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Relations between $\beta X \setminus X$ and a Certain Subspace of λX

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1. Introduction

The Čech-Stone compactification $\beta\mathbb{N}$ of the natural numbers $\mathbb{N}$ is extensively studied in general topology because of its nice topological properties and also because of its relations to set theory. This space has as underlying set the set of all ultrafilters on $\mathbb{N}$ topologized by the Wallman topology. Another type of extension, called the superextension $\lambda\mathbb{N}$ of $\mathbb{N}$, is the space with underlying set the set of all maximal *linked* systems in $\mathscr{P}(\mathbb{N})$ (i.e. systems maximal with respect to the property that every two of its members meet) also topologized by a natural Wallman topology.

Verbeek [19] has shown that $\lambda\mathbb{N}$ is a compact totally disconnected superspace of $\mathbb{N}$ which contains $\beta\mathbb{N}$ as a subspace but which is not homeomorphic to $\beta\mathbb{N}$. Also $\lambda\mathbb{N}$ possesses a countable dense set of isolated points and has weight $\mathfrak{c}$. In this paper we will derive more topological properties of $\lambda\mathbb{N}$ in order to show its relation and its difference with $\beta\mathbb{N}$.

We are indebted to Eric van Douwen for some helpful suggestions.

2. Superextensions

All topological spaces under discussion are assumed to be T_1. Let X be a topological space and let $\mathscr{S}$ be a closed subbase for X. Then $\mathscr{S}$ is defined to be a

(i) T_1-*subbase* if for each $x_0 \in X$ and $S \in \mathscr{S}$ such that $x_0 \notin S$ there exists a $T \in \mathscr{S}$ such that $x_0 \in T$ and $S \cap T = \varnothing$;

(ii) *normal* subbase if for each $S_0, T_0 \in \mathscr{S}$ with $S_0 \cap T_0 = \varnothing$ there exist $S_1, T_1 \in \mathscr{S}$ such that $S_0 \cap T_1 = \varnothing = T_0 \cap S_1$ and $S_1 \cup T_1 = X$.

A subsystem $\mathscr{M} \subset \mathscr{S}$ is called a *linked system* (ls), if every two of its members meet. A linked system $\mathscr{M} \subset \mathscr{S}$ is called a *maximal linked* system, or mls, if it is not properly contained in any other linked system $\mathscr{N} \subset \mathscr{S}$. By Zorn's lemma each linked system $\mathscr{M} \subset \mathscr{S}$ is contained in at least one maximal linked system $\mathscr{N} \subset \mathscr{S}$. The following simple lemma is due to Verbeek [19].

Lemma 2.1. *Let* $\mathcal{M}_0, \mathcal{M}_1$ *be* mls's *in* $\mathcal{S}$. *Then*

(a) $\varnothing \notin \mathcal{M}_0$;

(b) *If* $S \in \mathcal{M}_0, T \in \mathcal{S}$ *and* $S \subset T$ *then* $T \in \mathcal{M}_0$;

(c) *If* $S \in \mathcal{S} \setminus \mathcal{M}_0$ *then there is a* $T \in \mathcal{M}_0$ *such that* $S \cap T = \varnothing$;

(d) $\mathcal{M}_0 \neq \mathcal{M}_1$ *iff there are* $M_i \in \mathcal{M}_i$ $(i \in \{0, 1\})$ *such that* $M_0 \cap M_1 = \varnothing$;

(e) *If* $S, T \in \mathcal{S}$ *and* $S \cup T = X$ *then* $S \in \mathcal{M}_0$ *or* $T \in \mathcal{M}_0$.

The above lemma shows that maximal linked systems in some respect behave like ultrafilters.

Define: $\lambda_{\mathcal{S}}(X) = \{\mathcal{M} \subset \mathcal{S} \mid \mathcal{M}$ is an mls in $\mathcal{S}\}$. If $\mathcal{S}$ is a T_1-subbase then for each $x \in X$ we have that $\mathcal{M}_x = \{S \in \mathcal{S} \mid x \in S\}$ is an mls; the map i: $X \to \lambda_{\mathcal{S}}(X)$ defined by $\mathbf{i}(x) := \mathcal{M}_x$ is one to one. For $A \subset X$ we define

$$A^+ := \{\mathcal{M} \in \lambda_{\mathcal{S}}(X) \mid A \text{ contains a member of } \mathcal{M}\}.$$

Lemma 2.2. (a) *If* $A \subset B \subset X$ *then* $A^+ \subset B^+$;

(b) *If* $A, B \subset X$ *and* $A \cap B = \varnothing$ *then* $A^+ \cap B^+ = \varnothing$;

(c) *If* $S, T \in \mathcal{S}$ *then* $S \cap T = \varnothing$ *iff* $S^+ \cap T^+ = \varnothing$;

(d) *If* $S, T \in \mathcal{S}$ *then* $S \cup T = X$ *iff* $S^+ \cup T^+ = \lambda_{\mathcal{S}}(X)$;

(e) *If* $S \in \mathcal{S}$ *then* $S^+ \cup (X \setminus S)^+ = \lambda_{\mathcal{S}}(X)$.

For a proof of this simple lemma see Verbeek [19]. As a closed subbase for a topology $\lambda_{\mathcal{S}}(X)$ we take

$$\mathcal{S}^+ = \{S^+ \mid S \in \mathcal{S}\}.$$

With this topology $\lambda_{\mathcal{S}}(X)$ is called *the superextension of X relative the subbase* $\mathcal{S}$. In case $\mathcal{S}$ consists of all the closed subsets of X, then $\lambda_{\mathcal{S}}(X)$ is denoted by λX and is called *the* superextension of X. It is easy to see, using Lemma 2.2, that $\lambda_{\mathcal{S}}(X)$ always is T_1, is Hausdorff if $\mathcal{S}$ is normal and that in case $\mathcal{S}$ is a T_1-subbase the function **i** described above is a homeomorphism. Moreover it is easy to show that the closed subbase $\mathcal{S}^+$ has the property that each linked subsystem of it has a nonvoid intersection and hence that, by Alexander's subbase lemma, $\lambda_{\mathcal{S}}(X)$ is compact. A closed subbase with the property that each linked subsystem of it has a nonvoid intersection is called *binary* and a topological space which admits such a subbase is called *supercompact*. The class of supercompact spaces was introduced by de Groot [8]. Clearly each superextension is supercompact. It is known that many spaces are supercompact (for example all compact metric spaces, cf. Strok and Szymański [18]) and that many spaces are not supercompact (for example $\beta\mathbb{N}$, cf. Bell [1], or, more generally, all infinite F-spaces, cf. van Douwen and van Mill [6]).

An mls $\mathcal{M} \in \lambda X$ is said to be *defined on a set* $N \subset X$ if for all $M \in \mathcal{M}$ there exist an $M' \in \mathcal{M}$ such that $M' \subset M \cap N$ (Verbeek [19]). We say that N is a *defining set* for $\mathcal{M}$. Let $\lambda_f(X)$ denote the set of all $\mathcal{M} \in \lambda X$ which have a finite defining set. Verbeek [19] showed that $\lambda_f(X)$ is dense in X and that a point $\mathcal{M} \in \lambda X$ is isolated iff $\mathcal{M}$ has a defining set M consiting of finitely many isolated points of X. Combining these two results it follows that $\lambda\mathbb{N}$, the superextension of the natural numbers, possesses a dense set of countably many isolated points. Therefore

