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A NOTE ON WALLMAN COMPACTIFICATIONS

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ABSTRACT

It is shown that a compact tree-like space of weight less than or equal to $2^{\aleph_0}$ is regular Wallman. The same is true for the Čech-Stone compactification of a peripherally compact tree-like space which possesses at most $2^{\aleph_0}$ closed subsets.

1. INTRODUCTION

Every Tychonoff space X admits Hausdorff compactifications, obtainable as the ultra-filter space of some normal base on X. These compactifications are called *Wallman compactifications*. Until now the question, raised in [2] and [3], whether all Hausdorff compactifications are Wallman compactifications remains unanswered, although many well known compactifications turned out to be Wallman compactifications ([1],[4],[9],[10]).

In this note we will show that a compact tree-like space of weight less than or equal to $2^{\aleph_0}$ and the Čech-Stone compactification of a peripherally compact tree-like space, which possesses at most $2^{\aleph_0}$ closed subsets, are *regular Wallman* (in the sense of STEINER [10]; such a space is a Wallman compactification of each dense subspace).

2. REGULAR WALLMAN SPACES

Let X be a topological space and let S be a collection of subsets of X. We will write $\vee.S$ for the family of finite unions of elements of S and $\wedge.S$ for the family of finite intersections of elements of S. The family $\wedge.\vee.S = \vee.\wedge.S$ is closed both under finite intersections and finite unions; it is called the *ring* generated by S. We say that S is *separating* if for each closed subset $F \subset X$ and for each $x \in X\backslash F$ there exists $S_0, S_1 \in S$ such that $x \in S_0$, $F \subset S_1$ and $S_0 \cap S_1 = \emptyset$. A compact space is called *regular Wallman* if it possesses a separating ring of regular closed sets. It is known that each regular Wallman space is Wallman compactification of each dense subspace (STEINER [10]).

A connected space is called *tree-like* whenever every two points of X have a separation point. It is clear that all connected orderable spaces are tree-like, however, the class of tree-like spaces is much bigger. See, e.g., KOK [6]. Let X be a peripherally compact tree-like space. Let $a,b \in X$ $(a \neq b)$ and define $S(a,b) = \{x \in X \mid x \text{ separates } a \text{ and } b\} \cup \{a,b\}$. It is well known that $S(a,b)$ is an orderable connected subspace of X with two end points ([8],[6]) and, therefore, $S(a,b)$ is compact ([5]). In [8], V.V. PROIZVOLOV proved that any two disjoint closed sets A and B of X are separated by a closed discrete set $C = \{x_i \mid i \in I\}$. The set C is not uniquely determined. In fact, each x_i is a point arbitrarily chosen from $S(a_i,b_i)\backslash\{a_i,b_i\}$ for certain $a_i,b_i \in X$ $(i \in I)$. Hence it follows that for each x_i there are at least $2^{\aleph_0}$ different choices.

This observation will be used in the proof of the following theorem.

THEOREM 2.1. *Let* X *be a peripherally compact tree-like space. Suppose* X *has at most* $2^{\aleph_0}$ *closed subsets. Then* βX *is regular Wallman.*

PROOF. Let B be the collection of closed subsets of X. Define

$$A = \{(B_0,B_1) \mid B_0,B_1 \in B \text{ and } B_0 \cap B_1 = \emptyset\}.$$

Note that $\mathrm{card}(A) \leq 2^{\aleph_0}$. Assume that A is most economically well-ordered and denote the order by "<". Let (B_0,B_1) be the first element of A. Choose an open set U of X, with discrete boundary, such that

$B_0 \subset U$ and $\bar{U} \cap B_1 = \emptyset$. Define $U_{(B_0,B_1)} = U$. Let $(B_0',B_1') \in \mathcal{A}$ and suppose that all $U_{(B_0^*,B_1^*)}$ are constructed for all $(B_0^*,B_1^*) < (B_0',B_1')$. Note that

$$\text{card}\left(\{U_{(B_0^*,B_1^*)} \mid (B_0^*,B_1^*) < (B_0',B_1')\}\right) < 2^{\aleph_0},$$

since "<" is most economical. Define

$$\mathcal{H} = \wedge.\vee.\left\{U_{(B_0^*,B_1^*)} \mid (B_0^*,B_1^*) < (B_0',B_1')\right\}.$$

