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f-9 Topological Characterizations of Separable Metrizable Zero-Dimensional Spaces

A space X is called **zero-dimensional** if it is nonempty and has a base consisting of ***clopen*** sets, i.e., if for every point $x \in X$ and for every neighbourhood U of x there exists a clopen subset $C \subseteq X$ such that $x \in C \subseteq U$. It is clear that a nonempty subspace of a zero-dimensional space is again zero-dimensional and that products of zero-dimensional spaces are zero-dimensional.

We say that a space X is ***totally disconnected*** if for all distinct points x, y in X there exists a clopen set C in X such that $x \in C$ but $y \notin C$. It is clear that every zero-dimensional space is totally disconnected. The question naturally arises whether every totally disconnected space is zero-dimensional. If this were true then checking whether a given space is zero-dimensional would be simpler. However, the answer to this question is in the negative, as was shown by Sierpiński (cf. [KU, Chapter V, §46.VI, Footnote 2]).

In part 1 of this note, we are interested in theorems that state nontrivial and useful topological characterizations of zero-dimensional ***separable metrizable spaces***. So it will be convenient in part 1 to let 'space' denote 'separable metrizable space'. In part 2 we briefly mention a few results for general ***Tychonoff space***s that are in the same spirit.

1. Separable metrizable spaces

It is easy to see that a zero-dimensional space can be embedded in the real line $\mathbb{R}$, and that a nonempty subspace X of $\mathbb{R}$ is zero-dimensional if and only if it does not contain any nondegenerate interval. For example, the **space of rational numbers** $\mathbb{Q}$, the **space of irrational numbers** $\mathbb{P}$, the product $\mathbb{Q} \times \mathbb{P}$, etc., are all zero-dimensional.

A zero-dimensional space X is **strongly homogeneous** provided that all of its nonempty clopen sets are homeomorphic. It is easy to see that every strongly homogeneous space is ***homogeneous***, [5, 1.9.3]. It is tempting to conjecture that all homogeneous zero-dimensional spaces are strongly homogeneous. This is not true however, as was shown by van Douwen.

If $\mathcal{P}$ is a topological property then a space is called **nowhere** $\mathcal{P}$ provided that no nonempty open subset of it has $\mathcal{P}$. The characterizations theorems that we will mention below are all of the following form: up to homeomorphism, there is only one zero-dimensional space which has $\mathcal{P}$ but is nowhere $\mathcal{Q}$. Here $\mathcal{P}$ and $\mathcal{Q}$ can be quite complex topological properties. An interesting consequence will be that all nonempty clopen sets of the spaces considered share the same properties. This means that all these spaces are strongly homogeneous and hence homogeneous. So the characterization theorems give us homogeneity for free. (This phenomenon is not uncommon in topology.)

The following example of a zero-dimensional *compact* space is of particular interest. From $\mathbb{I} = [0, 1]$ remove the interval $(1/3, 2/3)$, i.e., the 'middle-third' interval. From the remaining two intervals, again remove their 'middle-thirds', and continue in this way infinitely often. What remains of $\mathbb{I}$ at the end of this process is called the **Cantor middle-third set**, C. A space homeomorphic to C is called a **Cantor set**.

It is easy to see that C is the subspace of $\mathbb{I}$ consisting of all points that have a triadic expansion in which the digit 1 does not occur, i.e., the set

$$\left\{x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \colon x_i \in \{0, 2\} \text{ for every } i\right\}.$$

The Cantor set was introduced by Cantor. It is clearly closed in $\mathbb{I}$, hence is compact. It also has no isolated points and is zero-dimensional because it does not contain any nontrivial interval. Interestingly, the mentioned properties topologically characterize C: up to homeomorphism, C is the only zero-dimensional compact space without isolated points. This is due to Brouwer [5, Theorem 1.5.5]. Observe that the characterization theorem implies that C is homeomorphic to the Cantor cube $\{0, 1\}^\infty$. Hence one can look upon Brouwer's Theorem also as a topological characterization of $\{0, 1\}^\infty$.