$\lambda\mathbb{N}$ can considered to be a compactification of $\mathbb{N}$ (Note that the set $\mathbf{i}[\mathbb{N}]$ is not dense in $\lambda\mathbb{N}$). Consequently $\lambda\mathbb{N}$ is a continuous image of $\beta\mathbb{N}$; however $\beta\mathbb{N}$ is not a continuous image of $\lambda\mathbb{N}$ (van Douwen and van Mill [6]). Also lemma 2.2 implies that $\lambda\mathbb{N}$ is totally disconnected and weight $(\lambda\mathbb{N})=\mathfrak{c}$. The space $\lambda\mathbb{N}\setminus\lambda_f\mathbb{N}$ is compact and possesses points with a countable neighborhood basis and points without a countable neighborhood basis. For example

$$\mathcal{M}=\{M\subset\mathbb{N}\mid\exists i>1:\{1,i\}\subset M \quad\text{or}\quad \{2,3,\ldots\}\subset M\}$$

easily can be seen to be an mls with a countable neighborhood basis. Theorem 2.9 will provide us with $2^{\mathfrak{c}}$ points in $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ without countable neighborhood basis. Hence $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ is not homogeneous and has less in common with $\beta\mathbb{N}\setminus\mathbb{N}$. Yet $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ contains a homeomorph of $\beta\mathbb{N}\setminus\mathbb{N}$. Verbeek [19] showed that $\mathbf{i}[\mathbb{N}]$ (which will be identified with $\mathbb{N}$ from now on) is C^*-embedded in $\lambda\mathbb{N}$ and hence that the closure of $\mathbb{N}$ in $\lambda\mathbb{N}$ is $\beta\mathbb{N}$. Also it is easily follows that $\mathrm{cl}_{\lambda\mathbb{N}}[\mathbb{N}]\setminus\mathbb{N}\subset\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ and hence that $\beta\mathbb{N}\setminus\mathbb{N}\subset\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$. It must be noticed that there also exists an other proof of this fact. It can easily be seen that each ultrafilter on $\mathbb{N}$ is a maximal linked system and hence that $\beta\mathbb{N}$, as a set, is contained in $\lambda\mathbb{N}$. Also the subspace topology for $\beta\mathbb{N}$ clearly coincides with the usual Wallman topology on $\beta\mathbb{N}$ so that the topological space $\beta\mathbb{N}$ is a subspace of $\lambda\mathbb{N}$. Therefore $\mathrm{cl}_{\lambda\mathbb{N}}[\mathbb{N}]=\beta\mathbb{N}$, since the compactness of $\beta\mathbb{N}$ implies that $\beta\mathbb{N}$ is closed in $\lambda\mathbb{N}$.

The following subspace of $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$, wich seems to be closer to $\beta\mathbb{N}\setminus\mathbb{N}$ than $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$, also fails to look like $\beta\mathbb{N}\setminus\mathbb{N}$; define

$$\sigma(\mathbb{N}):=\{\mathcal{M}\in\lambda\mathbb{N}\mid\mathcal{M}\text{ contains no finite set}\}.$$

Unfortunately $\sigma(\mathbb{N})$ is separable, because of the following lemma, while $\beta\mathbb{N}\setminus\mathbb{N}$ is not.

Lemma 2.3. *$\sigma(\mathbb{N})$ is a retract of $\lambda\mathbb{N}$.*

Proof. Let $\mathcal{A}=\{A\subset\mathbb{N}\mid|\mathbb{N}\setminus A|<\omega\}$. Then $\sigma(\mathbb{N})=\cap\{A^+\mid A\in\mathcal{A}\}$ and by theorem 3.2(ii) $\sigma(\mathbb{N})$ is a retract of $\lambda\mathbb{N}$. □

The subspace $\Sigma(\mathbb{N})=\{\mathcal{M}\in\lambda\mathbb{N}\mid$ for all $M_0,M_1\in\mathcal{M}:|M_0\cap M_1|=\omega\}$ of $\lambda\mathbb{N}\setminus\lambda_f(\mathbb{N})$ is another candidate for a substitute of $\beta\mathbb{N}\setminus\mathbb{N}$. This seems to be the right subspace. More general, for each topological space X define

$$\Sigma(X)=\{\mathcal{M}\in\lambda X\mid\text{for all } M_0,M_1\in\mathcal{M}: M_0\cap M_1 \text{ is not compact}\}.$$

Theorem 2.4. *Let X be a normal topological space. Then*

(i) $\Sigma(X)\subset\lambda X\setminus\lambda_f(X)$;

(ii) *$\Sigma(X)$ is compact iff X is locally compact;*

(iii) *If X is locally compact then $\Sigma(X)$ is homeomorphic to $\lambda(\beta X\setminus X)$.*

Proof. (i) is trivial. To prove (ii) assume that $\Sigma(X)$ is compact. Notice that βX is closed in λX and consequently $\beta X\setminus X$ is closed in $\lambda X\setminus\lambda_f(X)$. Therefore, as $\beta X\setminus X\subset\Sigma(X)$, $\beta X\setminus X$ is closed in $\Sigma(X)$ too. It follows that $\beta X\setminus X$ is compact and consequently X is locally compact.

The counterpart of (ii) follows from (iii), since $\lambda(\beta X \setminus X)$ is compact. To prove (iii), assume that X is locally compact. For each closed subset $M \subset X$ define $M^* = \mathrm{cl}_{\beta X}(M) \setminus M$. Then $\{M^* \mid M$ is closed in $X\}$ is a closed base for the topology of $\beta X \setminus X$, closed under finite intersections and finite unions. Define a mapping $\varphi: \lambda(\beta X \setminus X) \to \Sigma(X)$ by

$$\varphi(\mathcal{M}) := \{M \subset X \mid M^* \in \mathcal{M}\}.$$

First we will show that φ is well-defined. Clearly $\varphi(\mathcal{M})$ is a linked system for all $\mathcal{M} \in \lambda(\beta X \setminus X)$. Suppose that $\varphi(\mathcal{M})$ is not a *maximal* linked system for some $\mathcal{M} \in \lambda(\beta X \setminus X)$. Then there exists a closed set $A \subset X$ such that $\varphi(\mathcal{M}) \cup \{A\}$ is linked while $A \notin \varphi(\mathcal{M})$. Then $A^* \notin \mathcal{M}$ and consequently there exists an $M \in \mathcal{M}$ such that $A^* \cap M = \varnothing$. By the compactness of $\beta X \setminus X$ there is a closed subset $B \subset X$ such that $M \subset B^*$ and $B^* \cap A^* = \varnothing$. As $M \in \mathcal{M}$ it follows that $B^* \in \mathcal{M}$ and consequently $B \in \varphi(\mathcal{M})$. Therefore $B \cap A \neq \varnothing$. But $B^* \cap A^* = \varnothing$ implies that $B \cap A$ is compact. Choose a relatively compact neighborhood U of $A \cap B$ and define $C = B \setminus U$. Then $C^* = B^*$ and consequently also $C \in \varphi(\mathcal{M})$. This is a contradiction since $C \cap A = \varnothing$. Also it is clear that $\varphi(\mathcal{M}) \in \Sigma(X)$ for take $M, N \in \varphi(\mathcal{M})$ such that $M \cap N$ is compact. Then $M^* \cap N^* = \varnothing$ and consequently $\mathcal{M}$ is not linked. Contradiction.

Let B be a closed subset of X. Then

$$\begin{aligned} \mathcal{M} \in \varphi^{-1}[B^+ \cap \Sigma(X)] \quad &\text{iff } \varphi(\mathcal{M}) \in B^+ \cap \Sigma(X) \quad \text{iff } \varphi(\mathcal{M}) \in B^+ \\ &\text{iff } B^* \in \mathcal{M} \quad \text{iff } \mathcal{M} \in (B^*)^+. \end{aligned}$$

Therefore $\varphi^{-1}[B^+ \cap \Sigma(X)] = (B^*)^+$ (the first "plus" is taken in λX, the second in $\lambda(\beta X \setminus X)$!) showing that φ is continuous. Also it is not difficult to see that φ is one to one and surjective. Clearly $\lambda(\beta X \setminus X)$ and $\Sigma(X)$ both are Hausdorff spaces (Verbeek [19]). It follows that φ is a homeomorphism. □

Remark. The present proof of Theorem 2.4 (ii) is due to van Douwen; he discovered that our original proof was incorrect.

The space $\Sigma(\mathbb{N})$ hence is a homeomorph of $\lambda(\beta\mathbb{N} \setminus \mathbb{N})$. We prefer to study $\lambda(\beta\mathbb{N} \setminus \mathbb{N})$ as a subspace of $\lambda\mathbb{N}$. It must be noticed that the proof of the above theorem shows that $\{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$ is a binary closed subbase for $\Sigma(\mathbb{N})$ and hence that $\Sigma(\mathbb{N})$ is supercompact (for this fact there is also an elementary proof). This shows that $\Sigma(\mathbb{N})$ and $\beta\mathbb{N} \setminus \mathbb{N}$ are not homeomorphic since each infinite supercompact Hausdorff space contains nontrivial convergent sequences (van Douwen and van Mill [6]). We will now derive some properties of $\Sigma(\mathbb{N})$ (and hence of $\lambda(\beta\mathbb{N} \setminus \mathbb{N})$).

