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MATHEMATICS
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A Countable Space No Compactification of Which Is Supercompact

by

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Summary. We will show that the space $N \cup \{p\}$, the subspace of $\beta N \setminus N$ where p is a P-point of $\beta N \setminus N$, has no supercompact compactification.

All topological spaces under discussion are assumed to be Tychonoff. A topological space X is called *supercompact* provided that it possesses an open subbase $\mathscr{U}$ such that each covering of X by elements of $\mathscr{U}$ contains a subcover consisting of two elements of $\mathscr{U}$ (De Groot [6]). A supercompact space is compact; supercompactness is a topological invariant; it is productive and the class of all supercompact spaces contains the compact metric spaces [10], compact orderable spaces and compact tree-like spaces [4], [7]. Not all compact Hausdorff spaces are supercompact, as was shown by Bell [1].

Let X be a supercompact space. The supercompactness of X can also be described in terms of a closed subbase. A space is supercompact iff it possesses a closed subbase $\mathscr{S}$ such that each linked subsystem $\mathscr{M} \subset \mathscr{S}$ has a nonvoid intersection ($\mathscr{M}$ is called *linked* if every two of its members meet). Such a subbase is called *binary*.

A point p of a topological space X is called a *P-point* if the intersection of countably many neighborhoods of p is again a neighborhood of p.

If $\mathscr{S}$ is a closed subbase for the topological space X, then we will write $\vee.\mathscr{S}$ for the family of finite unions of elements of $\mathscr{S}$ and $\wedge.\mathscr{S}$ for the family of finite intersections of elements of $\mathscr{S}$. The family $\vee.\wedge.\mathscr{S} = \wedge.\vee.\mathscr{S}$ is closed both under finite intersections and finite unions; it is called the *ring* generated by $\mathscr{S}$. If X is compact, then for all open $U, V \subset X$ with $\mathrm{cl}_X(U) \subset V$ there exists a $T \in \vee.\wedge.\mathscr{S}$, such that $\mathrm{cl}_X(U) \subset T \subset V$, as can easily be seen. If $\mathscr{S}$ is binary, then for all $B \subset X$ define

$$I(B) := \cap \{S \in \mathscr{S} \mid B \subset S\}.$$

Notice that $B \subset \mathrm{cl}_X(B) \subset I(B) = I(I(B))$ for all $B \subset X$ and that $A \subset B$ implies that $I(A) \subset I(B)$.

THEOREM. *Let p be a P-point in $\beta N \setminus N$. Then the subspace $N \cup \{p\}$ of βN has the property that no compactification of it is supercompact.*

Proof. Define $X=N\cup\{p\}$. Let αX be a compactification of X and let $f\colon \beta X=\beta N\to\alpha X$ be the unique projection mapping which extends id_X. Assume that $\mathscr{S}$ is a binary closed subbase for αX; define

$$A:=\{n\in N\mid I(\{p,n\})\cap(\alpha X\setminus X)\neq\emptyset\}.$$

For all $n\in A$ choose an $x_n\in I(\{p,n\})\cap(\alpha X\setminus X)$ and let $B=\{x_n\mid n\in A\}$. As $f^{-1}(B)$ is an F_σ and as p is a P-point in $\beta N\setminus N$ it follows that

$$p\notin \mathrm{cl}_{\beta N} f^{-1}(B)$$

and consequently $p\notin f\big(\mathrm{cl}_{\beta N} f^{-1}(B)\big)$, for if not, then $f^{-1}(p)$ consists of more than one point, which is a contradiction. Now, as $B\subset f\big(\mathrm{cl}_{\beta N} f^{-1}(B)\big)$ and as f is a closed mapping, we conclude that $p\notin \mathrm{cl}_{\alpha X} B$. Choose open sets $U, V\subset\alpha X$ such that $p\in U\subset \mathrm{cl}_{\alpha X} U\subset V$ and $V\cap \mathrm{cl}_{\alpha X} B=\emptyset$. Let

$$T=\bigcup_{i=1}^{n}\bigcap_{j=1}^{n} S_{ij}$$

be an element of $\wedge.\vee.\mathscr{S}$ ($S_{ij}\in\mathscr{S}$, $i,j\in\{1,2,\ldots,n\}$) such that $\mathrm{cl}_{\alpha X} U\subset T\subset V$. Then

$$p\in \mathrm{cl}_{\alpha X} U=\mathrm{cl}_{\alpha X}(U\cap N)=\bigcup_{i=1}^{n}\mathrm{cl}_{\alpha X}\left(U\cap N\cap\bigcap_{j=1}^{n} S_{ij}\right),$$

and consequently there exists an $i_0\in\{1,2,\ldots,n\}$ such that

$$p\in \mathrm{cl}_{\alpha X}\left(U\cap N\cap\bigcap_{j=1}^{n} S_{i_0 j}\right).$$

