyball, a basketball or a tennis ball.

For all these objects, their symmetries can be viewed as isometries in Euclidean three-space. They form a group, the symmetry group of the object. It is interesting to enumerate all possible finite symmetry groups. This has been done by many authors in the past, and in many different ways. It turns out that there are exactly fourteen types of finite symmetry groups in three-space (see, e.g., Fejes Tóth (1965), p. 59 ff. or Coxeter (1969), p. 270 ff.).

In the literature many names and notations for these groups have been proposed, but one of the most illuminating is the signature notation, proposed some twenty years ago by John H. Conway and used by Conway and his co-authors Heidi Burgiel and Chaim Goodman-Strauss in their beautiful and lavishly illustrated book The Symmetries of Things. The first part of their book is devoted
to the symmetries of spherical and planar patterns. Conway’s signature notation\(^1\) has the advantage of reflecting in a very direct and simple way all symmetry features of a pattern. In this article, we shall use this notation, but in a slightly adapted form.

1.2 Symmetric Objects and Spherical Patterns

Any finite group \(G\) of isometries must have at least one fixed point, a point that is unchanged under all isometries of \(G\). Indeed, take any point \(P\) together with its images under all isometries of \(G\). Together they form a finite point set, the orbit of \(P\). Any isometry of \(G\) maps this orbit onto itself by permuting its points, so its centroid \(O\) must be invariant under all isometries of \(G\). In other words, \(O\) is a fixed point of all isometries in \(G\).

Moreover, since \(G\) consists of isometries, i.e., mappings that preserve distance between points, any sphere with center \(O\) is mapped onto itself by the elements of \(G\). Therefore, enumerating all finite groups of isometries in Euclidean three-space is the same as enumerating all finite groups of isometries on the surface of a sphere. Loosely spoken, any symmetric object in three-space can be associated with a symmetric pattern on a sphere around it by central projection from \(O\). Studying spherical patterns, which are two-dimensional, seems easier than studying three-dimensional objects. Moreover, this approach connects the study of (bounded) symmetric objects in three-space to the study of planar symmetric patterns like wallpaper patterns and strip patterns.

1.3 Examples of Spherical Patterns

In this subsection we give examples of spherical patterns, one for each type, each with its signature. On this moment, these signatures are just names, but later we shall explain how they are formed and how the signature of any symmetric spherical pattern can be found in a very easy way.

The first seven examples are called *parametric patterns* since they depend on a parameter, in this case an integer \(N \geq 1\). Each choice of \(N\) yields a different pattern. In our examples we have taken \(N = 7\), but it is immediately clear how other choices of \(N\) yield similar patterns. The other seven examples are so-called *Platonic patterns* since they may be associated with certain symmetry groups of the Platonic solids tetrahedron, cube, octahedron, dodecahedron and icosahedron.

In each example reflections (if any) are marked by the great circles, drawn in black, in which the reflection planes intersect the sphere. Rotations are indicated by dots in various colors, marking the points where the rotation axis intersects the sphere. Full explanations will be given later. Note that the coloring

\(^1\)also called *orbifold signature notation*. It was developed by Conway using ideas of William Thurston, see Conway et al. (2008), p. 7.
of the patterns is essential: symmetries must leave colors unchanged.

**Parametric patterns**

- \([g(7,7)]\)
- \([s(7,7)]\)
- \([g(7) s]\)
- \([g(7) x]\)

\([g(7,2,2)]\)

\([g(2) s(7)]\)

\([s(7,2,2)]\)

**Platonic patterns**

- \([g(3,3,2)]\)
- \([g(3) s(2)]\)
- \([s(3,3,2)]\)

\([g(4,3,2)]\)

\([s(4,3,2)]\)

\([g(5,3,2)]\)

\([s(5,3,2)]\)
1.4 Finite Spherical Groups of Isometries

It is well-known that any isometry on the sphere is either a rotation around an axis through its center, or a reflection in a plane through its center, or a rotatory reflection (a reflection followed by a rotation around an axis perpendicular to the plane of reflection). Rotations are direct isometries, (rotatory) reflections are opposite isometries. Opposite isometries change orientation while direct isometries leave the orientation unaffected.

Let \( G \) be a finite group of isometries on the sphere. Since the product of two rotations again is a rotation and the product of two (possibly rotatory) reflections also is a rotation, either the group \( G \) consists solely of rotations (including the identity) or the rotations in \( G \) form a normal subgroup \( H \) of index 2.

The axis of a nontrivial rotation intersects the sphere in two diametrically opposed points, its poles. The rotations in \( G \) with a common axis, including the identity, form a cyclic subgroup of \( G \). If \( p \) is the order of this subgroup, the two poles are called \( p \)-gonal poles, or \( p \)-fold centers, or \( p \)-poles for short. The smallest positive angle of a rotation in this subgroup is \( 2\pi/p \).

Two poles are called equivalent in \( G \) if there is an isometry in \( G \) that transforms one into the other. Note that antipodal poles always have the same order, but need not be equivalent. In the examples in subsection 1.3 equivalent poles are colored the same.

1.5 Chiral Patterns

A pattern on the sphere, or a bounded object in three-space, is called chiral if its only symmetries are direct isometries. In that case, its mirror image looks different from the original, just as a left hand and a right hand differ in appearance. (The Greek word "cheir" means hand.) If the pattern or the object permits at least one opposite isometry, it looks exactly the same as its mirror image. Then it is called achiral. Chirality of patterns or objects is easily recognized by using a mirror.

In the present subsection, we shall investigate the chiral patterns on the sphere. The symmetry group \( G \) of such a pattern consists entirely of rotations, including the identity. The following derivation of the enumeration of the finite rotation groups is taken from Coxeter (1969), pp. 274-275.

Let \( n \) be the order of \( G \) and let \( P \) be a \( p \)-gonal pole on the sphere. We claim that the number of poles that are equivalent to \( P \) (including \( P \) itself) equals \( n/p \). To prove this, take a point \( Q \) on the sphere near \( P \). The rotations with center \( P \) transform \( Q \) into a small regular \( p \)-gon with center \( P \). The other isometries in \( G \) transform this \( p \)-gon into \( p \)-gons around the poles that are equivalent to \( P \). In total, the vertices of these \( p \)-gons form a set of exactly \( n \) points on the sphere, divided into congruent regular \( p \)-gons. Thus there are \( n/p \) of these \( p \)-gons, and just as many equivalent \( p \)-poles.
Now let us count the \( n - 1 \) non-trivial rotations in \( G \) in a special way, grouping together the nontrivial rotations in classes of equivalent poles. For any \( p \)-gonal pole there are \( p - 1 \) nontrivial rotations, so for any set of \( n/p \) equivalent \( p \)-poles we have \((p - 1)n/p\) nontrivial rotations. In total this yields

\[
\sum (p - 1)n/p
\]

nontrivial rotations, where the summation is over all sets of equivalent poles, each with its own value for \( p \). But antipodal poles have the same set of rotations, so each nontrivial rotation occurs twice in this summation. Thus we get the equation

\[
2(n - 1) = n \sum \frac{p - 1}{p}
\]

which can also be written as

\[
2 - \frac{2}{n} = \sum \left( 1 - \frac{1}{p} \right)
\]

