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RECENT RESULTS ON SUPEREXTENSIONS

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1. INTRODUCTION

If \((X,d)\) is a compact metric space, then \(\mathcal{L}X\) denotes the space of all maximal linked systems of closed subsets of \(X\) (a system of closed subsets of \(X\) is called a linked system if every two of its members meet; a \textit{maximal linked system} or MLS is a linked system not properly contained in another linked system) topologized by the metric

\[
\tilde{d}(M,N) = \sup_{S \in M, T \in N} \min d_S(T)
\]

(VERBEKE [14]). A closed subbase for \(\mathcal{L}X\), which generates the same topology as \(\tilde{d}\), is the collection

\[
\{(m,\mathcal{L}X, m) \mid m \subseteq 2^X\}.
\]

By induction, it is easy to show that each linked system \(L \subseteq 2^X\) is contained in at least one maximal linked system \(L' \subseteq 2^X\). This implies that the closed subbase, described above, is both \textit{binary} (any of its linked subsystems has a nonvoid intersection) and \textit{normal} (two disjoint subbase elements are separated by disjoint complements of subbase sets).

The spaces \(\mathcal{L}X\) are called \textit{superextensions} (DE GROOT [9]); in this paper we announce some recent results on superextensions.

2. RECENT RESULTS ON SUPEREXTENSIONS

VERBEKE [14] has shown that \(\mathcal{L}X\) is a Peano continuum if and only if \(X\) is a metrizable continuum; he raised the question of whether \(\mathcal{L}X\) is an AR if and only if \(X\) is a metrizable continuum. Theorem 2.1 answers this question, cf. VAN MILL [10].
2.1. **THEOREM:** Let $X$ be a metrizable continuum that possesses a closed subbase which is both binary and normal. Then $X$ is an AR.

By a result of VERBEERK [14], the space $X$ in theorem 2.1 is a Peano continuum, and consequently $2^X$ is an AR, by the theorem of WOJDSLANSKI [16] (even $2^X \cong Q$, the Hilbert cube, if $X$ is nondegenerate, cf. CURTIS & SCHORI [7]). We prove that there is a retraction $r : 2^X \rightarrow X$, which shows that $X$ is an AR too. Notice that the normality of the subbase is essential, since each compact metric space possesses a binary closed subbase (cf. STROK & SZYMANSKI [13]).

DE GROOT [9] conjectured that $\lambda I$, the superextension of the closed unit interval $I = [-1,1]$ is homeomorphic to the Hilbert cube $Q$. This turned out to be the case, cf. VAN MILL [10].

2.2. **THEOREM:** $\lambda I$ is homeomorphic to the Hilbert cube.

We represent $\lambda I$ as an inverse limit $\lim (X_i, f_i)$ of an inverse sequence $(X_i, f_i)$ of Hilbert cubes such that the bonding maps are cellular. Then, by results of CHAPMAN [5], [6] and BROWN [3] it follows that $\lambda I \cong Q$. The spaces $X_i (i \in N)$ are first shown to be compact $Q$-manifolds; theorem 2.1 implies that they are contractible. Therefore $X_i \cong Q(i \in N)$, since a compact contractible $Q$-manifold is a Hilbert cube (cf. CHAPMAN [4]).

If $X$ is a compact metric space, then for each $A \subset X$ define

$$A^* := \{ M \in X \mid \exists M \in M : M \cap A \}.$$ 

It is easy to show that $(I^* \cap X \setminus 2^X)$ is an open subbase for the topology of $\lambda X$. We have the following theorem, cf. VAN MILL [12].

2.3. **THEOREM:** Let $X$ be a compact metric space for which $\lambda X$ is homeomorphic to the Hilbert cube $Q$. Then for all open $V_i \subset X (i \in \omega, n \in N)$ the closure (in $\lambda X$) of $U_0 \cap \cdots \cap U_i$ either is void or is a Hilbert cube.

To prove theorem 2.3, we use a compactification result of WEST [15] and the recent result of EDWARDS [8], that every AR is a Hilbert cube factor;
that is a space whose product with the Hilbert cube is homeomorphic to the Hilbert cube.