Define $M=U\cap N\cap\bigcap_{j=1}^{n} S_{i_0 j}$. Then M is infinite and

$$p\in \mathrm{cl}_{\alpha X} M\subset I(M)\subset\bigcap_{j=1}^{n} S_{i_0 j}\subset V.$$

Take $m\in M$. Then $I(\{p,m\})\subset I(M)$ and therefore $I(\{p,m\})\cap(\alpha X\setminus X)=\emptyset$, for if not, then $x_m\in I(\{p,m\})\cap B\subset I(M)\cap B\subset V\cap B=\emptyset$. Consequently, $I(\{p,m\})$ if finite, since if $I(\{p,m\})$ is infinite then $I(\{p,m\})\cap N$ is infinite and as $(I(\{p,m\})\cap N)\cup\{p\}$ is not a convergent sequence

$$\emptyset\neq \mathrm{cl}_{\alpha X}\big(I(\{p,m\})\cap N\big)\cap(\alpha X\setminus X)\subset I(\{p,m\})\cap(\alpha X\setminus X),$$

which is a contradiction.

For every $\kappa\leqslant\omega_1$ now define a finite subset $A(\kappa)$ of M such that

(i) If $p\in \mathrm{cl}_{\alpha X}\bigcup_{\mu<\kappa} A(\mu)$ then $A(\kappa)=\emptyset$.

(ii) If $p\notin \mathrm{cl}_{\alpha X}\bigcup_{\mu<\kappa} A(\mu)$ then $A(\kappa)\neq\emptyset$ and $I(A(\kappa)\cup\{p\})=A(\kappa)\cup\{p\}$ and $A(\kappa)\cap\bigcup_{\mu<\kappa} A(\mu)=\emptyset$.

Take a point $m\in M$ and define $A(0)=I(\{p,m\})\cap N$. Then $A(0)$ has all desired properties. Suppose that all $A(\mu)$ have been constructed for $\mu<\kappa\leqslant\omega_1$. Assume that $p \notin \mathrm{cl}_{\alpha X} \bigcup_{\mu<\kappa} A(\mu)$. Obviously, using the same technique as above, there exists an infinite $N_0 \subset M$ such that $p \in \mathrm{cl}_{\alpha X} N_0 \subset I(N_0)$ and $I(N_0) \cap \mathrm{cl}_{\alpha X} \bigcup_{\mu<\kappa} A(\mu) = \emptyset$. Take $n \in N_0$ and define $A(\kappa) = I(\{p, n\}) \cap N$. Then as $I(N_0) \subset I(M)$ the set $A(\kappa)$ has all desired properties.

As there are only countably many finite subsets of M there exists a $\kappa < \omega_1$ such that $p \in \mathrm{cl}_{\alpha X} \bigcup_{\mu<\kappa} A(\mu)$. As $\bigcup_{\mu<\kappa} A(\mu) \cup \{p\}$ is not a convergent sequence there exists a $q \in \mathrm{cl}_{\alpha X} \bigcup_{\mu<\kappa} A(\mu) \cap (\alpha X \setminus X)$; clearly $p \neq q$. Take an open neighborhood 0 of q such that $p \notin 0$ and choose $L \subset \bigcup_{\mu<\kappa} A(\mu)$ such that

$$q \in \mathrm{cl}_{\alpha X} L \subset I(L) \subset 0.$$

Then L is infinite and consequently there exists two different κ_0, κ_1 less than κ such that $L \cap A(\kappa_i) \neq \emptyset$ $(i=0, 1)$. Then the subsystem

$$\mathcal{M} = \{S \in \mathcal{S} \mid A(\kappa_0) \cup \{p\} \subset S \text{ or } A(\kappa_1) \cup \{p\} \subset S \text{ or } L \subset S\}$$

of $\mathcal{S}$ is linked, but has a void intersection. This is a contradiction. ■

Consequently, the existence of P-points in $\beta N \setminus N$, cf. [8], [3]; also [2] and [5], implies that there is a countable space no compactification of which is supercompact. Also, the scattered compactification of $N \cup \{p\}$, where p is a P-point in $\beta N \setminus N$, described by Ryll-Nardzewski and Telgársky [9] is not supercompact. This compactification provides for an interesting and presumably new example of a non-supercompact space.

We do not have an example of a metrizable space no compactification of which is supercompact.

Question *Is there a metrizable space no compactification of which is supercompact?*

Added in proof. Recently, Shelah, has shown that it is consistent to assume that there are no P-points in $\beta N \setminus N$. In addition, both van Douwen and Mills have found elementary proofs of the fact that every compact metric space is supercompact.

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Я. ван Милл, **Счетное пространство, никакая компактификация которого не является суперкомпактной**

Содержание. Показывается, что пространство $N \cup \{p\}$, подпространство $\beta N \setminus N$, где p является P-точкой $\beta N \setminus N$ не имеет суперкомпактной компактификации.