(1)

If \( n = 1 \) the group \( G \) is trivial: there are no poles and the sum is empty. For \( n \geq 2 \) we have

\[
1 \leq 2 - \frac{2}{n} < 2
\]

It follows from \( p \geq 2 \) that \( 1/2 \leq 1 - 1/p < 1 \) so there can only be 2 or 3 sets of equivalent poles. If there are 2 sets, we have

\[
2 - \frac{2}{n} = 1 - \frac{1}{p_1} + 1 - \frac{1}{p_2}
\]

that is

\[
\frac{n}{p_1} + \frac{n}{p_2} = 2
\]

But both terms on the left are positive integers, so both are equal to 1, and

\[
p_1 = p_2 = n
\]

Each of the two sets of equivalent poles then consists of one \( n \)-gonal pole and \( G \) is the cyclic group with one \( n \)-gonal pole at each end of its single axis. The signature of this group is \([g(n,n)]\). See section 1.3 for an example with \( n = 7 \). On page 10 a full explanation is given of this notation. To facilitate reading, we shall always include signatures in square brackets. The letter "\( g \)" is from the term "gyration point" used by Conway to denote a rotation center not lying in a reflection plane. (The Greek word "gyros" means round.)

If there are 3 sets of equivalent poles, we have

\[
2 - \frac{2}{n} = 1 - \frac{1}{p_1} + 1 - \frac{1}{p_2} + 1 - \frac{1}{p_3}
\]
whence
\[ \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + \frac{2}{n} > 1 \]

It follows that not all \( p_i \) can be 3 or more, so at least one of them, say \( p_3 \), is 2, and we have
\[ \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{2} \]

This can be rewritten as \( 2p_1 + 2p_2 > p_1p_2 \), or, equivalently, as
\[ (p_1 - 2)(p_2 - 2) < 4 \]

Taking \( p_1 \geq p_2 \) we find as solutions \( (p_1, p_2) = (p, 2) \) (for any \( p \geq 2 \)), \( (p_1, p_2) = (3, 3) \), \( (p_1, p_2) = (4, 3) \) and \( (p_1, p_2) = (5, 3) \).

Summarizing, we have the following possibilities for finite rotation groups on the sphere:

<table>
<thead>
<tr>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( n )</th>
<th>signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( p )</td>
<td>( p )</td>
<td>( 1 )</td>
<td>( [g(p,p)] )</td>
</tr>
<tr>
<td>( p )</td>
<td>2</td>
<td>2</td>
<td>( 2p )</td>
<td>( [g(p,2,2)] )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>12</td>
<td>( [g(3,3,2)] )</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>24</td>
<td>( [g(4,3,2)] )</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>60</td>
<td>( [g(5,3,2)] )</td>
</tr>
</tbody>
</table>

Each of the possibilities enumerated above can be realized as the symmetry group of an chiral spherical pattern (see section 1.3), which proves that there are exactly five types of chiral symmetry patterns on the sphere: two parametric patterns, with signatures \( [g(p,p)] \) and \( [g(p,2,2)] \) for any \( p \geq 2 \), and three Platonic patterns, with signatures \( [g(3,3,2)] \), \( [g(4,3,2)] \) and \( [g(5,3,2)] \). The last three are called \textit{Platonic} since their symmetry groups are the rotation groups of the Platonic solids: the tetrahedron \( (g(3,3,2)) \), the cube and the octahedron \( (g(4,3,2)) \) and the dodecahedron and the icosahedron \( (g(5,3,2)) \).

### 1.6 Achiral Patterns

We now suppose that the spherical pattern is achiral, in other words, that its symmetry group \( \mathcal{G} \) contains at least one opposite isometry (reflection or rotary reflection). Let \( n > 1 \) be the order of \( \mathcal{G} \). Then the rotations in \( \mathcal{G} \) form a subgroup \( \mathcal{H} \) of order \( n/2 \). Note that the plane of a reflection intersects the sphere in a great circle. For isometries of the sphere, it is convenient to speak of a reflection in this circle instead of a reflection in the plane of the circle. A reflection circle is sometimes called a \textit{mirror circle} or mirror, for short.

Again, we investigate the nontrivial rotations in \( \mathcal{G} \). There are \( n/2 - 1 \) of them. In \( \mathcal{G} \), there can be two different kinds of rotation centers: they may or may not be on a mirror. In the first case, if the order of rotation of such a center \( Q \) is \( q \), there must be \( q \) mirrors through \( Q \), equally spaced with angles \( \pi/q \).
For any point $T$ near to $Q$ but not on one of these mirrors, the rotations and reflections through $Q$ transform $T$ into a $2q$-gon. The other isometries in $\mathcal{G}$ transform this $2q$-gon into congruent $2q$-gons around rotation centers that are equivalent to $Q$. In total, these $2q$-gons have $n$ vertices, so the set of $q$-poles equivalent to $Q$ consists of $n/(2q)$ elements. For each $q$-pole, there are $q - 1$ nontrivial rotations, so in the set of $q$-poles equivalent to $Q$ we count $(q - 1)n/(2q)$ nontrivial rotations.

As in the former section, for a $p$-pole $P$ not on a mirror the set of $p$-poles equivalent to $P$ accounts for $(p - 1)n/p$ nontrivial rotations. Adding in this way all nontrivial rotations in all sets of $q$-poles on mirrors and $p$-poles not on mirrors, we count every rotation twice, since antipodal poles have the same set of rotations. Thus we get the following equation:

$$2 \left( \frac{n}{2} - 1 \right) = n \sum \frac{q - 1}{2q} + n \sum \frac{p - 1}{p}$$

where the first summation is over all sets of equivalent poles on mirrors and the second summation is over all sets of equivalent poles not on mirrors.

We write the last equation in the form

$$1 - \frac{2}{n} = \sum \left( \frac{1}{2} - \frac{1}{2q} \right) + \sum \left( 1 - \frac{1}{p} \right)$$

If $n = 2$, the left hand side is 0, so both sums on the right must be 0 and there are no nontrivial rotations. The group $\mathcal{G}$ then consists of the identity and either a single reflection or the single rotatory reflection with rotation angle $\pi$, which is the central inversion in the center of the sphere: the isometry that transforms each point in its antipodal point. In the first case, where $\mathcal{G}$ contains a single reflection, its signature is $[s]$, while in the second case, where $\mathcal{G}$ contains the central inversion, its signature is $[x]$.2

Now suppose that $n > 2$. Then

$$\frac{1}{3} \leq 1 - \frac{2}{n} < 1$$

For $p > 1$ we have $1 - 1/p \geq 1/2$, so the second sum contains at most one term.

**Case 1: All rotation centers are on mirrors**

Suppose that all rotation centers are on mirrors. The group $\mathcal{G}$ then is called a *reflection group*. Equation (2) then yields

$$1 - \frac{2}{n} = \sum \left( \frac{1}{2} - \frac{1}{2q} \right)$$

---

2Conway uses a star (*) and a multiplication sign ($\times$), but we thought that it is more convenient to use the letters “s” and “x”, respectively. See also page 25.
But if we put \( n' = n/2 \) (note that \( n \) is even!) this can be written as

\[
2 - \frac{2}{n'} = \sum \left( 1 - \frac{1}{q} \right)
\]

which is exactly the same as equation (1), the equation that describes all possible finite rotation groups. In this case, the equation describes the possibilities for the normal subgroup \( \mathcal{H} \) of all rotations in \( \mathcal{G} \).

