MATHEMATICS

# COMPACTIFICATIONS IN WHICH THE COLLECTION OF MULTIPLE POINTS IS LINDELÖF SEMI-STRATIFIABLE

BY

## J. VAN MILL

ABSTRACT

We show that every compactification of a topological space in which the collection
of multiple points is Lindelöf semi-stratifiable is a $z$-compactification. In particular
such a compactification is a Wallman compactification.

KEY WORDS AND PHRASES

Wallman compactification, $z$-compactification, semi-stratifiable, $\beta X$.

## 1. INTRODUCTION

Every Tychonoff space $X$ admits Hausdorff compactifications, obtainable as the ultra-filter space of some normal base on $X$. These compactifications are called *Wallman compactifications*. Until now the question, raised in [2] and [6], whether all Hausdorff compactifications are Wallman compactifications remains unanswered, although many well known compactifications turned out to be Wallman compactifications ([1], [4], [9], [10], [11], [12]).

In this paper we will show that any Hausdorff compactification of a topological space in which the collection of multiple points is Lindelöf semi-stratifiable is a $z$-compactification (a compactification obtainable as the ultra-filter space of a normal base consisting of zero-sets of $X$). This theorem generalizes results obtained by Aarts ([1]), Steiner and Steiner ([9], [11]) and Berney ([4]). Moreover we will show that every Hausdorff compactification of a Lindelöf semi-stratifiable space, in which the collection of multiple points also is Lindelöf semi-stratifiable, is *regular Wallman* (in the sense of Steiner ([12]); such a space is a Wallman compactification of each dense subspace.

Most of the results in this paper are also to be found (in a preliminary form) in the report [7a].

## 2. WALLMAN COMPACTIFICATIONS

All topological spaces, under discussion, are assumed to be Hausdorff. Let $X$ be a topological space and let $\mathscr{S}$ be a collection of subsets of $X$. We will write $\vee \cdot \mathscr{S}$ for the family of finite unions of elements of $\mathscr{S}$ and

$\wedge \cdot \mathscr{S}$ for the family of finite intersections of elements of $\mathscr{S}$. The family $\wedge \cdot \vee \cdot \mathscr{S} = \vee \cdot \wedge \cdot \mathscr{S}$ is closed both under finite unions and intersections; it is called the *ring* generated by $\mathscr{S}$. We say that $\mathscr{S}$ is *separating* if for each closed subset $F \subset X$ and for each $x \in X \backslash F$ there exist $S_0, S_1 \in \mathscr{S}$ such that $x \in S_0$, $F \subset S_1$ and $S_0 \cap S_1 = \emptyset$. The collection $\mathscr{S}$ is called *normal* if for each $S_0, S_1 \in \mathscr{S}$ with $S_0 \cap S_1 = \emptyset$ there exist $T_0, T_1 \in \mathscr{S}$ such that $S_0 \cap T_1 = \emptyset = T_0 \cap S_1$ and $T_0 \cup T_1 = X$. A *normal base* is a normal separating ring consisting of closed subsets of $X$. A compactification $\alpha X$ of $X$ is a Wallman compactification iff $\alpha X$ possesses a separating family $\mathscr{F}$ of closed subsets such that for all $F_i \in \mathscr{F}$ $(i = 1, 2, ..., n; n \in N)$ with $\bigcap_{i=1}^{n} F_i \neq \emptyset$, we have $\bigcap_{i=1}^{n} F_i \cap X \neq \emptyset$. (Steiner ([12])). A compactification $\alpha X$ of $X$ is called a z-compactification if $\mathscr{F}$ in the above characterization can be chosen in such a way that for all $F \in \mathscr{F}$ the set $F \cap X$ is a zero-set in $X$. A topological space $X$ is called *strongly $\aleph$ compact* if for each subset $A$ of $X$ with card $(A) > \aleph$ and for each total order $<$ on $A$ there exists a $y \in A$ such that for each open neighbourhood $U$ of $y$ both $U \cap \{x \in A | x < y\}$ and $U \cap \{x \in A | x > y\}$ are non void. It is very easy to show that each separable metric space is strongly $\aleph_1$ compact using the fact that such a space admits a countable base. More general Creede ([5]) showed that *each Lindelöf semi-stratifiable space is strongly $\aleph_1$ compact* (a topological space $X$ is *semi-stratifiable* if to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of $X$ such that (a) $\bigcup_{n=1}^{\infty} U_n = U$, (b) $U_n \subset V_n$ whenever $U \subset V$ (where $\{V_n\}_{n=1}^{\infty}$ is the sequence assigned to $V$)). Moreover *strongly $\aleph_1$ compact spaces are hereditarily seperable and hereditarily Lindelöf* (Berney ([3])). E. S. Berney has introduced the concept of strongly $\aleph$ compactness in the theory of Wallman compactifications. His techniques turn out to be very powerfull and will be used in this paper.