The Cantor set C topologically surfaces 'almost everywhere'. By a result of Souslin, every uncountable ***analytic space*** contains a topological copy of C [4, Exercise 14.13]. This result was partly generalized by van Douwen: if a ***topologically complete*** space X contains an uncountable family of pairwise disjoint homeomorphs of some compact space K then X contains a copy of the product $\mathsf{C} \times K$ [5, Corollary 1.5.15]. Observe that the uncountably many pairwise disjoint homeomorphs of K may be irregularly embedded. Van Douwen's result shows that they can be replaced by a Cantor set of 'regularly' embedded copies of K. Since Souslin's theorem works for analytic spaces, the question naturally arises whether van Douwen's theorem also holds for analytic spaces. This question was considered by Becker, van Engelen and van Mill; it is undecidable.

The Cantor set is a universal object for the class of all zero-dimensional spaces. By a result of Alexandroff and Urysohn, every compact space is a continuous image of C [5, Theorem 1.5.10]. This is also some sort of universal property.

Cantor sets are widely studied in geometric topology. It is easy to prove that all Cantor sets in $\mathbb{R}$ are topologically equivalent. The same result is also true in $\mathbb{R}^2$ but the proof is more difficult. In $\mathbb{R}^3$ there are 'wild' Cantor sets however, the most famous one of which is Antoine's necklace. Similar sets can be constructed in $\mathbb{R}^n$ for $n \geqslant 3$ and in the ***Hilbert cube*** Q. In contrast, in the **countable infinite product of lines** $\mathbb{R}^\infty$, all Cantor sets are tame. For details, and references, see [2, 5].

The rational numbers $\mathbb{Q}$ can also be characterized in topological terms. It is, up to homeomorphism, the only countable space without isolated points. This is due to Sierpiński [5, Theorem 1.9.6]. As is the case with Cantor sets, topological copies of $\mathbb{Q}$ surface 'almost everywhere'. It was shown for example by Hurewicz that if a space X is not a ***Baire space*** then X contains a closed copy of $\mathbb{Q}$ (the simple proof of this which was presented in [5, Theorem 1.9.12] is due to van Douwen).

Alexandroff and Urysohn proved that, up to homeomorphism, $\mathbb{P}$ is the only zero-dimensional topologically complete space which is nowhere compact [5, Theorem 1.9.8]. This implies for example that $\mathbb{P}$ is homeomorphic to $\mathbb{N}^\infty$, where $\mathbb{N}$ is the (discrete) space of natural numbers. They also proved that $\mathbb{Q} \times \mathsf{C}$ is up to homeomorphism the only σ-compact zero-dimensional space which is nowhere compact and nowhere countable. A similar result was obtained by the author: $\mathbb{Q} \times \mathbb{P}$ is the topologically unique zero-dimensional space which is a countable union of closed topologically complete subspaces and which in addition is nowhere topologically complete and nowhere σ-compact. It was subsequently shown by van Engelen [3] that $\mathbb{Q}^\infty$ is the only zero-dimensional absolute $F_{\sigma\delta}$ which is first category and nowhere an absolute $G_{\delta\sigma}$.

Observe that all the spaces considered so far are ***topological groups***. This is clear for $\mathbb{Q}$, being a subgroup of $\mathbb{R}$. It is also clear for C and $\mathbb{P}$ since their characterization theorems imply that they are homeomorphic to the topological groups $\{0, 1\}^\infty$ and $\mathbb{Z}^\infty$, respectively. Since products of topological groups are topological groups, we are also done for $\mathbb{Q} \times \mathsf{C}$, $\mathbb{Q} \times \mathbb{P}$ and $\mathbb{Q}^\infty$. These observations prompted van Douwen to ask whether every zero-dimensional homogeneous space admits the structure of a topological group. He proved that there is a (strongly homogeneous) zero-dimensional space T which is the union of a topologically complete and a countable subspace, and which is nowhere σ-compact and nowhere topologically complete. As to be expected, T is topologically characterized by these properties. Since a topological group containing a dense topologically complete subspace is topologically complete, we have that T is not a topological group. But it *is* homogeneous, being strongly homogeneous. The space T can easily be visualized. Indeed, let D be a countable dense subset of C, and put $P = \mathsf{C} \setminus D$. Then the subspace

$$T = (\mathsf{C} \times \mathsf{C}) \setminus (D \times P)$$

of $\mathsf{C} \times \mathsf{C}$ and T are homeomorphic. Van Douwen did not live long enough to publish his results on T. For details, see [3]. So not all homogeneous zero-dimensional ***absolute Borel sets*** are topological groups. One could revive van Douwen's problem by asking whether every zero-dimensional homogeneous absolute Borel set admits a ***transitive*** action by a topological group. It was recently shown by the author that the answer to this question is in the affirmative.