In the previous section we have proved that there are exactly five types of rotation groups, see the table on page 6. The examples in section 1.3 show that it is possible in each case to connect the rotation centers by mirror circles in such a way that through each \( q \)-pole there are exactly \( q \) mirror circles. In this way we obtain the five types of reflection groups. The following table shows their properties and their signatures.

<table>
<thead>
<tr>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q_3 )</th>
<th>( n )</th>
<th>signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>( q )</td>
<td>( \infty )</td>
<td>( 2q )</td>
<td>([s(q,q)])</td>
</tr>
<tr>
<td>( q )</td>
<td>2</td>
<td>2</td>
<td>( 4q )</td>
<td>([s(q,2,2)])</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>24</td>
<td>([s(3,3,2)])</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>48</td>
<td>([s(4,3,2)])</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>120</td>
<td>([s(5,3,2)])</td>
</tr>
</tbody>
</table>

Case 2: Not all rotation centers are on mirrors

Next we investigate the case that the second sum in equation 2 is not empty, in other words, that there is at least one \( p \)-pole not on a mirror \((p > 1)\). The symmetry group then is called a mixed group.

As we have seen, the second summation in equation (2) contains only one term, so all \( p \)-poles are equivalent and equation (2) can be written as

\[
\frac{1}{p} - \frac{2}{n} = \sum \left( \frac{1}{2} - \frac{1}{2q} \right)
\]

If \( n = 2p \) both sides are zero. Since \( n/p \) is the number of equivalent \( p \)-poles, there are \( n/p = 2 \) such poles, necessarily antipodal, counting for \( p \) rotations in \( \mathcal{G} \) (including the identity). If there are no mirrors then \( \mathcal{G} \) must also contain \( p \) rotatory reflections. These rotatory reflections must interchange the two antipodal \( p \)-gonal poles. This is the group with signature \([g(p) \times]\) \(g(p) \times\).

If \( n = 2p \) and there is a mirror, it must interchange the two poles since there are no mirrors through these poles. Then \( \mathcal{G} \) consists of \( p \) rotations (including the identity), one reflection and \( p - 1 \) rotatory reflections. This is the group with signature \([g(p) s]\) \(g(p) s\).

Now suppose that \( n > 2p \). Then there is at least one \( q \)-pole on a mirror circle with \( q \geq 2 \). Then \( 1/2 - 1/(2q) \geq 1/4 \) and since \( p \geq 2 \) it follows that

\[
\sum \frac{1}{4} \leq \sum \left( \frac{1}{2} - \frac{1}{2q} \right) = \frac{1}{p} - \frac{2}{n} < \frac{1}{p} \leq \frac{1}{2}
\]
Therefore there can only be one set of equivalent $q$-poles and, moreover, $p$ can only be 2 or 3.

If $p = 2$ equation (2) yields

\[
\frac{1}{2} \frac{2}{n} = \frac{1}{2} \frac{1}{2q}
\]

so $n = 4q$. There are $n/(2q) = 2$ antipodal $q$-poles lying on $q$ equally spaced mirror circles, bounding $2q$ ‘lunes’ (2-gons) with angles $\pi/q$. The centers of these lunes are 2-poles. The signature is $[g(2) s(q)]$.

If $p = 3$ then, since $1/2 - 1/(2q) < 1/p = 1/3$, only $q = 2$ is possible. Then equation (2) yields

\[
1 - \frac{2}{n} = \frac{1}{4} + \frac{2}{3}
\]

so $n = 24$. It follows that there are $n/p = 8$ poles $P$ of order 3 not on mirrors, and $n/(2q) = 6$ poles $Q$ of order 2 on mirrors. These mirrors intersect each other at right angles in the six points $Q$. This implies that there must be exactly three mirror circles, dividing the sphere into eight octants, which are spherical triangles with three right angles at the points $Q$. The eight 3-poles $P$ are the centers of these triangles. This pattern has signature $[g(3) s(2)]$.

Note that the groups of order 2 with signatures $[s]$ and $[x]$ may be seen as special cases of the groups with signatures $[g(p) s]$ and $[g(p) x]$ for $p = 1$, since a 1-gonal rotation (rotation angle $2\pi/1 = 2\pi$) is the same as the identity.

We summarize our results in the following table:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$n$</th>
<th>signature</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>2$p$</td>
<td>$2p$</td>
<td>$[g(p) x]$</td>
</tr>
<tr>
<td>$p$</td>
<td>1</td>
<td>$2p$</td>
<td>$[g(p) s]$</td>
</tr>
<tr>
<td>2</td>
<td>$q$</td>
<td>$4q$</td>
<td>$[g(2) s(q)]$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>24</td>
<td>$[g(3) s(2)]$</td>
</tr>
</tbody>
</table>

This completes our enumeration of the finite groups of isometries on the sphere. We have proved that there are exactly 14 types of finite symmetry groups on the sphere. Their signatures are given in the following table, where $p \geq 1$ and $q \geq 1$ are arbitrary parameters. Note that $[g(1,1)]$ is the trivial group consisting of the identity alone, that the signatures $[g(1)s]$ and $[s(1,1)]$ both can be simplified to $[s]$, that $[g(1)x]$ can be simplified to $[x]$ and that $[s(1,2,2)]$ is the same group as $[s(2,2)]$. 
1.7 Explaining Examples and Signatures

For each type of pattern we have given an example in subsection 1.3. For the parametric patterns we have taken \( p = 7 \) and \( q = 7 \), respectively. In each pattern the mirror circles are marked in black, while the rotation centers are marked with colored dots. Equivalent centers are colored the same.

In this way, our adaptation of Conway’s signature notation becomes almost self-evident. All equivalence classes of gyration points are enumerated between brackets after the symbol \( “g” \), while all equivalence classes of rotation centers on mirrors are enumerated after the symbol \( “s” \). So rotation centers of each color are entered only once in the signature. The order in which the numbers between brackets are given is immaterial; as a rule, we give them in non-ascending order. Also, \( g(\ldots) \) is put before \( s(\ldots) \). Finally, the occurrence of a rotatory reflection that is not the composition of a rotation and a reflection in the group, is marked by the symbol \( “x” \).

The reader is invited to check all mirror circles, gyration points and rotation centers on mirrors in the given examples, thus verifying the correctness of the given signatures. Also check that, indeed, equivalent poles are colored the same (but not all poles are visible in the drawings). Note that the coloring of the patterns is important: a symmetry of a pattern must leave all colors unchanged.

In fact, it is very easy to find the signature of any spherical pattern with a finite symmetry group. First find out whether it is chiral or achiral, then mark all mirror circles and all poles, giving equivalent poles the same color. The signature then can be read off by inspection, the only subtle point being the possibility that the pattern doesn’t possess mirror circles but nevertheless is achiral. Then there must be at least one rotatory reflection, and the signature must be \( [g(p) x] \) for some \( p \geq 1 \).