Lemma 2.5. *The cellularity of* $\Sigma(\mathbb{N})$ *is* $\mathfrak{c}$.

Proof. Let $\{A_\alpha \mid \alpha \in \mathfrak{c}\}$ be an almost disjoint collection of infinite subsets of N, i.e. for all $\alpha \in \mathfrak{c}$ we have $|A_\alpha| = \omega$ and $\alpha < \beta < \mathfrak{c}$ implies $|A_\alpha \cap A_\beta| < \omega$ (there is such a collection, see Gillman and Jerison [7]). Then $\{A_\alpha^+ \cap \Sigma(\mathbb{N}) \mid \alpha \in \mathfrak{c}\}$ is a collection of $\mathfrak{c}$ pairswise disjoint open subsets of $\Sigma(\mathbb{N})$. For take $\alpha < \beta < \mathfrak{c}$ and $\mathcal{M} \in A_\alpha^+ \cap A_\beta^+ \cap \Sigma(\mathbb{N})$. Then $|A_\alpha \cap A_\beta| = \omega$, since $\mathcal{M} \in \Sigma(\mathbb{N})$. Contradiction. Since the weight of $\lambda\mathbb{N}$ is $\mathfrak{c}$, the weight of $\Sigma(\mathbb{N})$ also equals $\mathfrak{c}$ ($\beta\mathbb{N}\setminus\mathbb{N}\subset\Sigma(\mathbb{N})$) and the result follows. □

Let κ be any cardinal. The following principle is called $\mathrm{P}(\kappa)$.

Let $\mathscr{A}$ be a collection of fewer than κ subsets of $\mathbb{N}$ such that each finite subcollection of $\mathscr{A}$ has infinite intersection. Then there is an infinite $F\subset\mathbb{N}$ such that $F\setminus A$ is finite for all $A\in\mathscr{A}$.

It is easy to show that $\mathrm{P}(\omega_1)$ holds in ZFC and moreover Martin's axiom (MA) implies $\mathrm{P}(\mathfrak{c})$ (Booth [2]). Moreover $\mathrm{P}(\kappa)$ implies that $2^\lambda=\mathfrak{c}$ for each infinite $\lambda<\kappa$ (Rothberger [14]). In particular, $\mathrm{P}(\omega_2)$ implies the negation of the Continuum Hypothesis.

It is easy to show that $\mathrm{P}(\kappa)$ is equivalent to the statement that each nonvoid intersection of fewer than κ open subsets of $\beta\mathbb{N}\setminus\mathbb{N}$ has nonempty interior. Unfortunately $\mathrm{P}(\kappa)$ does not imply the same property for $\Sigma(\mathbb{N})$. In fact we will show that there is a nonvoid countable intersection of clopen subsets of $\Sigma(\mathbb{N})$ with a void interior. The following lemma shows that $\mathrm{P}(\kappa)$ works for intersections of open sets in $\Sigma(\mathbb{N})$ containing an ultra-filter.

Lemma 2.6 [$\mathrm{P}(\kappa)$]. *Let A be an intersection of fewer than κ open subsets of $\Sigma(\mathbb{N})$. If $A\cap(\beta\mathbb{N}\setminus\mathbb{N})\neq\varnothing$ then there is an infinite $B\subset\mathbb{N}$ such that $B^+\cap\Sigma(\mathbb{N})\subset A$. In particular A has a nonvoid interior.*

Proof. Let $A=\cap\{O_\alpha\mid\alpha\in\beta\}$, where $\beta<\kappa$ and each O_α is open in $\Sigma(\mathbb{N})$. Take a point $\mathscr{F}\in A\cap(\beta\mathbb{N}\setminus\mathbb{N})$. For each $\alpha\in\beta$ choose an $F_\alpha\in\mathscr{F}$ such that $F_\alpha^+\cap\Sigma(\mathbb{N})\subset O_\alpha$. This is possible since it is easy to see that $\{F^+\cap\Sigma(\mathbb{N})\mid F\in\mathscr{F}\}$ is a neighborhood basis for $\mathscr{F}$ in $\Sigma(\mathbb{N})$. Then $\{F_\alpha\mid\alpha\in\beta\}$ is a collection of fewer than κ subsets of $\mathbb{N}$ each finite subcollection of which has infinite intersection. Choose an infinite $B\subset\mathbb{N}$ such that $|B\setminus F_\alpha|<\omega$ for all $\alpha\in\beta$. We will show that

$$B^+\cap\Sigma(\mathbb{N})\subset\cap\{F_\alpha^+\cap\Sigma(\mathbb{N})\mid\alpha\in\beta\}.$$

Indeed, choose a point $\mathscr{M}\in(B^+\cap\Sigma(\mathbb{N}))\setminus(F_\alpha^+\cap\Sigma(\mathbb{N}))$ for some $\alpha\in\beta$. Then $F_\alpha\notin\mathscr{M}$ and consequently $\mathbb{N}\setminus F_\alpha\in\mathscr{M}$. Hence $|B\cap(\mathbb{N}\setminus F_\alpha)|=\omega$, since $\mathscr{M}\in\Sigma(\mathbb{N})$. Contradiction. Therefore $B^+\cap\Sigma(\mathbb{N})\subset A$ and as B is infinite, $B^+\cap\Sigma(\mathbb{N})$ is a nonvoid open set. □

Remark. In the proof of Lemma 2.6 we showed that $A^+\cap\Sigma(\mathbb{N})\subset B^+\cap\Sigma(\mathbb{N})$ iff $|A\setminus B|<\omega$. This is a property of the binary subbase $\{A^+\cap\Sigma(\mathbb{N})\mid A\subset\mathbb{N}\}$. The binary subbase $\{A^+\mid A\subset\mathbb{N}\}$ does not have this property. For example let $A=\{1\}$ and $B=\{1,2\}$. Define an mls $\mathscr{M}\in\lambda\mathbb{N}$ by $\mathscr{M}:=\{C\subset\mathbb{N}\mid\{1,2\}\subset C$ or $\{1,3\}\subset C$ or $\{2,3\}\in C\}$. It is easy to see that $\mathscr{M}$ is an mls. Moreover $\mathscr{M}\in B^+\setminus A^+$ and yet $|B\setminus A|<\omega$.

We will now give an example showing that Lemma 2.6 cannot be sharpened.

Example 2.7. A countable nonvoid intersection of clopen subsets of $\Sigma(\mathbb{N})$ with a void interior.

Inductively we construct a collection $\{A_n\mid n\in\omega\}$ of infinite subsets of $\mathbb{N}$ such that for all $i\in\omega$

(i) $k \leqq l \leqq i \Rightarrow |A_k \cap A_l| = \omega$;

(ii) $k \leqq i \quad \Rightarrow |A_k \setminus \bigcup_{\substack{j \leqq i \\ j \neq k}} A_j| = \omega$;

(iii) $|\mathbb{N} \setminus \bigcup_{j \leqq i} A_j| = \omega$;

(iv) $k < l < m \leqq i \Rightarrow A_k \cap A_l \cap A_m = \varnothing$.