It is clear that $\mathcal{H}$ consists of open sets with discrete boundary. Let $C = \{x_i \mid i \in I\}$ be a discrete set separating B_0' and B_1', and, for each $i \in I$, let $S(a_i,b_i)$ be selected in such a way that $x_i \in S(a_i,b_i)\setminus\{a_i,b_i\}$ while, moreover, for any choice of $y_i \in S(a_i,b_i)$ $(i \in I)$ the set $D = \{y_i \mid i \in I\}$ is again a closed discrete set separating B_0' and B_1' (cf. the remark preceding this theorem). Since $S(a_i,b_i)$ is compact we have that

$$\text{card}\left(\partial H \cap S(a_i,b_i)\right) < \aleph_0 \quad \text{for all } H \in \mathcal{H},$$

and, consequently,

$$\text{card}\left(\bigcup_{H\in\mathcal{H}} [\partial H \cap S(a_i,b_i)]\right) < 2^{\aleph_0}.$$

For each $i \in I$ choose $x_i' \in S(a_i,b_i)\setminus\{a_i,b_i\}$ such that

$$x_i' \notin \bigcup_{H\in\mathcal{H}} [\partial H \cap S(a_i,b_i)].$$

It is clear that such a choice is possible. Define $C' = \{x_i' \mid i \in I\}$ and let U be an open subset of X such that $B_0' \subset \bar{U} \subset (U \cup C')$ and $(U \cup C') \cap B_1' = \emptyset$. Define

$$U_{(B_0',B_1')} = U.$$

Finally define

$$\mathcal{V} = \wedge.\vee.\left\{U_{(B_0,B_1)} \mid (B_0,B_1) \in A\right\}.$$

As the intersection of two regular closed sets, with disjoint boundaries, is again regular closed it immediately follows that $\{\bar{V} \mid V \in \mathcal{V}\}$ is a ring consisting of regular closed subsets of X while, moreover, it separates the closed subsets of X. Since X is normal, βX is regular Wallman (MISRA [7], theorem 3.4). □

This theorem only proves that βX is regular Wallman, even in case X is peripherally compact tree-like, for a rather small class of spaces. It includes, for instance, the fact that $\beta\mathbb{R}$ is regular Wallman. It is clear that with the same technique it follows that

COROLLARY 2.2. *A compact tree-like space of weight less than or equal to* $2^{\aleph_0}$ *is regular Wallman.*

This suggests the following question.

QUESTION 2.3. *Is any compact tree-like space regular Wallman?*

[1] AARTS, J.M., *Every metric compactification is a Wallman-type compactification*, Proc. Intern. Symp. on Topology and its Applications, Herceg-Novi (Yugoslavia) (1968).

[2] BANACHSCHEWSKI, B., *On Wallman's method of compactification*, Math. Nachrichten, 27, 105-114 (1963).

[3] FRINK, O., *Compactifications and semi-normal spaces*, Amer. J. Math., 86, 602-607 (1964).

[4] HAMBURGER, P., *On Wallman-type, regular Wallman-type and z-compactifications*, Per. Math. Hung., 1, 303-309 (1971).

[5] KELLEY, J.L., *General topology*, Van Nostrand, New York (1955).

[6] KOK, H., *Connected orderable spaces*, Mathematical Centre Tracts, 49, Mathematisch Centrum, Amsterdam (1974).

[7] MISRA, A.K., *Some regular Wallman* βX, Indag. Math., 35, 237-242 (1973).

[8] PROIZVOLOV, V.V., *On the hereditary and collectiv normality of a peripherally compact tree-like space*, Soviet Math. Dokl., 11, 1072-1075 (1970).

[9] STEINER, A.K. & E.F. STEINER, *Products of compact metric spaces are regular Wallman*, Indag. Math., 30, 428-430 (1968).

[10] STEINER, E.F., *Wallman spaces and compactifications*, Fund. Math., 61, 295-304 (1968).

[11] WALLMAN, H., *Lattices and topological spaces*, Annals of Math., 39, 112-126 (1938).

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