Note that equations (1) and (2) are satisfied by all chiral and achiral spherical patterns, respectively. The order \( n \) of the corresponding finite symmetry group follows from these equations. Also, recall that the number of equivalent \( p \)-gonal gyration points equals \( n/p \) while the number of equivalent \( q \)-gonal
rotation centers on mirrors equals \( n/(2q) \).

Equations (1) and (2) enabled us to enumerate all 14 types of symmetric patterns in a rather simple manner and to prove that there are no other types. As a matter of fact, equations (1) and (2) are essentially the same as Conway’s ‘Magic Theorem’ for spherical patterns (Conway et al. (2008), p. 53). However, our derivation (for chiral patterns based on Coxeter (1969), pp. 274-275) is more elementary, not making use of any advanced topological results.

2 Planar Rosette Patterns and Strip Patterns

2.1 Rosette Patterns

Now we turn to patterns in the Euclidean plane. First we consider patterns with a finite number of symmetries. For reasons that will become clear in a moment, these are called rosette patterns.

A rosette pattern may have various kinds of symmetry, but there must be at least one fixed point, a point that is unchanged under all elements of its group \( G \) of symmetries. Indeed, since the orbit of any point \( P \) is a finite point set, its centroid \( O \) is a fixed point of all symmetries in \( G \).

If there are two distinct fixed points \( O_1 \) and \( O_2 \), the line \( \ell \) through \( O_1 \) and \( O_2 \) must consist entirely of fixed points, since any symmetry is an isometry. If there is also a fixed point \( O_3 \) not on \( \ell \), any point in the plane is a fixed point under all elements of \( G \), so the group only consists of the identity, and the pattern has no symmetry at all. If only the points of a line \( \ell \) are fixed points of all symmetries, \( G \) has only two elements: the identity and the reflection in \( \ell \). In common language, the pattern has the reflection in \( \ell \) as its only symmetry.

In all other cases, there is exactly one point \( O \) that is fixed under all elements of \( G \). The symmetries then can only be rotations with center \( O \) or reflections in lines through \( O \).

If the pattern is chiral, its only symmetries are rotations. The group \( G \) then is cyclic. If its order is \( p \), then the smallest positive rotation angle is \( 2\pi/p \) and all orbits are regular \( p \)-gons with center \( O \). We denote its signature by \([g(p)]\).

If the pattern is achiral, there must be a finite number, say \( q \), of mirror lines through \( O \), equally spaced with angles \( \pi/q \) between adjacent lines. The group then is the so-called dihedral group of order \( 2q \), consisting of \( q \) reflections and \( q \) rotations (including the identity). We denote its signature by \([s(q)]\).

On page 12 two examples of rosette patterns with their signatures are given. Note that any rosette pattern may be pasted on a big sphere, say at its north pole, thus yielding a parametric spherical pattern with signature \([g(p, p)]\) for the cyclic group or \([s(q, q)]\) for the dihedral group. In this way any rosette pattern may be identified with a spherical pattern, having the ‘same’ group of
symmetries, which in the spherical case are spherical rotations and reflections in circles through the antipodal poles and in the planar case rotations around its center and line reflections in lines through its center.

\[
\text{[g(11)]} \quad \text{[s(11)]}
\]

### 2.2 Strip Patterns

Strip patterns (also called frieze patterns) are planar patterns in which a motif is repeated in a row, say, a horizontal row. Although a strip pattern in real life is always finite, we shall imagine that it is continued infinitely in both directions, so that its symmetries include horizontal translations. If \( T \) is the translation taking each motif to its neighbour to the right, any other translation in the symmetry group of the pattern can be written as \( T^n \) for some integer \( n \). But there may also be other symmetries in the pattern.

It is well-known that there are exactly seven types of symmetry groups for strip patterns, and in fact we already know them, since they can be seen as limiting cases of the seven types of symmetry groups for \textit{parametric spherical patterns}. As Conway observed: take a finite segment of a strip pattern consisting of \( N \) copies of the motif, and wrap it around the equator of a sphere of a suitably chosen radius. Then a parametric spherical pattern results with one of the seven signatures \([g(N,N)], [s(N,N)], [g(N) s], [g(N) x], [g(N,2,2)], [g(2) s(N)]\) and \([s(N,2,2)]\). Conversely, any parametric spherical pattern can be unwrapped into the plane, yielding an \( N \)-segment of a strip pattern.

Below, parametric spherical patterns with \( N = 7 \) are drawn, restricting the sphere to a disk around the equator.
Unwrapping the patterns into the plane leads to segments of length 7 of the corresponding infinite strip patterns. It is natural, letting \( N \to \infty \), to denote the signatures of these strip patterns by \([g(\infty,\infty)]\), \([s(\infty,\infty)]\), \([g(\infty) s]\), \([g(\infty) x]\), \([g(\infty,2,2)]\), \([g(2) s(\infty)]\) and \([s(\infty,2,2)]\), respectively.

Since the symmetry group of any strip pattern is infinite, equations (1) and (2) on pages 5 and 7, by letting \( n \to \infty \), yield

\[
2 = \sum \left( 1 - \frac{1}{p} \right)
\]  
\( (3) \)

for chiral strip patterns and

\[
1 = \sum \left( \frac{1}{2} - \frac{1}{2q} \right) + \sum \left( 1 - \frac{1}{p} \right)
\]  
\( (4) \)

for achiral strip patterns, where the \( p \)-summations in (3) and (4) are over equivalence classes of gyration points, while the \( q \)-summation in (4) is over equivalence classes of rotation centers on mirror lines.
Note that the ‘north pole’ and the ‘south pole’ have gone to infinity in the vertical direction, and that the rotations around these poles now have become horizontal translations. These translations may be seen as ‘rotations’ with ‘rotation angle’ \(2\pi/\infty = 0\) and ‘center’ at plus or minus infinity in the vertical direction.

For these centers ‘at infinity’ we must take \(p = \infty\) or \(q = \infty\), respectively, so \(1/p = 1/(2q) = 0\), contributing a term 1 or 1/2 in the above summations. In this way, equations (3) and (4) are easily verified for all strip patterns.

Rotatory reflections on the sphere become glide reflections of strip patterns in the plane, a glide reflection being the combination of a reflection in a mirror line (its axis) and a translation in the direction of the axis. Note that applying a glide reflection twice results in a translation along the axis.

See pattern \([g(\infty)x]\) above for an example with a horizontal glide axis that is not a mirror line of the pattern. In our example, the glide axis is indicated by a dashed line with alternating half arrows.

Pattern \([g(\infty)s]\) also possesses horizontal glide reflections, but these are composed of a horizontal reflection and a translation already present in the group, so the glide reflections are not mentioned in the signature. Conway’s signature notation never contains redundant information. This is also the reason why equivalent poles are only entered once between the brackets.

A horizontal glide reflection also results when combining a vertical reflection and a 2-gonal rotation (a half-turn) with center not on the mirror line. Then the axis of the glide reflection is the line through the 2-center perpendicular to the mirror line. Conversely, combining a glide reflection with a half-turn with center on the glide axis results in an ordinary reflection with mirror axis perpendicular to the glide axis.