To define A_0 just pick an infinite subset of $\mathbb{N}$ with an infinite complement. Suppose that $\{A_j \mid 0 \leqq j \leqq i\}$ are defined satisfying (i)–(iv). For each $k \leqq i$ choose an infinite $C_k \subset A_k \setminus \bigcup_{\substack{j \leqq i \\ j \neq k}} A_j$ such that also $(A_k \setminus \bigcup_{\substack{j \leqq i \\ j \neq k}} A_j) \setminus C_k$ is infinite. Choose an infinite subset D of $\mathbb{N} \setminus \bigcup_{j \leqq i} A_j$ such that also $(\mathbb{N} \setminus \bigcup_{j \leqq i} A_j) \setminus D$ is infinite. Define $A_{i+1} := \bigcup_{j=0}^{i} C_j \cup D$. Then clearly (i), (ii) and (iii) are satisfied. Take $k, l \leqq i$ such that $k < l$. Then $A_k \cap A_l \cap A_{i+1} = A_k \cap A_l \cap \bigcup_{j=0}^{i} C_j = C_k \cap C_l = \varnothing$. Hence (iv) also is satisfied. We will show that the nonvoid set $\cap \{A_n^+ \mid n \in \omega\}$ has a void interior (that $\cap \{A_n^+ \mid n \in \omega\}$ is nonvoid is trivial since $|A_i \cap A_j| = \omega$ for all $i, j \in \omega$). We prove one more simple lemma.

Lemma 2.8. *Let $M_\alpha \subset \mathbb{N}$ $(\alpha \in \beta)$ such that $\bigcap_{\alpha \in \beta} M_\alpha^+ \cap \Sigma(\mathbb{N}) \neq \varnothing$. Then for all $B \subset \mathbb{N}$ we have $\bigcap_{\alpha \in \beta} M_\alpha^+ \cap \Sigma(\mathbb{N}) \subset B^+ \cap \Sigma(\mathbb{N})$ iff $|M_{\alpha_0} \setminus B| < \omega$ for some $\alpha_0 \in \beta$.*

Proof. If $|M_\alpha \setminus B| < \omega$ for some $\alpha \in \beta$ then $M_\alpha^+ \cap \Sigma(\mathbb{N}) \subset B^+ \cap \Sigma(\mathbb{N})$ (cf. the proof of Lemma 2.6) and consequently $\bigcap_{\alpha \in \beta} M_\alpha^+ \cap \Sigma(\mathbb{N}) \subset B^+ \cap \Sigma(\mathbb{N})$. On the other hand if $|M_\alpha \setminus B| = \omega$ for all $\alpha \in \beta$, then the linked system $\{M_\alpha \mid \alpha \in \beta\} \cup \{\mathbb{N} \setminus B\}$ is a linked system any two members of which meet in an infinite set. Hence this linked system can be extended to a maximal linked system $\mathscr{M} \in \bigcap_{\alpha \in \beta} M_\alpha^+ \cap (\mathbb{N} \setminus B)^+ \cap \Sigma(\mathbb{N})$. Contradiction. □

Now suppose there exists a nonvoid open (in $\Sigma(\mathbb{N})$) set $U \subset \cap \{A_n^+ \mid n \in \omega\} \cap \Sigma(\mathbb{N})$. Without loss of generality $U = \bigcap_{i \leqq n} M_i^+ \cap \Sigma(\mathbb{N})$ for some infinite $M_i \subset \mathbb{N} (i \leqq n)$. Now Lemma 2.8 implies that for each $m \in \omega$ there is a $k(m) \leqq n$ such that $|M_{k(m)} \setminus A_m| < \omega$. Hence there must be a $i \leqq n$ such that $B = \{m \in \omega \mid k(m) = i\}$ is infinite. Choose three elements m_1, m_2, $m_3 \in B$ such that $m_1 < m_2 < m_3$. Then clearly M_i is finite since $A_{m_1} \cap A_{m_2} \cap A_{m_3} = \varnothing$, which is a contradiction. □

Van Douwen has pointed out to me that Lemma 2.6 and Example 2.7 imply that $\Sigma(\mathbb{N})$ is not homogeneous. For take $\mathscr{F} \in \beta\mathbb{N} \setminus \mathbb{N}$ and $\mathscr{L}$ in a nonvoid countable intersection of clopen subsets of $\Sigma(\mathbb{N})$ with a void interior. Then lemma 2.6 implies that there is no autohomeomorphism φ of $\Sigma(\mathbb{N})$ which map $\mathscr{F}$ onto $\mathscr{L}$. The above example shows that nonvoid countable intersections of open sets need not have nonvoid interiors in $\Sigma(\mathbb{N})$. The following theorem in any case implies that such intersections have cardinality $2^{\mathfrak{c}}$ and hence are rather big.

Theorem 2.9. *Let A be a nonvoid countable intersection of open sets in $\Sigma(\mathbb{N})$. Then A contains a homeomorph of $\beta\mathbb{N} \setminus \mathbb{N}$.*

Proof. Take $\mathcal{M} \in A = \bigcap_{i \in \omega} O_i$, where O_i is open in $\Sigma(\mathbb{N})$, and let $M_0^i, \ldots, M_{k(i)}^i$ be elements of $\mathcal{M}$ such that

$$\bigcap_{j \leq k(i)} M_j^{i+} \cap \Sigma(\mathbb{N}) \subset O_i$$

for all $i \in \omega$. Then $\mathcal{B} = \{M_j^i \mid j \leq k(i),\ i \in \omega\}$ is a countable collection of subsets of $\mathbb{N}$ any two members of which meet in an infinite set. If $|\mathbb{N} \setminus B| < \omega$ for all $B \in \mathcal{B}$ then $\Sigma(\mathbb{N}) = \cap \{B^+ \cap \Sigma(\mathbb{N}) \mid B \in \mathcal{B}\} \subset A$ and hence clearly A contains a homeomorph of $\beta\mathbb{N} \setminus \mathbb{N}$. Therefore we may assume that there is a $B_0 \in \mathcal{B}$ such that $|\mathbb{N} \setminus B_0| = \omega$. Define $\mathcal{C} = \{B \cap B_0 \mid B \in \mathcal{B}\}$ Then $\mathcal{C}$ consists of countably many infinite subsets of B_0. List $\mathcal{C}$ as $\{C_i \mid i \in \omega\}$ such that each $C \in \mathcal{C}$ is listed countably many times. Now, by induction, for each $i \in \omega$ pick $p_i, q_i \in C_i$ such that

(i) $p_i \neq q_i$;
(ii) $\{p_i, q_i\} \cap \{p_0, \ldots, p_{i-1}, q_0, \ldots, q_{i-1}\} = \varnothing$.

Define $P = \{p_i \mid i \in \omega\}$ and $Q = \{q_i \mid i \in \omega\}$. Then P and Q are two disjoint infinite sets such that $|P \cap C_i| = |Q \cap C_i| = \omega$ for all $i \in \omega$. Define a retraction $r\colon \Sigma(\mathbb{N}) \to \cap \{B^+ \mid B \in \mathcal{B}\} \cap \Sigma(\mathbb{N})$ by

$$r(\mathcal{N}) := \cap \{N^+ \cap \Sigma(\mathbb{N}) \mid N \in \mathcal{N} \text{ and } |N \cap B| = \omega \text{ for all } B \in \mathcal{B}\} \cap \{B^+ \cap \Sigma(\mathbb{N}) \mid B \in \mathcal{B}\}.$$

That r is a retraction will be shown in Theorem 3.2(ii).

Let $D = \mathbb{N} \setminus B_0$. We will show that $r \restriction \beta D \setminus D$ is a homeomorphism (notice that $\beta D \setminus D \subset \beta\mathbb{N} \setminus \mathbb{N} \subset \Sigma(\mathbb{N})$). Take two ultra-filters $\mathcal{F}_0, \mathcal{F}_1 \in \beta D \setminus D$ such that $\mathcal{F}_0 \neq \mathcal{F}_1$. Then there exist $F_i \in \mathcal{F}_i$ such that $F_i \subset D (i \in \{0, 1\})$ and $F_0 \cap F_1 = \varnothing$. Clearly $F_0 \cup P \in \mathcal{F}_0$, $F_1 \cup Q \in \mathcal{F}_1$ and $(F_0 \cup P) \cap (F_1 \cup Q) = \varnothing$. Also $|(F_0 \cup P) \cap B| = \omega = |(F_1 \cup Q) \cap B|$ for all $B \in \mathcal{B}$. Hence $r(\mathcal{F}_0) \in (F_0 \cup P)^+$ and $r(\mathcal{F}_1) \in (F_1 \cup Q)^+$. But $(F_0 \cup P)^+ \cap (F_1 \cup Q)^+ = \varnothing$ and consequently $r(\mathcal{F}_0) \neq r(\mathcal{F}_1)$. Hence $r \restriction \beta D \setminus D$ is one to one and consequently $r \restriction \beta D \setminus D$ is a homeomorphism. □

Corollary 2.10. *No $p \in \Sigma(\mathbb{N})$ admits a countable neighborhood basis.*

A well-known property of $\beta\mathbb{N} \setminus \mathbb{N}$ is, under P($\mathfrak{c}$), that each nonvoid open set contains $2^{\mathfrak{c}}$ $P_{\mathfrak{c}}$-points (see e.g. van Douwen [4]). Recall that a point p of a topological space is called a *$P_{\mathfrak{c}}$-point* if the intersection of less than $\mathfrak{c}$ neighborhoods of p is again a neighborhood of p. We will show that each nonvoid open set in $\Sigma(\mathbb{N})$ also contains $2^{\mathfrak{c}}$ $P_{\mathfrak{c}}$-points.