The difficult problem to topologically characterize *all* zero-dimensional homogeneous absolute Borel sets was solved by van Engelen in [3]. He used the hierarchy of small Borel classes in Δ^0_3 to characterize all homogeneous zero-dimensional absolute Borel sets of ambiguous class 2. However, when extended to the classes Δ^0_α for $\alpha < \omega_1$, this hierarchy turned out to be too coarse to distinguish between all zero-dimensional homogeneous absolute Borel sets. Instead, he used the so-called Wadge hierarchy of Borel sets developed by Wadge (see [4]) and powerful results of Louveau, Martin, and Steel. He concluded that there are precisely ω_1 homogeneous zero-dimensional absolute Borel sets and that they can all be topologically characterized. The question which of those spaces admits the structure of a topological group is not completely solved. It is known that if a first category zero-dimensional absolute Borel set is homeomorphic to its own square then it is a topological group (van Engelen). The conjecture is that the converse holds.

Call a space ***rigid*** if the identity is its only homeomorphism. It is a natural question related to the results mentioned here whether there exist rigid zero-dimensional absolute Borel sets. This question was posed by van Douwen. The zero-dimensionality is essential for there are simple examples of rigid continua. Van Engelen, Miller and Steel answered it in the negative, [3]. So any zero-dimensional absolute Borel set admits a nontrivial homeomorphism. As far as we know, it is unknown whether the same result can be proved for zero-dimensional analytic spaces.

2. General spaces

There are only a few results known for general topological spaces that are in the same spirit as the results presented in Section 1. An important result to be mentioned is the characterization of $\beta\mathbb{N} \setminus \mathbb{N}$ by Parovičenko, [E, p. 236]. Here $\beta\mathbb{N} \setminus \mathbb{N}$ is the ***Čech-Stone remainder*** of the discrete space $\mathbb{N}$. Parovičenko's characterization states that under the ***Continuum Hypothesis***, $\beta\mathbb{N} \setminus \mathbb{N}$ is the topologically unique zero-dimensional compact F***-space*** of weight $\mathfrak{c}$ in which nonempty G_δ's have infinite interior. It is known that the assumption of the Continuum Hypothesis is essential in Parovičenko's characterization. There are some alternative and useful characterizations under weaker axioms, but it would lead us too far to go into that. The space $\beta\mathbb{N} \setminus \mathbb{N}$ was and is widely studied, see e.g., the articles on ultrafilters and βX in this volume for more information.

We saw above that C is 'co-universal' for second-countable compact spaces. The space $\beta\mathbb{N} \setminus \mathbb{N}$ has a similar property. It was proved by Parovičenko that every compact space of weight at most ω_1 is a continuous image of $\beta\mathbb{N} \setminus \mathbb{N}$, [E,

p. 236]. Only recently similar results for other classes of spaces were obtained. For example, Dow and Hart proved that every continuum of weight at most ω_1 is a continuous image of the space $\beta\mathbb{H} \setminus \mathbb{H}$; here $\mathbb{H} = [0, \infty)$.

In part 1, we stated Brouwer's characterization of the Cantor cube $\{0, 1\}^\infty$. It is a natural question to ask whether a similar characterization can be found for Cantor cubes of larger weight, i.e., for spaces of the form $\{0, 1\}^\kappa$, where κ is some uncountable cardinal. This is indeed the case. Call a compact space X an ***Absolute Extensor*** in dimension 0 if for each zero-dimensional compactum Z and for each closed subspace Z_0 of Z, any continuous function $f : Z_0 \to X$ can be extended over Z. Ščepin proved that if $\kappa > \omega$ then a zero-dimensional compact space X of weight κ is homeomorphic to $\{0, 1\}^\kappa$ if and only if X is an Absolute Extensor in dimension 0 while moreover all points in X have the same ***character*** [1, Theorem 8.1.6]. This result has nice applications. It can be used for example to prove Sirota's Theorem that $\{0, 1\}^{\omega_1}$ is homeomorphic to its own ***hyperspace***. (It is known that there is no corresponding result for $\{0, 1\}^{\omega_2}$.)

There are similar topological characterizations of various other zero-dimensional spaces. For example, it is possible to generalize the topological characterization of $\mathbb{P}$ stated in part 1 to topological characterizations of all spaces of the form $\mathbb{N}^\kappa$, where $\kappa > \omega$. For this result and others we refer the reader to Chigogidze [1].

References

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