Patterns \([g(2)s(\infty)]\) and \([s(\infty,2,2)]\) yield examples with vertical mirrors and 2-gonal rotation centers on a horizontal line. Since the vertical reflections and the 2-gonal rotations are already present in the group, the glide reflections are not mentioned in the signature.

This completes our treatment of strip patterns.

3 Wallpaper Patterns

3.1 Introduction

While rosette patterns admit no translations and strip patterns admit translations in one direction only, in the so-called wallpaper patterns translations in more than one direction are present. We shall suppose that the symmetry group \(\mathcal{G}\) of such a pattern is discrete, meaning that for any point \(P\) there is a circle around \(P\) which contains no other points of the orbit of \(P\). This implies
that there is a bounded motif that is repeated infinitely all over the plane by the
translations in \( \mathcal{G} \). Note that, although we didn’t mention it explicitly, we tacitly
assumed that also for spherical patterns, rosette patterns and strip patterns the
symmetry groups are discrete. For spherical patterns and rosette patterns this
was guaranteed by demanding that the symmetry groups be finite, thus ex-
cluding, e.g., cases in which the pattern only consists of one (unmarked) circle,
which has infinitely many rotations around its center.

The discreteness of the symmetry group \( \mathcal{G} \) of a wallpaper pattern implies that
there exists a translation \( T_1 \) with minimum positive translation distance \( d_1 \) and
a second translation \( T_2 \), independent of \( T_1 \), with a minimum positive transla-
tion distance \( d_2 \geq d_1 \). By reversing if necessary the direction of \( T_2 \), we may
suppose that the angle \( \alpha \) between \( T_1 \) and \( T_2 \) is non-obtuse. Furthermore, if
\( \alpha < \pi/3 \) would hold, the translation distance of \( T_1 - T_2 \) would be less than \( d_2 \)
by the rule of sines, contradicting the choice of \( T_2 \). So \( \pi/3 \leq \alpha \leq \pi/2 \).

It is easy to prove that any translation \( T \) in \( \mathcal{G} \) can be written as \( T = T_1^n T_2^m \)
for certain integers \( n \) and \( m \), in other words, that the normal subgroup \( \mathcal{T} \) of
all translations in \( \mathcal{G} \) is generated by \( T_1 \) and \( T_2 \). For any point \( P \), the orbit of \( P \)
under \( T \) is a lattice spanned by vectors representing \( T_1 \) and \( T_2 \).

But wallpaper patterns may also have symmetries other than translations. In
fact, it is well-known, and we shall prove it again in this article, that there are 17
distinct types of wallpaper patterns. Each type has its own Conway signature,
which in a very concise way describes the symmetry features of the pattern.
As was the case with spherical patterns, rosette patterns and strip patterns,
the signature of wallpaper patterns is easily read off from the pattern as soon
as one has identified its chirality and, if present, its mirror lines, its rotation
centers and its glide reflection axes that are not mirror lines.

### 3.2 Examples of the Seventeen Types of Wallpaper Patterns

In this subsection for each of the 17 types of wallpaper patterns we shall present
a simple example together with its signature. In each instance, mirror lines
will be marked in black and rotation centers by colored dots, where equivalent
centers get the same color. Glide reflection axes that are not mirror lines will
be marked with dashed lines.

The reader will have no difficulty interpreting and verifying the given signa-
tures. Following Conway, we use the signature [O] for the pattern in which
translations are the only symmetries. All other patterns have signatures which
should look familiar by now.

In pattern [s s] there are two non-equivalent types of mirror lines, both vertical.
In pattern [s x] vertical mirror lines and glide reflection axes alternate at equal
distances. All mirror lines are equivalent, as are all glide reflection axes. The
pattern [x x] has two non-equivalent types of glide reflection axes, both vertical
and both with the same vertical translation distance.
Special mention deserves pattern \([g(2,2) \times]\), in which two non-equivalent types of two-fold gyration points occur, colored blue and green, and two types of glide reflection axes that are no mirror lines. Note that a horizontal glide reflection, combined with a half turn around a center not on its axis, yields a glide reflection with a vertical axis. This is why in the signature \([g(2,2) \times]\) only one symbol “\(x\)” occurs.

In the next subsections we shall give more details on the various patterns and we also shall prove that there are no more than 17 types.

*Seventeen wallpaper patterns with their signatures:*

- \([O]\)
- \([s \ s]\)
- \([s \times]\)
- \([\times \times]\)
- \([g(2,2) \ s]\)
- \([g(2,2) \times]\)
- \([g(2,2,2,2)]\)
- \([g(2) \ s(2,2)]\)
- \([s(2,2,2,2)]\)
3.3 Isometries in the Plane

In this and the following sections we use ideas from Fejes Tóth (1965, p. 29 ff.). It is well-known that every isometry in the Euclidean plane is either a translation, or a rotation, or a reflection in a line, or a glide reflection. Translations and rotations are direct isometries, line reflections and glide reflections are opposite isometries. Any group $\mathcal{G}$ of isometries either consists of direct isometries only, or its direct isometries form a normal subgroup $\mathcal{H}$ of index 2 in $\mathcal{G}$.

A pattern in the plane is called chiral if its group of symmetries consists of direct isometries only, otherwise it is called achiral. Chirality of patterns can easily be determined by using a mirror.

A rotation has a center and a rotation angle, which we shall always measure anti-clockwise in radians modulo $2\pi$. Sometimes it is convenient to consider
translations as rotations with zero angle and center at infinity in a direction perpendicular to the translation direction.

In a discrete group \( G \) the rotations with a common center form a cyclic subgroup of finite order \( p \), generated by a rotation with angle \( 2\pi/p \). Its center is called a \( p \)-center. Likewise, the translations in \( G \) in a given direction form an infinite cyclic group generated by a translation \( T \) with distance \( d > 0 \).

If \( R_1 \) and \( R_2 \) are reflections in mirror lines \( m_1 \) and \( m_2 \), respectively, then \( R_2R_1 \) (\( R_1 \) followed by \( R_2 \)) is a translation with distance \( 2d \) if \( m_1 \) and \( m_2 \) are parallel with distance \( d \) and a rotation with center \( A \) and rotation angle \( 2\alpha \) if \( m_1 \) and \( m_2 \) intersect at \( A \) with intersection angle \( \alpha \).

If \( R_1 \) and \( R_2 \) are rotations with centers \( A_1 \) and \( A_2 \) and rotation angles \( \alpha_1 \) and \( \alpha_2 \), respectively, then \( R_2R_1 \) is a rotation with angle \( \alpha_1 + \alpha_2 \), or a translation if \( \alpha_1 + \alpha_2 = 0 \mod 2\pi \). If both \( R_1 \) and \( R_2 \) are half-turns, so that both rotation angles are equal to \( \pi \), then \( R_2R_1 \) is a translation with distance \( 2d(A_1, A_2) \) in the direction from \( A_1 \) to \( A_2 \).