Theorem 2.11 [P($\mathfrak{c}$)]. *Each nonvoid open set in $\Sigma(\mathbb{N})$ contains $2^{\mathfrak{c}}$ $P_{\mathfrak{c}}$-points*

Proof. Let $A = \{\mathcal{F} \in \beta\mathbb{N} \setminus \mathbb{N} \mid \mathcal{F}$ is a $P_{\mathfrak{c}}$-point$\}$. Define

$$B = \{\mathcal{M} \in \Sigma(\mathbb{N}) \mid \exists \mathcal{F}_i \in A (i \leq n,\ n \in \omega)\ \exists \mathcal{L} \in \lambda\{0, 1, \ldots, n\}$$
$$: \mathcal{M} = \{F \subset \mathbb{N} \mid \exists L \in \mathcal{L} : F \in \mathcal{F}_i (i \in L)\}\}.$$

We will show that B consists of $P_{\mathfrak{c}}$-points of $\Sigma(\mathbb{N})$ and that each nonvoid open set contains $2^{\mathfrak{c}}$ elements of B. Indeed, take $\mathcal{M} \in B$ and let $\{O_\alpha \mid \alpha \in \beta\}$ be a collection of less than $\mathfrak{c}$ neighborhoods of $\mathcal{M}$. Without loss of generality we may assume that each O_α is of the form M_α^+ with $M_\alpha \in \mathcal{M}(\alpha \in \beta)$. Choose $\mathcal{F}_i \in A(i \leqq n, n \in \omega)$ and $\mathcal{L} \in \lambda\{0,1,\ldots,n\}$ such that $\mathcal{M} = \{F \subset \mathbb{N} \mid \exists L \in \mathcal{L} : F \in \mathcal{F}_i(i \in L)\}$. For each M_α choose $L_\alpha \in \mathcal{L}$ such that $M_\alpha \in \mathcal{F}_i$ for all $i \in L_\alpha$. For each $L \in \mathcal{L}$ define $A(L) = \{\alpha \in \beta \mid L = L_\alpha\}$. Fix $L \in \mathcal{L}$. For each $i \in L$ choose $F_i(L) \in \mathcal{F}_i$ such that $|F_i(L) \setminus M_\alpha| < \omega$ for all $\alpha \in A(L)$. This is possible since $\mathcal{F}_i$ is a $P_\mathfrak{c}$-point of $\beta\mathbb{N} \setminus \mathbb{N}$. Moreover for each $i \in \{0, 1, \ldots, n\}$ define $\mathcal{L}_i = \{L \in \mathcal{L} \mid i \in L\}$. Then let

$$F_i = \bigcap_{L \in \mathcal{L}_i} F_i(L).$$

Clearly $F_i \in \mathcal{F}_i(i \leqq n)$. Finally define

$$U = \bigcap_{L \in \mathcal{L}} (\bigcup_{i \in L} F_i)^+ \cap \Sigma(\mathbb{N}).$$

It is easy to see that U is a neighborhood of $\mathcal{M}$ such that $U \subset \bigcap_{\alpha \in \beta} O_\alpha$. This shows that B consists of $P_\mathfrak{c}$-points.

Now, let U be a nonvoid open set in $\Sigma(\mathbb{N})$. Take $\mathcal{M} \in U$ and let $M_i \in \mathcal{M}(i \leqq n)$ such that $\bigcap_{i \leqq n} M_i^+ \cap \Sigma(\mathbb{N}) \subset U$. For each $i, j \in \{0, 1, \ldots, n\}$ take a $P_\mathfrak{c}$-point $\mathcal{F}_{ij} = \mathcal{F}_{ji} \in A$ such that $M_i \cap M_j \in \mathcal{F}_{ij}$. This is possible since $|M_i \cap M_j| = \omega$. Take a maximal linked system $\mathcal{L} \in \lambda(\{0, 1, \ldots, n\}^2)$ such that for all $i \leqq n$ the set $L_i = \{(i, j) \mid j \leqq n\}$ is an element of $\mathcal{L}$. Notice that $\{L_i \mid i \leqq n\}$ is linked. Now define

$$\mathcal{N} := \{F \subset N \mid \exists L \in \mathcal{L}: F \in \mathcal{F}_{ij}(i, j) \in L\}.$$

We will show that $\mathcal{N}$ is a maximal linked system. Clearly $\mathcal{N}$ is linked. Now suppose $\mathcal{N}$ is not maximally linked. Take $M \subset \mathbb{N}$ such that $\mathcal{N} \cup \{M\}$ is linked while $M \notin \mathcal{N}$. Define $E = \{(i, j) \mid M \in \mathcal{F}_{ij}\}$. Clearly $E \neq \varnothing$ and also $\{E\} \cup \mathcal{L}$ is linked. Hence, as $\mathcal{L}$ is a maximal linked system, $E \in \mathcal{L}$ and consequently $M \in \mathcal{N}$. Contradiction. Since each $\mathcal{F}_{ij}$ is an ultra-filter, $\mathcal{N}$ is a maximal linked system any two members of which meet in an infinite set and hence $\mathcal{N} \in \Sigma(\mathbb{N})$. Also it is clear that $\mathcal{N} \in U$ and that there are $2^\mathfrak{c}$ different choices for $\mathcal{N}$. □

Remark. The technique used in the proof of Theorem 2.11 is based on a technique due to Verbeek [19].

3. Spaces with a Binary Normal Closed Subbase

Let X be a set and let $\mathcal{S}$ be a collection of subsets of X. For each $A \subset X$ define

$$\mathrm{cl}_\mathcal{S}(A) = \cap \{S \mid S \in \mathcal{S} \text{ and } A \subset S\}$$

and

$$\mathrm{int}_\mathcal{S}(A) = \cup \{X \setminus S \mid S \in \mathcal{S} \text{ and } X \setminus S \subset A\}$$

respectively.

The sets $\mathrm{cl}_{\mathscr{S}}(A)$ and $\mathrm{int}_{\mathscr{S}}(A)$ are called the *$\mathscr{S}$-closure* and the *$\mathscr{S}$-interior* of A. In addition a set $B \subset X$ is called *$\mathscr{S}$-closed* (*$\mathscr{S}$-convex*) if $B = \mathrm{cl}_{\mathscr{S}}(B)$ (for all $b_0, b_1 \in B$: $\mathrm{cl}_{\mathscr{S}}(\{b_0, b_1\}) \subset B$, respectively). Clearly each $\mathscr{S}$-closed set also is $\mathscr{S}$-convex. Simple examples show that the converse need not be true. If $\mathscr{S}$ is a binary normal closed subbase for the topological space X then these two concepts coincide as the following theorem shows. We will often use, without reference, the following simple lemma

Lemma 3.1. *Let $\mathscr{S}$ be a binary subbase for X. Then*

(i) *$\mathscr{S}$ is a T_1-subbase;*

(ii) *If $\mathscr{S}$ is normal then for each $x, y \in X$ with $x \neq y$ there exist $S_0, S_1 \in \mathscr{S}$ with $x \in S_0 \cap (X \setminus S_1)$ and $y \in S_1 \cap (X \setminus S_0)$ and $S_0 \cup S_1 = X$;*

(iii) *Let $B \subset X$ be $\mathscr{S}$-closed. Then $\{S \cap B \mid S \in \mathscr{S}\}$ is binary.*

Theorem 3.2. *Let $\mathscr{S}$ be a binary normal closed subbase for the topological space X. Then*

(i) *If B is closed in X, then*

$B = \mathrm{cl}_{\mathscr{S}}(B)$ iff for all $x, y \in B$ we have $\mathrm{cl}_{\mathscr{S}}(\{x, y\}) \subset B$;

(ii) *If $B = \mathrm{cl}_{\mathscr{S}}(B)$ then the mapping $r : X \to B$ defined by*

$\{r(x)\} = \cap \{S \in \mathscr{S} \mid x \in S$ and $S \cap B \neq \varnothing\} \cap B$ *is a retraction;*

(iii) (Verbeek [19]) *If X is connected then X also is locally connected.*

Proof. (i) "$\Rightarrow$" trivial.

