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RECENT RESULTS ON SUPEREXTENSIONS

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1. INTRODUCTION

If (X,d) is a compact metric space, then λX denotes the space of all maximal linked systems of closed subsets of X (a system of closed subsets of X is called a *linked system* if every two of its members meet; a *maximal linked system* or *mls* is a linked system not properly contained in another linked system) topologized by the metric

$$\overline{d}(M,N) = \sup_{S\in M} \min_{T\in N} d_H(S,T)$$

(VERBEEK [14]). A closed subbase for λX, which generates the same topology as $\overline{d}$, is the collection

$$\{\{M\in\lambda X \mid M\in M\} \mid M\in 2^X\}.$$

By induction, it is easy to show that each linked system $L \subset 2^X$ is contained in at least one maximal linked system $L' \subset 2^X$. This implies that the closed subbase, described above, is both *binary* (any of its linked subsystems has a nonvoid intersection) and *normal* (two disjoint subbase elements are separated by disjoint complements of subbase sets).

The spaces λX are called *superextensions* (DE GROOT [9]); in this paper we announce some recent results on superextensions.

2. RECENT RESULTS ON SUPEREXTENSIONS

VERBEEK [14] has shown that λX is a Peano continuum if and only if X is a metrizable continuum; he raised the question of whether λX is an AR if and only if X is a metrizable continuum. Theorem 2.1 answers this question, cf. VAN MILL [10].

2.1. THEOREM: *Let* X *be a metrizable continuum that possesses a closed subbase which is both binary and normal. Then* X *is an* AR.

By a result of VERBEEK [14], the space X in theorem 2.1 is a Peano continuum, and consequently 2^X is an AR, by the theorem of WOJDYSLAWSKI [16] (even $2^X \approx Q$, the Hilbert cube, if X is nondegenerate, cf. CURTIS & SCHORI [7]). We prove that there is a retraction $r: 2^X \to X$, which shows that X is an AR too. Notice that the normality of the subbase is essential, since each compact metric space possesses a binary closed subbase (cf. STROK & SZYMANSKI [13]).

DE GROOT [9] conjectured that λI, the superextension of the closed unit interval $I = [-1,1]$ is homeomorphic to the Hilbert cube Q. This turned out to be the case, cf. VAN MILL [10].

2.2. THEOREM: λI *is homeomorphic to the Hilbert cube.*

We represent λI as an inverse limit $\varprojlim (X_i, f_i)$ of an inverse sequence (X_i, f_i) of Hilbert cubes such that the bonding maps are cellular. Then, by results of CHAPMAN [5], [6] and BROWN [3] it follows that $\lambda I \approx Q$. The spaces X_i ($i \in \mathbb{N}$) are first shown to be compact Q-manifolds; theorem 2.1 implies that they are contractible. Therefore $X_i \approx Q (i \in \mathbb{N})$, since a compact contractible Q-manifold is a Hilbert cube (cf. CHAPMAN [4]).

If X is a compact metric space, then for each $A \subset X$ define

$$A^+ := \{\mathcal{M} \in \lambda X \mid \exists M \in \mathcal{M} : M \subset A\}.$$

It is easy to show that $\{U^+ \mid X \setminus U \in 2^X\}$ is an open subbase for the topology of λX. We have the following theorem, cf. VAN MILL [12].

2.3. THEOREM: *Let* X *be a compact metric space for which* λX *is homeomorphic to the Hilbert cube* Q. *Then for all open* $U_i \subset X$ $(i \le n, n \in \mathbb{N})$ *the closure (in* λX*) of* $\bigcap_{i \le n} U_i^+$ *either is void or is a Hilbert cube.*

To prove theorem 2.3, we use a compactification result of WEST [15] and the recent result of EDWARDS [8], that every AR is a *Hilbert cube factor*;

that is a space whose product with the Hilbert cube is homeomorphic to the Hilbert cube.

If $f:X \to Y$ (X and Y are compact metric) is continuous, then there is a natural extension $\lambda(f) : \lambda Y \to \lambda X$ of f (cf. VERBEEK [14]) defined by

$$\lambda(f)(M) := \{f[M] \mid M \in M\}$$

($\lambda(f)$ can considered to be an extension of f since there are natural embeddings $i_X : X \to \lambda X$ and $i_Y : Y \to \lambda Y$ such that the diagram

$$\begin{array}{ccc} \lambda X & \xrightarrow{\lambda(f)} & \lambda Y \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle i_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes). We have the following remarkable result:

2.4. <u>THEOREM</u>: *Let* X *and* Y *be metrizable continua and let* $f: X \to Y$ *be a continuous surjection. Then* $\lambda(f): \lambda X \to \lambda Y$ *is cellular.*

2.5. <u>COROLLARY</u>: *Let* $X \approx \varprojlim (X_i, f_i)$ *where each* $f_i: X_{i+1} \to X_i$ *is surjective and* $\lambda X_i \approx Q (i \in N)$. *Then* $\lambda X \approx Q$.

Corollary 2.5 implies that the superextension of a space such as $Y = \{(0,y) \mid -1 \le y \le 1\} \cup \{(x, \sin \frac{1}{x}) \mid 0 < x \le 1\}$ is homeomorphic to the Hilbert cube.

If Y is a closed subset of X then there is a natural embedding $j_{YX}: \lambda Y \to \lambda X$ defined by

$$j_{YX}(M) := \{A \in 2^X \mid A \cap Y \in M\}$$

(cf. VERBEEK [14]). We will always identify λY and $j_{YX}[\lambda Y]$.

A closed subset M of a metric space (X,d) is called a *Z-set* (cf. ANDERSON [1]) provided that for each $\varepsilon > 0$ there is a continuous $f_\varepsilon: X \to X \backslash M$ such that $d(f_\varepsilon, id) < \varepsilon$.

2.6. <u>THEOREM</u>: *Let* X *be a metrizable continuum and let* $A \in 2^X$. *Then*
(i) A^+ *is a* Z-*set in* λX *iff* A *has a void interior in* X;
(ii) *if* $A \neq X$ *then* λA *is a* Z-*set in* λX.

This theorem can be used to construct capsets of λI. A subset $A \subset Q$ is called a *capset* (cf. ANDERSON [2]) if there is an autohomeomorphism $\phi : Q \to Q$ such that $\phi[A] = B(Q) = \{x \in Q \mid \exists\, i \in N: |x_i| = 1\}$. An mls $\mathcal{M} \in \lambda X$ is said to be *defined* on $A \in 2^X$ if $M \cap A \in \mathcal{M}$ for all $M \in \mathcal{M}$ (VERBEEK [14]). Define

$$W := \{\mathcal{M} \in \lambda I \mid \mathcal{M} \text{ is defined on some } M \in 2^I \backslash \{I\}\}.$$

2.7. <u>THEOREM</u>: W *is a capset of* λI.

The proof is in two steps. First we prove, using theorem 2.2 and theorem 2.6, that

$$V := \{\mathcal{M} \in \lambda I \mid \mathcal{M} \text{ is defined on some closed set } M \subset (-1,1)\}$$

is a capset of λI. By theorem 2.6, W is a countable union of Z-sets of λI. This implies that W is a capset of λI, since the union of a capset and a countable union of Z-sets is again a capset (cf. ANDERSON [2]).

The space $V \subset \lambda I$ defined above was conjectured by VERBEEK [14] to be homeomorphic to ℓ_2, the separable Hilbert space. This is not true however, since $V \approx B(Q)$ (cf. VAN MILL [11]).

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