Let $\alpha X$ be a compactification of $X$ and let $\xi$ denote the unique projection map of $\beta X$, the Čech-Stone compactification of $X$, onto $\alpha X$ which on $X$ is the identity. We say that a point $p \in \alpha X \backslash X$ is a *multiple point* of $\alpha X$ if $\xi^{-1}(p)$ consists of more than one point. Let $Z(X)$ denote the ring of zero-sets of $X$.

LEMMA 2.1: *Let $Y$ be a subspace of $\beta X$ such that $X \subset Y \subset \beta X$ If. $Z_0, Z_1 \in Z(X)$ then $\mathrm{cl}_Y (Z_0) \cap \mathrm{cl}_Y (Z_1) = \mathrm{cl}_Y (Z_0 \cap Z_1)$.*

PROOF: $\mathrm{cl}_{\beta X} (Z_0) \cap \mathrm{cl}_{\beta X} (Z_1) = \mathrm{cl}_{\beta X} (Z_0 \cap Z_1)$. $\square$

For the remainder of this section, let $\alpha X$ be a compactification of the topological space $X$ and let $M$ denote the set of mutiple points of $\alpha X$.

LEMMA 2.2: *Let $Z \in Z(X)$. If $\delta \, \mathrm{cl}_{\alpha X} (Z) \cap M = \emptyset$, then $\xi^{-1}(\mathrm{cl}_{\alpha X} (Z)) = \mathrm{cl}_{\beta X} (Z)$.*

PROOF: Assume there exists an

$$x_0 \in \xi^{-1}(\mathrm{cl}_{\alpha X} (Z)) \backslash \mathrm{cl}_{\beta X} (Z).$$

Then $\xi(x_0) \in \mathrm{cl}_{\alpha X}(Z) \cap M$ and consequently $\xi(x_0) \in \mathrm{int}_{\alpha X}\, \mathrm{cl}_{\alpha X}(Z)$ since $\delta\, \mathrm{cl}_{\alpha X}(Z) \cap M = \emptyset$. Therefore

$$x_0 \in \xi^{-1}(\mathrm{int}_{\alpha X}\, \mathrm{cl}_{\alpha X}(Z)) \subset \mathrm{int}_{\beta X}\, \xi^{-1}(\mathrm{cl}_{\alpha X}(Z)).$$

Let $0$ be an open neighbourhood of $x_0$ in $\beta X$. Then

$$0 \cap \mathrm{int}_{\beta X}\, \xi^{-1}(\mathrm{cl}_{\alpha X}(Z)) \neq \emptyset$$

so that

$$0 \cap \mathrm{int}_{\beta X}\, \xi^{-1}(\mathrm{cl}_{\alpha X}(Z)) \cap X \neq \emptyset.$$

As $\xi$ is the identity on $X$ it follows that $0 \cap Z \neq \emptyset$. Therefore $x_0 \in \mathrm{cl}_{\beta X}(Z)$, which is a contradiction. $\square$

If $f \in C(\alpha X, I)$ then we will write $U(\delta, f)$ instead of $f^{-1}([0, \delta))$.

LEMMA 2.3: *Let $f \in C(\alpha X, I)$ and assume that $M$ is strongly $\aleph_1$ compact. Then* $\mathrm{card}\, \{\delta \in (0, 1)|\mathrm{cl}_{\alpha X}(f^{-1}[0, \delta] \cap X) \cap M \neq \mathrm{cl}_{\alpha X}(U(\delta, f)) \cap M\} < \aleph_0$.