"$\Leftarrow$" Suppose there exists an $x \in \mathrm{cl}_{\mathscr{S}}(B) \setminus B$.
We will show that $\{x\} = \bigcap_{y \in B} \mathrm{cl}_{\mathscr{S}}(\{x, y\})$.

Indeed, assume there exists a $z \in \bigcap_{y \in B} \mathrm{cl}(\{x, y\})$ such that $x \neq z$.

Choose $S_0, S_1 \in \mathscr{S}$ such that $x \in S_0 \cap (X \setminus S_1)$, $y \in S_1 \cap (X \setminus S_0)$ and $S_0 \cup S_1 = X$. Clearly $B \not\subset S_1$, for otherwise $B \subset \mathrm{cl}_{\mathscr{S}}(B) \subset S_1$, which is a contradiction since $x \in \mathrm{cl}_{\mathscr{S}}(B)$. Therefore there exists a $b_0 \in S_0 \cap B$. Then $\mathrm{cl}_{\mathscr{S}}(\{x, b_0\}) \subset S_0$ which implies that $y \in \bigcap_{b \in B} \mathrm{cl}_{\mathscr{S}}(\{x, b\}) \subset \mathrm{cl}_{\mathscr{S}}(\{x, b_0\}) \subset S_0$ which also is a contradiction. Define $\mathscr{T} := \{\mathrm{cl}_{\mathscr{S}}(\{x, b\}) \cap B \mid b \in B\}$. Clearly $\mathscr{T}$ consists of closed subsets of B (and hence of X). We will show that $\mathscr{T}$ is linked. Choose $b_0, b_1 \in B$. Then as $\mathscr{S}$ is binary, $\mathrm{cl}_{\mathscr{S}}(\{b_0, b_1\}) \cap \mathrm{cl}_{\mathscr{S}}(\{b_0, x\}) \cap \mathrm{cl}_{\mathscr{S}}(\{b_1, x\}) \neq \varnothing$ and as $\mathrm{cl}_{\mathscr{S}}(\{b_0, b_1\}) \subset B$ the sets $\mathrm{cl}_{\mathscr{S}}(\{b_0, x\}) \cap B$ and $\mathrm{cl}_{\mathscr{S}}(\{b_1, x\}) \cap B$ must intersect. Therefore $\mathscr{T}$ is a linked system with an empty intersection. By the compactness of B there are already finitely many elements of $\mathscr{T}$ which have a void intersection. Let $n \in \omega$ be the smallest element for which there are n elements of $\mathscr{T}$ with an empty intersection. Clearly $n \geq 3$ since $\mathscr{T}$ is linked. Take n elements $T_0, \ldots, T_{n-1} \in \mathscr{T}$ such that $\bigcap_{i \leq n-1} T_i = \varnothing$. Choose $b_l \in \bigcap_{i=0}^{l-1} T_i \cap \bigcap_{i=l+1}^{n-1} T_i$ $(l \in \{0, 1, 2\})$. Then

$$\emptyset \neq \text{cl}_{\mathscr{S}}(\{b_0, b_1\}) \cap \text{cl}_{\mathscr{S}}(\{b_0, b_2\}) \cap \text{cl}_{\mathscr{S}}(\{b_1\ b_2\}) \subset \bigcap_{i \leq n-1} T_i,$$

which is a contradiction.

(ii) Take $x \in X$. Then $\cap \{S \in \mathscr{S} \mid x \in S \text{ and } S \cap B \neq \emptyset\} \cap B \neq \emptyset$, since $B = \text{cl}_{\mathscr{S}}(B)$ and $\mathscr{S}$ is binary. Assume that this intersection contains two distinct points, say b_0 and b_1. There are $S_0, S_1 \in \mathscr{S}$ such that $b_0 \in S_0 \cap (X \setminus S_1)$ and $b_1 \in S_1 \cap (X \setminus S_0)$ and $S_0 \cup S_1 = X$. One of the sets $\{S_0, S_1\}$ must intersect x; assume $x \in S_0$. Then, since $S_0 \cap B \neq \emptyset$, the set $\cap \{S \in \mathscr{S} \mid x \in S \text{ and } S \cap B \neq \emptyset\} \cap B \subset S_0 \cap B$ which is a contradiction since $b_1 \notin S_0 \cap B$. Therefore r is well-defined. To prove the continuity of r, take $S \in \mathscr{S}$ and suppose that $x \notin r^{-1}[S]$ (consider r to be a mapping of X into X). Also assume that $r^{-1}[S] \neq \emptyset$. Then $r(x) \notin S$ and as $\{r(x)\} = \cap \{S \in \mathscr{S} \mid x \in S \text{ and } S \cap B \neq \emptyset\} \cap B$ either $B \cap S = \emptyset$ or there is an $S_0 \in \mathscr{S}$ such that $x \in S_0$ and $S_0 \cap B \neq \emptyset$ and $S_0 \cap S = \emptyset$ (notice that $B = \text{cl}_{\mathscr{S}}(B)$ and that $\mathscr{S}$ is binary). In the first case $r^{-1}[S] = \emptyset$, which is a contradiction. In the second case choose $S'_0, S'_1 \in \mathscr{S}$ such that $S_0 \cap S'_1 = \emptyset = S'_0 \cap S_1$ and $S'_0 \cup S'_1 = X$. Then $U = X \setminus S'_1$ is a neighborhood of x which misses $r^{-1}[S]$, for $X \setminus S'_1 \subset S_0$. Hence $r^{-1}[S]$ is closed. Therefore r is continuous. Clearly r is a retraction.

(iii) Take $x \in X$ and let U be a neighborhood of x. Choose fintely many $S_1, \ldots, S_n \in \mathscr{S}$ such that $x \notin \bigcup_{i=1}^{n} S_i \supset X \setminus U$. For each $i \in \{1, 2, \ldots, n\}$ choose $S'_i \in \mathscr{S}$ such that $x \in \text{int}\, S'_i \subset S'_i$ and $S'_i \cap S_i = \emptyset$. This is possible since $\mathscr{S}$ is T_1 and normal. Then $V = \bigcap_{i \leq 1}^{n} S'_i$ is a neighborhood of x, contained in U. Moreover it is clear that $V = \text{cl}_{\mathscr{S}}(V)$. Therefore V is connected, being a retract of a connected space. □

Remark. The circle, S_1, admits a binary subbase for which (i) and (ii) of the above theorem do not hold. Hence the normality of the subbase is essential. It is well-known that $\beta\mathbb{N} \setminus \mathbb{N}$ is an F-space (Gillman and Jerison [7]) and hence that each cozeroset of $\beta\mathbb{N} \setminus \mathbb{N}$ is C^*-embedded. In particular it follows that a countable union of clopen sets is C^*-embedded. On the other hand $\Sigma(\mathbb{N})$ cannot be an F-space, since no infinite compact F-space is supercompact (van Douwen and van Mill [6]). We give an example of a countable union of clopen subsets of $\Sigma(\mathbb{N})$ that is not C^*-embedded.

Negrepontis ([12]) has shown that the closure of a countable union of clopen sets in $\beta\mathbb{N} \setminus \mathbb{N}$ is a retract of $\beta\mathbb{N} \setminus \mathbb{N}$. The following theorem shows that a similar assertion holds in $\Sigma(\mathbb{N})$ for suitable countable unions of clopen sets taken from the "canonical" closed subbase $\{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$.