If \( ABC \) is a clockwise oriented triangle with angles \( \alpha, \beta, \gamma \), then a rotation with center \( A \) and angle \( 2\alpha \) followed by a rotation with center \( B \) and angle \( 2\beta \) brings \( C \) back to itself, so it is the rotation with center \( C \) (and angle \( 2\alpha + 2\beta = 2\pi - 2\gamma \)). In particular, if \( A \) and \( B \) are rotation centers of a pattern, then the same must hold for \( C \).

### 3.4 Chiral Patterns

Let be given a chiral wallpaper pattern with a discrete symmetry group \( G \). Then all isometries in \( G \) are translations or rotations. We have already seen that the normal subgroup \( T \) of all translations in \( G \) may be generated by two independent translations \( T_1 \) and \( T_2 \) with minimum translation distances \( d_1 \leq d_2 \) and angle \( \pi/3 \leq \alpha \leq \pi/2 \). For any point \( P \) the orbit of \( P \) under the translation group is a lattice. If there are no rotations, \( G \) coincides with its translation group and its signature is \([O]\).

Now suppose that all rotations in \( G \) are half-turns, in other words, that there are only 2-centers. Let \( A \) be a 2-center and let \( ABCD \) be the parallelogram with \( B = T_1A \), \( C = T_2T_1A \) and \( D = T_2A \). Then the other vertices of the parallelogram, the midpoints of its sides and the center of the parallelogram must also be 2-centers, nine centers in total.
But centers that differ by a translation in $\mathcal{T}$ are equivalent, so there are only four non-equivalent 2-centers in the parallelogram, which can be taken at $A$ and at the midpoints of $AB$, $AC$ and $AD$.

We claim that all half-turns in $G$ are equivalent to one of these four. In fact, for any other half-turn there is an equivalent one with center $E$ inside or on the boundary of $ABCD$. But if $E$ is not one of the nine 2-centers mentioned above, a combination with the half-turn with center $A$ would result in a translation not generated by $T_1$ and $T_2$, which is impossible. Therefore there are exactly four non-equivalent 2-centers, and the signature of this group thus is $[g(2,2,2,2)]$ 

If not all rotation centers are 2-centers, we may take two centers $A$ and $B$ of order $p$ and $q$ both greater than 2 and with minimum distance $d$. Let $C$ be such that triangle $ABC$ has angles $\pi/p$ and $\pi/q$ at $A$ and $B$, respectively. Then $C$ is the center of a rotation in $G$ with angle $2\pi/p + 2\pi/q = 2\pi - 2\pi/r$ for some positive rational number $r$, so

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad (5)$$

If $p = q = 3$ then also $r = 3$ must hold, so $A$, $B$ and $C$ are centers of 3-fold rotations. Applying these rotations repeatedly we obtain an equilateral lattice of 3-centers. The composition of rotations with angles $\pm 2\pi/3$ around $A$ and $\mp 2\pi/3$ around $B$ yields two translation $T_1$ and $T_2$, generating the translation group $\mathcal{T}$.

By the choice of $A$ and $B$ at minimum distance, no other centers of order greater than 2 occur inside a circle with radius $d$ and center $A$ covering a regular hexagon. The translations of $\mathcal{T}$ produce a tiling of the plane by these hexagons, so no other centers of order greater than 2 can occur anywhere in the plane.

But then also 2-centers are impossible, since they would imply the existence of rotations with angle $2\pi/2 - 2\pi/3 = 2\pi/6$, contradicting the former observation. Since $A$, $B$ and $C$ are non-equivalent 3-centers, it follows that the group we just have described must have signature $[g(3,3,3)]$.

If, in the same notations as above, $p$ and $q$ are not both equal to 3 but still greater than 2, then $C$ (which is a rotation center of the pattern) is closer to $A$ than $B$, so $C$ must be a 2-center. It follows that $r = 2$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \quad (6)$$

which can be written as $(p - 2)(q - 2) = 4$. Its only solutions with $p \geq q \geq 3$ are $(p,q) = (4,4)$ and $(p,q) = (6,3)$. This implies in particular that in a
wallpaper pattern only 2-centers, 3-centers, 4-centers and 6-centers can occur. This is the famous crystallographic restriction.

If \( p = q = 4 \), then \( A \) and \( B \) must be adjacent 4-centers, and, again, \( C \) is a 2-center. Triangle \( ABC \) then is an isosceles right-angled triangle, which can be seen as part of a square with \( A \) as one vertex, \( B \) as its center and \( C \) as the midpoint of a side adjacent to \( A \). Combining rotations around \( A \) and \( B \) with angles \( \pm 2\pi/4 \) and \( \mp 2\pi/4 \) yields translations along the sides of the square. These translations generate \( T \). It follows that \( A \) and \( B \) are non-equivalent 4-centers, so the signature of this group is \([g(4,4,2)]\).

In a similar way, \( p = 6, q = 3 \) leads to a pattern of 6-centers, 3-centers and 2-centers forming a lattice of equilateral triangles, with 6-centers at their vertices, 3-centers at their centers and 2-centers at the midpoints of their sides. The signature of this pattern then is \([g(6,3,2)]\).

Note that equation (5) can be written as \( 2 = \left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{r}\right) \) or, more concisely, as

\[
2 = \sum \left(1 - \frac{1}{p}\right)
\]

which has the same form as equation (3) on page 13. Also pattern \([g(2,2,2,2)]\] satisfies this equation (with four 2-centers of ‘weight’ \(\frac{1}{2} = 1 - \frac{1}{2}\)). Even patterns with signature \([O]\) satisfy this equation if we consider the two independent translations \(T_1\) and \(T_2\) as ‘rotations’ with rotation angle 0 and ‘order’ \( p = q = \infty \). Each then contributes a ‘weight’ of \(1 - 1/\infty = 1\) to the right-hand side.

### 3.5 Achiral Patterns

Next we turn to achiral patterns. Then the symmetry group \( G \) must contain at least one line reflection or glide reflection. The direct isometries in \( G \) form a normal subgroup \( H \) of index 2. The translations form a subgroup \( T \) of \( H \) which is also normal in \( G \). We consider the following cases.
Case 1: No Rotations

First, suppose that there are no rotations in \( \mathcal{G} \), so \( \mathcal{H} = \mathcal{T} \). Then mirror lines and glide reflection axes can occur in only one direction, since intersecting mirrors or glide reflection axes yield nontrivial rotations.

If there is a mirror line \( m \), then there are infinitely many equally spaced mirror lines parallel to \( m \). Let \( m' \) be a mirror line at minimum distance from \( m \). If \( m \) and \( m' \) are not equivalent, there are just two types of mirror lines, either equivalent to \( m \) or to \( m' \). The signature of the group then is \([s\ s]\). If \( m \) and \( m' \) are equivalent, they must be related by a glide reflection with the midparallel \( n \) of \( m \) and \( m' \) as its axis. Then mirror lines and glide axes alternate at equal distances, and the group has signature \([s\ x]\).

Finally, if there are no mirror lines, all glide axes must be parallel and equidistant. If \( n \) and \( n' \) are adjacent glide axes, they cannot be equivalent, so the signature is \([x\ x]\).