PROOF: Assume to the contrary that

$$\mathrm{card}\, \{\delta \in (0, 1)|\mathrm{cl}_{\alpha X}(f^{-1}[0, \delta] \cap X) \cap M \neq \mathrm{cl}_{\alpha X}(U(\delta, f)) \cap M\} \geqslant \aleph_1.$$

If $\mathrm{cl}_{\alpha X}(U(\delta, f)) \cap M \neq \mathrm{cl}_{\alpha X}(f^{-1}[0, \delta] \cap X) \cap M$ then there exists an

$$a(\delta) \in (\mathrm{cl}_{\alpha X}(f^{-1}[0, \delta] \cap X) \backslash \mathrm{cl}_{\alpha X}(U(\delta, f))) \cap M.$$

Let $B$ be the collection of $a(\delta)$ chosen in this way. Since $f(a(\delta)) = \delta$ it follows that $\delta_1 \neq \delta_2$ implies that $a(\delta_1) \neq a(\delta_2)$ and therefore $B$ is uncountable. Also, a total order "$<$" is defined on $B$ by putting $a(\delta_0) < a(\delta_1) \Leftrightarrow \delta_0 < \delta_1$. Since $B \subset M$ and since $M$ is strongly $\aleph_1$ compact it follows that $B$ has a limit point $a(\delta_0)$ from below. Let $U$ be an arbitrary open neighbourhood of $a(\delta_0)$. Since $a(\delta_0)$ is a limit point from below there is an $a(\delta_1) \in U \cap B$ such that $a(\delta_1) < a(\delta_0)$. This shows that $a(\delta_1) \in U(\delta_0, f) \cap U$ and in particular $U \cap U(\delta_0, f) \neq \emptyset$. Hence $a(\delta_0) \in \mathrm{cl}_{\alpha X}(U(\delta_0, f)) \cap M$, which is a contradiction. $\square$

LEMMA 2.4: *Let $f \in C(\alpha X, I)$ and let $U$ be open in $\alpha X$. If $A \subset \alpha X$ is strongly $\aleph_1$ compact then*

$$\mathrm{card}\, \{\delta \in (0, 1)|\mathrm{cl}_{\alpha X}(U) \cap \mathrm{cl}_{\alpha X}(U(\delta, f)) \cap A \neq \mathrm{cl}_{\alpha X}(U \cap U(\delta, f)) \cap A\} < \aleph_0.$$

PROOF: Use the same technique as in lemma 2.3 or see Berney ([4]). $\square$

THEOREM 2.6: *Any compactification of a topological space $X$ in which the collection of mutiple points is strongly $\aleph_1$ compact is a z-compactification.*

COROLLARY 2.7: *Any compactification of a topological space in which the collection of multiple points is Lindelöf semi-stratifiable is a z-compactification.*

PROOF: Let $M^*$ denote the closure of $M$ in $\alpha X$. Then $M^*$ is a compactification of $M$ and since $M$ is separable, the weight of $M^*$ is less than or equal to $2^{\aleph_0}$. Let $\mathscr{B}$ be an open basis for the topology of $M^*$ such that $\operatorname{card}(\mathscr{B}) \leqslant 2^{\aleph_0}$ and define

$$\mathscr{C} = \{(\operatorname{cl}_{\alpha X}(B_0), \operatorname{cl}_{\alpha X}(B_1)) | B_0, B_1 \in \mathscr{B} \text{ and } \operatorname{cl}_{\alpha X}(B_0) \cap \operatorname{cl}_{\alpha X}(B_1) = \emptyset\}.$$

For each $(\operatorname{cl}_{\alpha X}(B_0), \operatorname{cl}_{\alpha X}(B_1)) \in \mathscr{C}$, choose an $f \in C(\alpha X, I)$ such that $f(\operatorname{cl}_{\alpha X}(B_0)) = 0$ and $f(\operatorname{cl}_{\alpha X}(B_1)) = 1$. Let $\mathscr{F}$ denote the set of mappings obtained in this way and assume that $\mathscr{F}$ is most economically well-ordered (denote the order by $<$). Note that $\operatorname{card}(\mathscr{F}) \leqslant 2^{\aleph_0}$. By transfinite induction we will construct for each $f \in \mathscr{F}$ a $\delta_f \in (0, 1)$ such that

(i)  $\operatorname{cl}_{\alpha X}(f^{-1}[0, \delta_f] \cap X) \cap M = \operatorname{cl}_{\alpha X}(U(\delta_f, f)) \cap M$,
(ii) $\operatorname{cl}_{\alpha X}(U(\delta_f, f)) \cap \operatorname{cl}_{\alpha X}(V) \cap M = \operatorname{cl}_{\alpha X}(U(\delta_f, f) \cap V) \cap M$, for all $V \in \wedge \cdot \vee \cdot \{U(\delta_g, g) | g < f\}$.