From now on let $\mathscr{S} = \{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$. This subbase is binary and for all $S \in \mathscr{S}$ the set $X \setminus S$ is also in $\mathscr{S}$. In particular, $\mathscr{S}$ is normal.

Theorem 3.3. *Let* $\{A_\alpha \mid \alpha \in \beta\}$ *be a collection of* $\mathscr{S}$*-closed sets such that* $A_\alpha \subset A_\gamma$ *iff* $\alpha < \gamma$. *Then* $(\bigcup_{\alpha \in \beta} A_\alpha)^-$ *equals* $\text{cl}_{\mathscr{S}}(\bigcup_{\alpha \in \beta} A_\alpha)$. *In particular* $(\bigcup_{\alpha \in \beta} A_\alpha)^-$ *is supercompact and is a retract of* $\Sigma(\mathbb{N})$.

Proof. Clearly $(\bigcup_{\alpha \in \beta} A_\alpha)^- \subset \text{cl}_{\mathscr{S}}(\bigcup_{\alpha \in \beta} A_\alpha)$. Take two distinct points $\mathscr{M}_0, \mathscr{M}_1 \in (\bigcup_{\alpha \in \beta} A_\alpha)^-$ and assume there exists a point $\mathscr{P}$ such that

$$\mathscr{P} \in \mathrm{cl}_{\mathscr{S}}(\{\mathscr{M}_0, \mathscr{M}_1\}) \setminus (\bigcup_{\alpha \in \beta} A_\alpha)^-.$$

Take finitely many $P_i \in \mathscr{P}$ ($i \leqq n$, $n \in \omega$) such that $\bigcap_{i \leqq n} P_i^+ \cap \bigcup_{\alpha \in \beta} A_\alpha = \varnothing$. Now suppose that for some $l \leqq n$ we have that $P_l \notin \mathscr{M}_0$ and $P_l \notin \mathscr{M}_1$. Then take $M_i \in \mathscr{M}_i$ such that $M_i \cap P_l = \varnothing$ ($i \in \{0, 1\}$). Clearly $P_l \cap (M_0 \cup M_1) = \varnothing$ and also $\mathrm{cl}_{\mathscr{S}}(\{\mathscr{M}_0, \mathscr{M}_1\}) \subset (M_0 \cup M_1)^+$. However $P_l^+ \cap (M_0 \cup M_1)^+ = \varnothing$, which is a contradiction since $\mathscr{P} \in \mathrm{cl}_{\mathscr{S}}(\{\mathscr{M}_0, \mathscr{M}_1\})$. Therefore each P_l either belongs to $\mathscr{M}_0$ or belongs to $\mathscr{M}_1$. Define $C_i = \{l \leqq n \mid P_l \in M_i\}$ ($i \in \{0, 1\}$). Then $\bigcap_{l \in C_i} P_l^+$ is a neighboorhood of $\mathscr{M}_i$ and hence intersects $\bigcup_{\alpha \in \beta} A_\alpha$ ($i \in \{0, 1\}$). Choose $\alpha_i \in \beta$ such that $\bigcap_{l \in C_i} P_l^+ \cap A_{\alpha_i} \neq \varnothing$ ($i \in \{0,1\}$). Without loss of generality assume that $\alpha_0 \leqq \alpha_1$. Then $\{\bigcap_{l \in C_0} P_l^+, \bigcap_{l \in C_1} P_l^+, A_{\alpha_1}\}$ is a linked system consisting of $\mathscr{S}$-closed sets (Theorem 3.2). Consequently, by the fact that $\mathscr{S}$ is binary,

$$\bigcap_{i \leqq n} P_i^+ \cap A_{\alpha_1} = \bigcap_{l \in C_0} P_l^+ \cap \bigcap_{l \in C_1} P_l^+ \cap A_{\alpha_1} \neq \varnothing,$$

which is a contradiction.

It now follows that $(\bigcup_{\alpha \in \beta} A_\alpha)^-$ is $\mathscr{S}$-convex and hence $\mathscr{S}$-closed by Theorem 3.2. Therefore $(\bigcup_{\alpha \in \beta} A_\alpha)^- = \mathrm{cl}_{\mathscr{S}}(\bigcup_{\alpha \in \beta} A_\alpha)^-$. Hence $Z = (\bigcup_{\alpha \in \beta} A_\alpha)^-$ is supercompact (Lemma 3.1) and is a retract of $\Sigma(\mathbb{N})$ (Theorem 3.2). □

Corollary 3.4. *Let $S_i \in \mathscr{S}$ such that $S_i \subset S_{i+1}$ and $S_{i+1} \setminus S_i \neq \varnothing$ ($i \in \omega$). Then $\bigcup_{i \in \omega} S_i$ is not C^*-embedded in $\Sigma(\mathbb{N})$.*

Proof. Notice that $A = \bigcup_{i \in \omega} S_i$ is not compact for $\{S_i \mid i \in \omega\}$ is a countable open covering of A without finite subcover. Suppose that A is C^*-embedded in $\Sigma(\mathbb{N})$. Then A^- is a compactification of A which is equivalent to the Čech-Stone compactification of A. Hence, by Theorem 3.3, βA $(= A^-)$ is supercompact. This implies, by a theorem of Bell [1], that A is pseudocompact. However A is not pseudocompact as can easily be seen. Contradiction. □

There are still many questions to be asked concerning $\Sigma(\mathbb{N})$. For example Theorem 2.9 says that each nonvoid countable intersection of open sets in $\Sigma(\mathbb{N})$ contains a homeomorph of $\beta\mathbb{N} \setminus \mathbb{N}$. Hence such an intersection contains many countable subspaces that are C^*-embedded. On the other hand $\Sigma(\mathbb{N})$ is supercompact and hence for each countable subspace K it follows that at least one cluster point of K is the limit of a nontrivial convergent sequence in $\Sigma(\mathbb{N})$ (van Douwen and van Mill [6]). Hence there are also many countable subspaces of $\Sigma(\mathbb{N})$ that are not C^*-embedded.

This suggests the following question:

Question 3.5. When is a countable $A \subset \Sigma(\mathbb{N})$ C^*-embedded?

Also we have said nothing about normality in $\Sigma(\mathbb{N})$. It is well-known that CH implies that $\beta\mathbb{N} \setminus \mathbb{N} \setminus \{p\}$ is not normal for all $p \in \beta\mathbb{N} \setminus \mathbb{N}$ (see Comfort and Negrepontis [3], Warren [20]). Hence if for each $p \in \Sigma(\mathbb{N})$ there exists a copy of

$\beta\mathbb{N}\setminus\mathbb{N}$ in $\Sigma(\mathbb{N})$ containing p, then CH also implies that $\Sigma(\mathbb{N})\setminus\{p\}$ is not normal. Theorem 2.9 suggests such a fact.

Question 3.6. Is there for each $p\in\Sigma(\mathbb{N})$ a homeomorph of $\beta\mathbb{N}\setminus\mathbb{N}$ containing p?

Question 3.7. Is it true that $\Sigma(\mathbb{N})\setminus\{p\}$ is not normal for all $p\in\Sigma(\mathbb{N})$?

4. A Characterization of $\Sigma(\mathbb{N})$

In [13], Parovičenko characterized $\beta\mathbb{N}\setminus\mathbb{N}$ in terms of its Boolean algebra of of clopen subsets. We will show that Parovičenko's result allows us to give a characterization of $\Sigma(\mathbb{N})$, not in terms of its Boolean algebra of clopen subsets but in terms of the Boolean algebra $\{M^+\cap\Sigma(\mathbb{N})\mid M\subset\mathbb{N}\}$. Clearly $\mathscr{S}=\{M^+\cap\Sigma(\mathbb{N})\mid M\subset\mathbb{N}\}$ is not a Boolean subalgebra of the Boolean algebra of clopen subsets of $\Sigma(\mathbb{N})$. Therefore we define for $\mathscr{S}$ new Boolean operations and show that, under the Continuum Hypothesis (CH), the Boolean algebra thus obtained characterizes $\Sigma(\mathbb{N})$ and hence $\lambda(\beta\mathbb{N}\setminus\mathbb{N})$. Parovičenko also uses the Continuum Hypothesis and from an example given by van Douwen [4] it follows that the Continuum Hypothesis is essential in this characterization. There is a locally compact, σ-compact and separable space M such that $\beta\mathbb{N}\setminus\mathbb{N}$ and $\beta M\setminus M$ are homeomorphic under CH but not under MA + $\neg$CH. Van Douwen's example also shows that in our characterization CH is essential. The spaces $\Sigma(\mathbb{N})$ and $\Sigma(M)$ are homeomorphic under CH but not under MA + $\neg$CH. One moment one might think that this immediately follows from van Douwen's result using the equalities $\Sigma(\mathbb{N})\simeq\lambda(\beta\mathbb{N}\setminus\mathbb{N})$ and $\Sigma(M)\simeq\lambda(\beta M\setminus M)$. This is not true, however. We give an example of two compact metric spaces X and Y which are not homeomorphic while yet λX and λY are homeomorphic.

