A NOTE ON WALLMAN COMPACTIFICATIONS

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ABSTRACT

It is shown that a compact tree-like space of weight less than or equal to $2^\aleph_0$ is regular Wallman. The same is true for the Čech-Stone compactification of a peripherally compact tree-like space which possesses at most $2^\aleph_0$ closed subsets.

1. INTRODUCTION

Every Tychonoff space $X$ admits Hausdorff compactifications, obtainable as the ultra-filter space of some normal base on $X$. These compactifications are called Wallman compactifications. Until now the question, raised in [2] and [3], whether all Hausdorff compactifications are Wallman compactifications remains unanswered, although many well known compactifications turned out to be Wallman compactifications ([1],[4],[9],[10]).

In this note we will show that a compact tree-like space of weight less than or equal to $2^\aleph_0$ and the Čech-Stone compactification of a peripherally compact tree-like space, which possesses at most $2^\aleph_0$ closed subsets, are regular Wallman (in the sense of STEINER [10]; such a space is a Wallman compactification of each dense subspace).

2. REGULAR WALLMAN SPACES

Let $X$ be a topological space and let $S$ be a collection of subsets of $X$. We will write $v.S$ for the family of finite unions of ele-
ments of $S$ and $\Lambda S$ for the family of finite intersections of elements of $S$. The family $\Lambda \cdot \Lambda S = \Lambda \cdot S$ is closed both under finite intersections and finite unions; it is called the ring generated by $S$. We say that $S$ is separating if for each closed subset $F \subset X$ and for each $x \in X \setminus F$ there exists $S_0, S_1 \in S$ such that $x \in S_0$, $F \subset S_1$ and $S_0 \cap S_1 = \emptyset$. A compact space is called regular Wallman if it possesses a separating ring of regular closed sets. It is known that each regular Wallman space is Wallman compactification of each dense subspace (Steiner [10]).

A connected space is called tree-like whenever every two points of $X$ have a separation point. It is clear that all connected orderable spaces are tree-like, however, the class of tree-like spaces is much bigger. See, e.g., KOK [6]. Let $X$ be a peripherally compact tree-like space. Let $a, b \in X$ ($a \neq b$) and define $S(a, b) = \{x \in X \mid x$ separates $a$ and $b\} \cup \{a, b\}$. It is well known that $S(a, b)$ is an orderable connected subspace of $X$ with two end points ([8],[6]) and, therefore, $S(a, b)$ is compact ([5]). In [8], V.V. Proizvolov proved that any two disjoint closed sets $A$ and $B$ of $X$ are separated by a closed discrete set $C = \{x_i \mid i \in I\}$. The set $C$ is not uniquely determined. In fact, each $x_i$ is a point arbitrarily chosen from $S(a_i, b_i) \setminus \{a_i, b_i\}$ for certain $a_i, b_i \in X$ ($i \in I$). Hence it follows that for each $x_i$ there are at least $2^{N_0}$ different choices.

This observation will be used in the proof of the following theorem.

**Theorem 2.1.** Let $X$ be a peripherally compact tree-like space. Suppose $X$ has at most $2^{N_0}$ closed subsets. Then $\mathcal{F}X$ is regular Wallman.

**Proof.** Let $\mathcal{B}$ be the collection of closed subsets of $X$. Define

$$A = \{(B_0, B_1) \mid B_0, B_1 \in \mathcal{B} \text{ and } B_0 \cap B_1 = \emptyset\}.$$ 

Note that $\text{card}(A) \leq 2^{N_0}$. Assume that $A$ is most economically well-ordered and denote the order by "$\prec$". Let $(B_0, B_1)$ be the first element of $A$. Choose an open set $U$ of $X$, with discrete boundary, such that
B_0 \subset U and \bar{U} \cap B_1 = \emptyset. Define U(B_0^*,B_1^*) = U. Let (B_0^*,B_1^*) \in A and suppose that all U(B_0^*,B_1^*) are constructed for all (B_0^*,B_1^*) < (B_0^*,B_1^*). Note that
\[
\text{card}\left(\{U(B_0^*,B_1^*) \mid (B_0^*,B_1^*) < (B_0^*,B_1^*)\}\right) < 2^{\aleph_0},
\]
since "<" is most economical. Define
\[
H = \forall \forall \forall \left\{U(B_0^*,B_1^*) \mid (B_0^*,B_1^*) < (B_0^*,B_1^*)\right\}.
\]

It is clear that \(H\) consists of open sets with discrete boundary. Let \(C = \{x_i \mid i \in I\}\) be a discrete set separating \(B_0^*\) and \(B_1^*\), and, for each \(i \in I\), let \(S(a_i,b_i)\) be selected in such a way that \(x_i \in S(a_i,b_i) \setminus \{a_i,b_i\}\) while, moreover, for any choice of \(y_i \in S(a_i,b_i)\) (\(i \in I\)) the set \(D = \{y_i \mid i \in I\}\) is again a closed discrete set separating \(B_0^*\) and \(B_1^*\) (cf. the remark preceding this theorem). Since \(S(a_i,b_i)\) is compact we have that
\[
\text{card}\left(\exists H \cap S(a_i,b_i)\right) < \aleph_0 \quad \text{for all } H \in H,
\]
and, consequently,
\[
\text{card}\left(\bigcup_{H \in H} [\exists H \cap S(a_i,b_i)]\right) < 2^{\aleph_0}.
\]
For each \(i \in I\) choose \(x'_i \in S(a_i,b_i) \setminus \{a_i,b_i\}\) such that
\[
x'_i \notin \bigcup_{H \in H} [\exists H \cap S(a_i,b_i)].
\]
It is clear that such a choice is possible. Define \(C' = \{x'_i \mid i \in I\}\) and let \(U\) be an open subset of \(X\) such that \(B_0^* \subset \bar{U} \subset (U \cup C')\) and \((U \cup C') \cap B_1^* = \emptyset\). Define
\[
U(B_0^*,B_1^*) = U.
\]
Finally define
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\[ V = \bigwedge_{V} \left\{ U(B_0, B_1) \mid (B_0, B_1) \in A \right\}. \]

As the intersection of two regular closed sets, with disjoint boundaries, is again regular closed it immediately follows that 
\( \{ \bar{V} \mid V \in V \} \) is a ring consisting of regular closed subsets of \( X \) while, moreover, it separates the closed subsets of \( X \). Since \( X \) is normal, \( \beta X \) is regular Wallman (MISRA [7], theorem 3.4).

This theorem only proves that \( \beta X \) is regular Wallman, even in case \( X \) is peripherally compact tree-like, for a rather small class of spaces. It includes, for instance, the fact that \( \beta \mathbb{R} \) is regular Wallman. It is clear that with the same technique it follows that

**Corollary 2.2.** A compact tree-like space of weight less than or equal to \( 2^\alpha \) is regular Wallman.

This suggests the following question.

**Question 2.3.** Is any compact tree-like space regular Wallman?


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