Let $f_0$ be the first element of $\mathscr{F}$. Choose $\delta_{f_0} \in (0, 1)$ such that

$$\operatorname{cl}_{\alpha X}(f_0^{-1}[0, \delta_{f_0}] \cap X) \cap M = \operatorname{cl}_{\alpha X}(U(\delta_{f_0}, f_0)) \cap M.$$

Such a choice for $\delta_{f_0}$ is possible (lemma 2.3). Next, let $g \in \mathscr{F}$ and assume $\delta_f$ to be defined for all $f < g$ ($f \in \mathscr{F}$). Note that $\operatorname{card}\{\delta_f | f < g\} < 2^{\aleph_0}$, since "$<$" is most economical. Define $\mathscr{V} = \wedge \cdot \vee \cdot \{U(\delta_f, f) | f < g\}$. Then if $V \in \mathscr{V}$,

$$\operatorname{card}\{\delta \in (0, 1) | \operatorname{cl}_{\alpha X}(U(\delta, g)) \cap \operatorname{cl}_{\alpha X}(V) \cap M \neq \operatorname{cl}_{\alpha X}(U(\delta, g) \cap V) \cap M\} < \aleph_0$$

by lemma 2.4 and consequently

$$\operatorname{card}\bigcup_{V \in \mathscr{V}} \{\delta \in (0, 1) | \operatorname{cl}_{\alpha X}(U(\delta, g)) \cap \operatorname{cl}_{\alpha X}(V) \cap M \neq$$
$$\neq \operatorname{cl}_{\alpha X}(U(\delta, g) \cap V) \cap M\} < 2^{\aleph_0}.$$

From lemma 2.3 it now follows that there exists a $\delta_0 \in (0, 1)$ such that for all

$$V \in \mathscr{V} : \operatorname{cl}_{\alpha X}(U(\delta_0, g)) \cap \operatorname{cl}_{\alpha X}(V) \cap M = \operatorname{cl}_{\alpha X}(U(\delta_0, g) \cap V) \cap M$$

and $\operatorname{cl}_{\alpha X}(g^{-1}[0, \delta_0] \cap X) \cap M = \operatorname{cl}_{\alpha X}(U(\delta_0, g)) \cap M$. Define $\delta_g = \delta_0$. This completes the construction of the $\delta_f (f \in \mathscr{F})$. Now define for each $f \in \mathscr{F}$ $H_f = f^{-1}[0, \delta_f] \cap X$. Notice that $H_f \in Z(X)$ for all $f \in \mathscr{F}$. Finally define $\mathscr{H} = \{H_f | f \in \mathscr{F}\}$ and

$$\mathscr{L} = \{Z \in Z(X) | \operatorname{cl}_{\alpha X}(Z) \cap M^* = \emptyset \text{ or } M^* \subset \operatorname{int}_{\alpha X} \operatorname{cl}_{\alpha X}(Z)\} \cup \mathscr{H}.$$

Using the compactness of $\alpha X$ it is easy to show that

$$\wedge \cdot \vee \cdot \{\operatorname{cl}_{\alpha X}(L) | L \in \mathscr{L}\}$$

is a separating ring. We will show that for each finite number of elements $L_0, L_1, ..., L_n \in \mathscr{L}$ the equality

(*) $$\mathrm{cl}_{\alpha X} \left( \bigcap_{i=0}^{n} L_i \right) = \bigcap_{i=0}^{n} \mathrm{cl}_{\alpha X} (L_i)$$

holds, which then proves our theorem.

If $L_i \notin \mathscr{H}$ $(i = 0, 1, ..., n)$ then apply lemma 2.2 and use the analogous equality

(**) $$\mathrm{cl}_{\beta X} \left( \bigcap_{i=0}^{n} L_i \right) = \bigcap_{i=0}^{n} \mathrm{cl}_{\beta X} (L_i)$$

in $\beta X$. Note that equality (**) holds because $L_i \in Z(X)$ $(i = 0, 1, ..., n)$. So it suffices to prove equality (*) in case $L_1, L_2, ..., L_n \in \mathscr{H}$ and $L_0 \notin \mathscr{H}$ (if all $L_i \in \mathscr{H}$ then enlarge $\{L_0, L_1, ..., L_n\}$ with $L_{n+1} = X$ and renumber them). Suppose that equality (*) does not hold; then there exists an

$$x_0 \in \bigcap_{i=0}^{n} \mathrm{cl}_{\alpha X} (L_i) \setminus \mathrm{cl}_{\alpha X} \left( \bigcap_{i=0}^{n} L_i \right).$$

We have to consider two cases:

case 1: $\mathrm{cl}_{\alpha X} (L_0) \cap M^* = \emptyset$.