If \( f: X \rightarrow Y \) (\( X \) and \( Y \) are compact metric) is continuous, then there is a natural extension \( \lambda(f) : \lambda X \rightarrow \lambda X \) of \( f \) (cf. VERBEKE [14]) defined by

\[
\lambda(f)(M) := [f(M)]_{\text{lex}} M
\]

(\( \lambda(f) \)) can considered to be an extension of \( f \) since there are natural embeddings \( i_X : X \rightarrow \lambda X \) and \( i_Y : Y \rightarrow \lambda Y \) such that the diagram

\[
\begin{array}{ccc}
\lambda X & \rightarrow & \lambda Y \\
\downarrow i_X & & \downarrow i_Y \\
X & \rightarrow & Y \\
\rightarrow & f & \rightarrow
\end{array}
\]

commutes. We have the following remarkable result:

2.4. THEOREM: Let \( X \) and \( Y \) be metricable continua and let \( f: X \rightarrow Y \) be a continuous surjection. Then \( \lambda(f) : \lambda X \rightarrow \lambda Y \) is cellular.

2.5. COROLLARY: Let \( X = \lim_{\rightarrow} (X_1, f_1) \) where each \( f_1 : X_1 \rightarrow X_1 \) is surjective and \( \lambda X_1 \approx Q('ic N) \). Then \( \lambda X \approx Q \).

Corollary 2.5 implies that the superextension of a space such as

\[ Y = \{(0, y) \mid -1 < y < 1\} \cup \{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\} \]

is homeomorphic to the Hilbert cube.

If \( Y \) is a closed subset of \( X \) then there is a natural embedding

\[ j_{XX} : \lambda Y \rightarrow \lambda X \text{ defined by} \]

\[
j_{XX}(M) := \{A \in 2^X \mid A \cap Y = M\}
\]

(cf. VERBEKE [14]). We will always identify \( \lambda Y \) and \( j_{XX}(\lambda Y) \).

A closed subset \( M \) of a metric space \((X, d)\) is called a \( \kappa \)-set (cf. ANDERSON [1]) provided that for each \( \epsilon > 0 \) there is a continuous \( f_{\epsilon} : X \rightarrow \chi \setminus M \) such that \( d(f_{\epsilon}, \text{Id}) < \epsilon \).
2.6. **Theorem**: Let $X$ be a metrizable continuum and let $A \subset X$. Then

(i) $A^\circ$ is a $Z$-set in $\mathcal{X}$ if and only if $A$ has an $\mathcal{X}$-null interior in $X$;
(ii) $\emptyset \neq A \neq X$ then $\lambda A$ is a $Z$-set in $\mathcal{X}$.

This theorem can be used to construct capsets of $\lambda I$. A subset $A \subset Q$ is called a capset (cf. Anderson [2]) if there is an autohomeomorphism $\phi : Q \to Q$ such that $\phi(A) = B(Q) = \{ x \in Q | \exists i \in N: |x_i| = 1 \}$. An $M \in \lambda X$ is said to be defined on $A \subset X$ if $M \cap A = M$ for all $M \in \lambda$ (Verbeek [14]). Define

$$W := \{ M \in \lambda I | M \text{ is defined on some } M \in 2^I \setminus \{ i \} \}.$$

2.7. **Theorem**: $W$ is a capset of $\lambda I$.

The proof is in two steps. First we prove, using theorem 2.2 and theorem 2.6, that

$$V := \{ M \in \lambda I | M \text{ is defined on some closed set } M \subset (-1,1) \}$$

is a capset of $\lambda I$. By theorem 2.6, $W$ is a countable union of $Z$-sets of $\lambda I$. This implies that $W$ is a capset of $\lambda I$, since the union of a capset and a countable union of $Z$-sets is again a capset (cf. Anderson [2]).

The space $V \subset \lambda I$ defined above was conjectured by Verbeek [14] to be homeomorphic to $L_2$, the separable Hilbert space. This is not true however, since $V \not= B(Q)$ (cf. Van Mill [11]).

**References**


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