Case 2: All Rotation Centers are on Mirrors

If in an achiral pattern rotations occur, there may be rotation centers on mirror lines and rotation centers not on mirror lines (the so-called gyration points). We first consider the case that all rotation centers are on mirror lines. The normal subgroup \( \mathcal{H} \) of index 2 consisting of all direct isometries in \( \mathcal{G} \) then is one of the four types with signatures \([g(2,2,2,2)]\), \([g(3,3,3)]\), \([g(4,4,2)]\) and \([g(6,3,2)]\), and the examples in subsection 3.2 show that in the last three cases it is possible to connect all rotation points by mirror lines such that through every \( q \)-center exactly \( q \) mirror lines pass with angles \( \pi/q \). In this way we obtain the reflection groups with signatures \([s(3,3,3)]\), \([s(4,4,2)]\) and \([s(6,3,2)]\).

For patterns with signature \([g(2,2,2,2)]\) this is only possible if the lattice of 2-centers is rectangular. The mirror lines then form a rectangular lattice with a 2-center at each lattice point. The signature of this reflection group is \([s(2,2,2,2)]\).

Note that the groups with signatures \([s(2,2,2,2)]\), \([s(3,3,3)]\), \([s(4,4,2)]\) and \([s(6,3,2)]\) all are generated by their reflections. Also note that equation (7) now yields

\[
2 = \sum \left( 1 - \frac{1}{q} \right) \quad \text{or, after division by 2}
\]

\[
1 = \sum \left( \frac{1}{2} - \frac{1}{2q} \right)
\]

which is similar to equation (4) on page 13 (there are no gyration points).

Case 3: Gyration Points and No Mirror Lines

Next we consider the case that there are gyration points but no mirror lines. Again, the normal subgroup \( \mathcal{H} \) of all direct isometries in \( \mathcal{G} \) has as its signature
[g(2,2,2)], [g(3,3,3)], [g(4,4,2)] or [g(6,3,2)]. Then there must be a glide reflection $G$ in $G$. Like all isometries in $G$, the glide reflection $G$ must map $p$-centers on $p$-centers. But if it maps a $p$-center $A$ on a $p$-center $A'$ that is equivalent to $A$ in $H$, then there is a direct isometry $D$ mapping $A'$ back to $A$. Therefore the opposite isometry $DG$ has a fixed point $A$, so it must be a reflection in a line through $A$, contradicting the assumption that the pattern has no mirror lines.

But if $G$ cannot map gyration points on equivalent gyration points (equivalent in $H$), the cases $[g(4,4,2)]$ or $[g(6,3,2)]$ for $H$ are excluded since, e.g., 2-centers must be mapped on (necessarily equivalent) 2-centers. Also $[g(3,3,3)]$ is impossible, for suppose that a 3-center $A$ is mapped to a non-equivalent 3-center $B$, then, since $G^2$ is a translation, $B = GA$ must be mapped to a 3-center $A' = G^2A$ equivalent to $A$. But then any 3-center $C$ not equivalent to $A$ or $B$ is mapped to a 3-center $C'$ equivalent to $C$, contradiction.

However, if the signature of $H$ is $[g(2,2,2,2)]$ it may happen that there are glide axes but no mirror lines, provided the lattice of 2-centers is rectangular. If $A$, $B$, $C$ and $D$ are 2-centers forming a rectangle $ABCD$ of minimum side lengths, then there is a glide reflection $G$ with $GA = C$ and $GB = D$. In fact, there are two such glide reflections with perpendicular axes parallel to the sides of the rectangle, both passing through its center. There are two non-equivalent types of 2-centers and the pattern has signature $[g(2,2) x]$.

**Case 4: Gyration Points and Mirror Lines**

Finally we consider achiral patterns with gyration points and mirror lines. It may happen that all mirror lines are parallel. We take them vertical. Gyration points then can only be of order 2 and they must be situated halfway between two adjacent mirror lines. Let $A$ be such a gyration point. Since there must be vertical translations in $G$, there is a vertical translation $T$ with minimum translation distance $2d$. Then a point $B$ above $A$ at distance $d$ is also a gyration point. All other gyration points are equivalent to either $A$ or $B$ and the signature of this pattern is $[g(2,2) s]$.

If there are mirror lines in more than one direction, it may happen that all rotation centers on mirrors are of order 2. Then all mirror lines form a rectangular lattice. The gyration points then must occur at the centers of the rectangles in this lattice and they are all equivalent by reflections in mirrors. If the gyration points are 2-centers, then there are two non-equivalent types of 2-centers on mirrors and the signature is $[g(2) s(2,2)]$. If the gyration points are 4-centers, the lattice of mirror lines must be a square lattice and the signature is $[g(4) s(2)]$.

Finally, if there is are $q$-centers of order $q \geq 3$ on mirror lines then the subgroup of all isometries in $G$ that is generated by all reflections must be $[s(3,3,3), [s(4,4,2)] or [s(6,3,2)]$. But in the last two cases the ‘cells’ are rectangular triangles, in which gyration points are impossible. Only if the mirror lines form a lattice of equilateral triangles, gyration points (of order 3) are possible. They must occur at the centers of the triangles and the signature is $[g(3) s(3)]$. 

$[g(2,2) x]$
This completes our enumeration of all types of wallpaper patterns. In view of Conway’s ‘magic theorems’, to be treated later, we note that one easily verifies that all achiral wallpaper patterns in which rotation centers occur, satisfy the following equation:

\[ 1 = \sum \left( \frac{1}{2} - \frac{1}{2q} \right) + \sum \left( 1 - \frac{1}{p} \right) \tag{9} \]

which is the same as equation (4) for achiral strip patterns. Again, the \( p \)-summation is over all equivalence classes of gyration points while the \( q \)-summation is over all classes of centers on mirror lines.

The following table summarizes our results on wallpaper patterns. The symmetry groups of chiral patterns are called rotation groups (bearing in mind that translations can be viewed as rotations with angle 0 and center at infinity). Groups of achiral patterns in which all rotations centers (if any) are on mirror lines, are called reflection groups and groups of achiral patterns with gyration points mixed groups.

<table>
<thead>
<tr>
<th>Rotation groups (chiral patterns)</th>
<th>Mixed groups (achiral patterns)</th>
<th>Reflection groups (achiral patterns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[O]</td>
<td>[g(2,2) s]</td>
<td>[g(2,2) x]</td>
</tr>
<tr>
<td>[g(2,2,2,2)]</td>
<td>[g(2,2,2,2)]</td>
<td>[s(2,2,2,2)]</td>
</tr>
<tr>
<td>[g(3,3,3)]</td>
<td>[g(3,3,3)]</td>
<td>[s(3,3,3)]</td>
</tr>
<tr>
<td>[g(4,4,2)]</td>
<td>[g(4,4,2)]</td>
<td>[s(4,4,2)]</td>
</tr>
<tr>
<td>[g(6,3,2)]</td>
<td>[g(6,3,2)]</td>
<td>[s(6,3,2)]</td>
</tr>
</tbody>
</table>

4 Conway’s ‘Magic Theorems’

One of the most intriguing points in the treatment of symmetry patterns in Conway et al. (2008) is his use of ‘magic theorems’ to facilitate the enumeration of symmetric patterns. These theorems are used early in the book, but the proofs are postponed many times, which is caused by the fact that they make use of rather advanced topological tools like the concept of ‘orbifold’, Euler characteristic and the classification of surfaces. In our treatment of symmetric spherical and planar patterns, however, these ‘magic theorems’ appear as by-products that are obtained in a more elementary way, as we shall show presently. It must, however, be stressed that Conway’s proof of his ‘magic theorems’ provides deeper insight and applications in many other symmetry questions, as is amply illustrated in Parts II and III of his book (Conway et al. (2008)).