Since

$$x_0 \in \bigcap_{=0}^{n} \mathrm{cl}_{\alpha X} (L_i) \subset \mathrm{cl}_{\alpha X} (L_0)$$

it follows that $x_0 \notin M^*$. Let $Y = \alpha X \setminus M$. Notice that $Y$ is homeomorphic to $\xi^{-1}(Y)$. As

$$x_0 \in \bigcap_{i=0}^{n} \mathrm{cl}_{\alpha X} (L_i) \cap Y = \bigcap_{i=0}^{n} \mathrm{cl}_Y (L_i) \text{ (lemma 2.1)}$$

$$= \mathrm{cl}_{\alpha X} \left( \bigcap_{i=0}^{n} L_i \right) \cap Y \subset \mathrm{cl}_{\alpha X} \left( \bigcap_{i=0}^{n} L_i \right),$$

this is a contradiction.

case 2: $M^* \subset \mathrm{int}_{\alpha X} \mathrm{cl}_{\alpha X} (L_0)$.

Let $L_i = f_i^{-1}[0, \delta_{f_i}] \cap X$ $(i = 1, 2, ..., n)$.

If $x_0 \notin M$ then use the same technique as in case 1 in order to derive a contradiction. Next, suppose $x_0 \in M$; then

$$x_0 \in \bigcap_{i=1}^{n} \mathrm{cl}_{\alpha X} (f_i^{-1}[0, \delta_{f_i}]) \cap X) \cap \mathrm{cl}_{\alpha X} (L_0) \cap M$$

and consequently ((i))

$$x_0 \in \bigcap_{i=1}^{n} \mathrm{cl}_{\alpha X}\left(U(\delta_{f_i}, f_i)\right) \cap \mathrm{cl}_{\alpha X}(L_0) \cap M = \qquad ((ii))$$

$$= \mathrm{cl}_{\alpha X}\left(\bigcap_{i=1}^{n} U(\delta_{f_i}, f_i)\right) \cap \mathrm{cl}_{\alpha X}(L_0) \cap M =$$

$$= \mathrm{cl}_{\alpha X}\left(\bigcap_{i=1}^{n} U(\delta_{f_i}, f_i)\right) \cap \mathrm{int}_{\alpha X}\,\mathrm{cl}_{\alpha X}(L_0) \cap M =$$

$$\subset \mathrm{cl}_{\alpha X}\left(\bigcap_{i=1}^{n} U(\delta_{f_i}, f_i) \cap \mathrm{int}_{\alpha X}\,\mathrm{cl}_{\alpha X}(L_0)\right) \cap M =$$

$$= \mathrm{cl}_{\alpha X}\left(\bigcap_{i=1}^{n} U(\delta_{f_i}, f_i) \cap \mathrm{int}_{\alpha X}\,\mathrm{cl}_{\alpha X}(L_0) \cap X\right) \cap M \subset$$

$$\subset \mathrm{cl}_{\alpha X}\left(\bigcap_{i=0}^{n} L_i\right) \cap M \subset$$

$$\subset \mathrm{cl}_{\alpha X}\left(\bigcap_{i=0}^{n} L_i\right),$$

which is a contradiction. $\qquad \square$

Since separable metric spaces and countable spaces are Lindelöf semi-stratifiable we have the following corollaries:

COROLLARY 2.8 ([1], [9]): *Every metric compactification is a Wallman compactification.*

COROLLARY 2.9 ([11]): *Every countable multiple point compactification is a z-compactification.*

## 3. REGULAR WALLMAN SPACES

Let $X$ be a topological space which is strongly $\aleph_1$ compact and let $\alpha X$ be a compactification of $X$ such that the collection of mutiple points of $\alpha X$ is also strongly $\aleph_1$ compact. We will show that $X$ is regular Wallman (a topological space is called *regular Wallman* if it is compact and possesses a separating ring consisting of regular closed sets. It is known that a regular Wallman space is a Wallman compactification of each dense subspace ([12])). If $B \subset X$, let $\bar{B}$ denote the closure of $B$ in $X$. We need a simple lemma.

LEMMA 3.1: *Let $U$ and $V$ be open subsets of $\alpha X$ such that*

(i) $\overline{U \cap X} \cap \overline{V \cap X} = \overline{U \cap V \cap X}$

(ii) $\mathrm{cl}_{\alpha X}(U) \cap \mathrm{cl}_{\alpha X}(V) \cap M = \mathrm{cl}_{\alpha X}(U \cap V) \cap M$

*then also*

$$\mathrm{cl}_{\alpha X}(U) \cap \mathrm{cl}_{\alpha X}(V) = \mathrm{cl}_{\alpha X}(U \cap V).$$

PROOF: Suppose to the contrary there exists an

$$x_0 \in (\mathrm{cl}_{\alpha X}(U) \cap \mathrm{cl}_{\alpha X}(V)) \backslash \mathrm{cl}_{\alpha X}(U \cap V)).$$

Let $Y = \alpha X \backslash M$. Since $X$ is hereditarily Lindelöf, every closed subset of $X$ is a $G_\delta$ and consequently

$$\mathrm{cl}_{\beta X}(\overline{U \cap X}) \cap \mathrm{cl}_{\beta X}(\overline{V \cap X}) = \mathrm{cl}_{\beta X}(\overline{U \cap X} \cap \overline{V \cap X}) = \mathrm{cl}_{\beta X}(\overline{U \cap V \cap X}).$$

Hence it follows that

$$\mathrm{cl}_Y(U \cap X) \cap \mathrm{cl}_Y(V \cap X) = \mathrm{cl}_Y(U \cap V \cap X)$$

and therefore $x_0 \notin Y$. However it is clear that $x_0 \notin M$. Contradiction. □

THEOREM 3.2: *Any compactification of a strongly $\aleph_1$ compact space in which the collection of multiple points is also strongly $\aleph_1$ compact, is regular Wallman.*

COROLLARY 3.3: *Any compactification of a Lindelöf semi-stratifiable space in which the collection of multiple points is also Lindelöf semi-stratifiable, is regular Wallman.*

PROOF: Since $X$ is separable it follows that the weight of $\alpha X$ is less than or equal to $2^{\aleph_0}$. Let $\mathscr{B}$ be an open basis for $\alpha X$ such that card $(\mathscr{B}) \leqslant \leqslant 2^{\aleph_0}$. Define

$$\mathscr{C} = \{(\mathrm{cl}_{\alpha X}(B_0), \mathrm{cl}_{\alpha X}(B_1)) | B_0, B_1 \in \mathscr{B} \text{ and } \mathrm{cl}_{\alpha X}(B_0) \cap \mathrm{cl}_{\alpha X}(B_1) = \emptyset\}.$$

For each $(\mathrm{cl}_{\alpha X}(B_0), \mathrm{cl}_{\alpha X}(B)) \in \mathscr{C}$ choose an $f \in C(\alpha X, I)$ such that

$$f(\mathrm{cl}_{\alpha X}(B_0)) = 0 \text{ and } f(\mathrm{cl}_{\alpha X}(B_1)) = 1.$$

Let $\mathscr{F}$ denote the set of mappings obtained in this way and assume that $\mathscr{F}$ is most economically well-ordered (denote the order by $<$). Note that card $(\mathscr{F}) \leqslant 2^{\aleph_0}$. By transfinite induction we can construct, in a similar manner as in theorem 2.6, for each $f \in \mathscr{F}$ a $\delta_f \in (0, 1)$ such that

(i) $\mathrm{cl}_{\alpha X}(U(\delta_f, f)) \cap \mathrm{cl}_{\alpha X}(V) \cap M = \mathrm{cl}_{\alpha X}(U(\delta_f, f) \cap V) \cap M$
    for all $V \in \wedge \cdot \vee \cdot \{U(\delta_g, g) | g < f\}$.
(ii) $\overline{U(\delta_f, f) \cap V \cap X} = \overline{U(\delta_f, f) \cap X} \cap \overline{V \cap X}$
    for all $V \in \wedge \cdot \vee \cdot \{U(\delta_g, g) | g < f\}$.

Here we use lemma 2.4 in case $A = X$. From lemma 3.1 we deduce that

$$\wedge \cdot \vee \cdot \{\mathrm{cl}_{\alpha X}(U(\delta_f, f)) | f \in \mathscr{F}\}$$

is a separating ring of regular closed sets in $\alpha X$. □

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*Free University*
*De Boelelaan 1081*
*Amsterdam*

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