First, for any symmetric feature appearing in the signature of a pattern, Conway defines its ‘cost’ in the following way. Instead of ‘cost’, we rather prefer to speak of ‘weight’.
• For an equivalence class of \( p \)-fold gyration points, its weight is \( 1 - \frac{1}{p} \).

• For an equivalence class of \( q \)-fold rotation centers on mirrors, its weight is \( \frac{1}{2} - \frac{1}{2q} \).

• For a symbol "s" (with or without brackets), its weight is 1.

• For a symbol "x", its weight is 1.

• For the symbol "g", its weight is 0.

• For the symbol “O”, its weight is 2.

Note that for \( \infty \)-fold gyration points or \( \infty \)-fold rotation centers on mirrors (as occur in strip patterns) the weight is 1 or \( \frac{1}{2} \), respectively.

The magic theorem for spherical patterns: The total weight of the signature of a pattern with a symmetry group of order \( n \) equals \( 2 - \frac{2}{n} \).

Proof: For chiral patterns, this is equation (1) on page 5. For achiral patterns this is equation (2) on page 7, adding 1 to both sides to account for the symbol "s" or the symbol "x".

The magic theorem for strip and wallpaper patterns: The total weight of the signature of a strip pattern or a wallpaper pattern equals 2.

Proof: For chiral strip patterns, this is equation (3) on page 13. For achiral strip patterns this is equation (4) on page 13, adding 1 to both sides to account for the symbol "s" or the symbol "x". For chiral wallpaper patterns, see equation (7) on page 20 and the remarks following it. For achiral wallpaper patterns see equations (8) on page 21 and (9) on page 23, where one should add 1 to both sides to account for the symbols "s" or "x". The remaining cases [s s], [s x] and [x x] are trivial.

A Note on Notations

Concerning our adapted signature notation, we note that Conway’s signature notation is much more compact than ours, omitting all commas, brackets and the letter “g” for gyration points. For our letters “s” and “x”, Conway uses the symbols + and \( \times \), respectively. His signatures are put in bold typeface font. So, for instance, for our notations [g(2) s(2,2)] and [s x], Conway uses 2*22 and *\( \times \).

Note that Conway uses the asterisk (*) not only to denote mirrors, but also as a separation sign: every number before it describes an equivalence class of gyration points, while every number after it describes an equivalence class of rotation centers on mirrors. We adapted Conway’s notation because we thought
that perhaps for beginners it might be difficult to fully grasp all its subtleties at first sight.

Below we compare our signature notation (JvdC) with Conway’s notation (Con) and, for the strip and wallpaper patterns, the notation of the International Tables for X-ray Crystallography (Int). For finite symmetry groups, we also list the order \( n \) of the group. For a spherical pattern, the total weight of its signature equals \( 2 - \frac{2}{n} \), which the reader might like to check.

### Notations for symmetry patterns

<table>
<thead>
<tr>
<th></th>
<th>JvdC</th>
<th>Con</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rosette patterns</strong></td>
<td>([g(N)])</td>
<td>(N\bullet)</td>
<td>(N)</td>
</tr>
<tr>
<td></td>
<td>([s(N)])</td>
<td>(*N\bullet)</td>
<td>(2N)</td>
</tr>
<tr>
<td><strong>Spherical patterns</strong></td>
<td>([g(N, N)])</td>
<td>(NN)</td>
<td>(N)</td>
</tr>
<tr>
<td></td>
<td>([s(N, N)])</td>
<td>(*NN)</td>
<td>(2N)</td>
</tr>
<tr>
<td></td>
<td>([g(N)])</td>
<td>(N\times)</td>
<td>(2N)</td>
</tr>
<tr>
<td></td>
<td>([g(N,2,2)])</td>
<td>(22N)</td>
<td>(2N)</td>
</tr>
<tr>
<td></td>
<td>([s(N,2,2)])</td>
<td>(*22N)</td>
<td>(4N)</td>
</tr>
<tr>
<td></td>
<td>([g(2) s(N)])</td>
<td>(+N)</td>
<td>(4N)</td>
</tr>
<tr>
<td><strong>Spherical patterns</strong></td>
<td>([g(3,3,2)])</td>
<td>(3\times)</td>
<td>(12)</td>
</tr>
<tr>
<td></td>
<td>([s(3,3,2)])</td>
<td>(*332)</td>
<td>(24)</td>
</tr>
<tr>
<td></td>
<td>([g(3)])</td>
<td>(3\times)</td>
<td>(24)</td>
</tr>
<tr>
<td></td>
<td>([s(3)])</td>
<td>(*332)</td>
<td>(24)</td>
</tr>
<tr>
<td></td>
<td>([g(4,2)])</td>
<td>(4\times)</td>
<td>(24)</td>
</tr>
<tr>
<td></td>
<td>([s(4,2)])</td>
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<td>(48)</td>
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<tr>
<td></td>
<td>([g(5,3,2)])</td>
<td>(532)</td>
<td>(60)</td>
</tr>
<tr>
<td></td>
<td>([s(5,3,2)])</td>
<td>(*532)</td>
<td>(120)</td>
</tr>
<tr>
<td><strong>Wallpaper patterns</strong></td>
<td>([O])</td>
<td>(O)</td>
<td>(p1)</td>
</tr>
<tr>
<td></td>
<td>([s])</td>
<td>(*)</td>
<td>(pm)</td>
</tr>
<tr>
<td></td>
<td>([x])</td>
<td>(*\times)</td>
<td>(cm)</td>
</tr>
<tr>
<td></td>
<td>([g(2,2)])</td>
<td>(22\times)</td>
<td>(pg)</td>
</tr>
<tr>
<td></td>
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<td>(pg)</td>
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<td></td>
<td>([g(2,2,2,2)])</td>
<td>(2222)</td>
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<td>(*22)</td>
<td>(cmm)</td>
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<td>([g(2)])</td>
<td>(*2222)</td>
<td>(pmm)</td>
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<tr>
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<td>(p31m)</td>
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<td>(4\times)</td>
<td>(p4)</td>
</tr>
<tr>
<td></td>
<td>([g(4)])</td>
<td>(*42)</td>
<td>(p4)</td>
</tr>
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<td>(*442)</td>
<td>(p4m)</td>
</tr>
<tr>
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<td>([s(6)])</td>
<td>(632)</td>
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<tr>
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<td>([s(6,3,2)])</td>
<td>(*632)</td>
<td>(p6m)</td>
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</table>

On p. 416 of Conway et al. (2008) tables are given comparing Conway’s signature notation with older notations by other authors for the spherical and wallpaper groups.

### Literature