Parovičenko [13] has also shown that, without using the Continuum Hypothesis, each compact Hausdorff space of weight less than or equal to ω_1 is a continuous image of $\beta\mathbb{N}\setminus\mathbb{N}$. We will show that for $\Sigma(\mathbb{N})$ this is not true. There is a compact Hausdorff space with ω_1 points which is not the continuous image of $\Sigma(\mathbb{N})$.

Let $\mathscr{B}=\langle B,0,1,',\wedge,\vee\rangle$ be a boolean algebra. $\mathscr{B}$ is called *Cantor separable* if no strictly increasing sequence has a least upper bound, i.e. if

$$a_0<\cdots<a_n<\cdots<b,$$

there exists an element $c<b$ such that $a_n<c$ for all $n\in\omega$. In addition $\mathscr{B}$ is called *DuBois-Reymond separable* if a strictly increasing sequence can be separated from a strictly decreasing sequence dominating the increasing one, i.e. if

$$a_0<\cdots<a_n<\cdots<b_n<\cdots<b_0,$$

there exists a $c\in B$ such that $a_n<c<b_n$ for all $n\in\omega$. Finally $\mathscr{B}$ is called *dense in itself* if for all $a,c\in B$ with $a<c$ there is a $b\in B$ such that $a<b<c$. A subset $M\subset B$ is called a *linked system* if $m_0\wedge m_1\neq 0$ for all $m_0,m_1\in M$. A *maximal linked system* is a linked system not properly contained in any other linked system. By Zorn's Lemma each linked system is contained in at least one maximal linked system.

This fact, however, also follows from the Order Extension Principle *(each partial ordering on a set can be extended to a total ordering)*, which is strictly weaker than Stone's representation theorem which is, in turn, strictly weaker than Zorn's lemma, see Schrijver [17]. Parovičenko [13] has shown that, under CH, a compact Hausdorff totally disconnected space of weight $\mathfrak{c}$ which possesses no isolated points *is homeomorphic to* $\beta\mathbb{N} \setminus \mathbb{N}$ *iff the Boolean algebra of clopen subsets of* X *is both Cantor and DuBois-Reymond separable.* In fact he showed that all Boolean algebra's of cardinality $\mathfrak{c}$ which are dense in itself and which are both Cantor and DuBois-Reymond separable are isomorphic under CH. We will use Parovičenko's result in this form.

For technical reasons we will assume from now on that each closed subbase $\mathscr{S}$ for a topological space X contains $\varnothing$ and X.

Theorem 4.1 [CH]. *Let X be a compact Hausdorff space of weight $\mathfrak{c}$ which possesses no isolated points. Then X is homeomorphic to $\Sigma(\mathbb{N})$ (and hence to $\lambda(\beta\mathbb{N} \setminus \mathbb{N})$) iff X possesses a binary closed subbase $\mathscr{S}$ satisfying:*

(i) *For all $S \in \mathscr{S}$ also $X \setminus S \in \mathscr{S}$.*
(ii) *For all $S_0, S_1 \in \mathscr{S}$ also $\mathrm{cl}_{\mathscr{S}}(S_0 \cup S_1) \in \mathscr{S}$.*
(iii) *For all $S_0, S_1 \in \mathscr{S}$: $\mathrm{cl}_{\mathscr{S}}(S_0 \cup S_1) = X \Leftrightarrow S_0 \cup S_1 = X$.*
(iv) *For all $S_0, S_1, S_2 \in \mathscr{S}$:*

$$\mathrm{int}_{\mathscr{S}}(S_0 \cap \mathrm{cl}_{\mathscr{S}}(S_1 \cup S_2)) = \mathrm{cl}_{\mathscr{S}}(\mathrm{int}_{\mathscr{S}}(S_0 \cap S_1) \cap \mathrm{int}_{\mathscr{S}}(S_0 \cap S_2)).$$

(v) *If $S_n \in \mathscr{S}$, $S_n \supset S_{n+1}$ $(n \in \omega)$ then $\bigcap_{n\in\omega} S_n$ contains a nonvoid element of $\mathscr{S}$.*
(vi) *Disjoint countable unions of elements of $\mathscr{S}$ have disjoint $\mathscr{S}$-closures.*

Proof. "$\Rightarrow$" Define $\mathscr{S} = \{M^+ \cap \Sigma(\mathbb{N}) \mid M \subset \mathbb{N}\}$. Then $\mathscr{S}$ is a binary closed subbase for $\Sigma(\mathbb{N})$ which satisfies (i). It satisfies (ii), (iii) and (iv) because of the equalities

$$\mathrm{cl}_{\mathscr{S}}((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N}))) = (M_0 \cup M_1)^+ \cap \Sigma(\mathbb{N}) \qquad (*)$$

$$\mathrm{int}_{\mathscr{S}}((M_0^+ \cap \Sigma(\mathbb{N})) \cap (M_1^+ \cap \Sigma(\mathbb{N}))) = (M_0 \cap M_1)^+ \cap \Sigma(\mathbb{N}). \qquad (**)$$

Let us proof (*) only.

Clearly $\mathrm{cl}_{\mathscr{S}}((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N}))) \subset (M_0 \cup M_1)^+ \cap \Sigma(\mathbb{N})$. Suppose there exists an $\mathscr{M} \in ((M_0 \cup M_1)^+ \setminus \mathrm{cl}_{\mathscr{S}}((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N})))) \cap \Sigma(\mathbb{N})$. Choose $L \subset \mathbb{N}$ such that $\mathrm{cl}_{\mathscr{S}}((M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N})) \subset L^+ \cap \Sigma(\mathbb{N})$ such that $\mathscr{M} \notin L^+ \cap \Sigma(\mathbb{N})$. Then $M_i^+ \cap \Sigma(\mathbb{N}) \subset L^+ \cap \Sigma(\mathbb{N})$ implies that $|M_i \setminus L| < \omega$ $(i \in \{0, 1\})$ and hence that $|(M_0 \cup M_1) \setminus L| < \omega$. Choose $M \in \mathscr{M}$ such that $|M \cap L| < \omega$. Then $|M \cap (M_0 \cup M_1)| < \omega$, which is a contradiction since $\mathscr{M} \in (M_0 \cup M_1)^+$.

This shows that $\mathscr{S}$ satisfies (iii) for take $S_0, S_1 \in \mathscr{S}$. If $S_0 \cup S_1 = \Sigma(\mathbb{N})$ then clearly $\mathrm{cl}_{\mathscr{S}}(S_0 \cup S_1) = \Sigma(\mathbb{N})$. Now assume that $\mathrm{cl}_{\mathscr{S}}(S_0 \cup S_1) = \Sigma(\mathbb{N})$. Let $S_i = M_i^+ \cap \Sigma(\mathbb{N})$ $(i \in \{0, 1\})$. Then $\Sigma(\mathbb{N}) = (M_0 \cup M_1)^+ \cap \Sigma(\mathbb{N})$ by (*). Hence $|\mathbb{N} \setminus (M_0 \cup M_1)| < \omega$ and consequently $(M_0^+ \cap \Sigma(\mathbb{N})) \cup (M_1^+ \cap \Sigma(\mathbb{N})) = \Sigma(\mathbb{N})$ (notice that in general $|\mathbb{N} \setminus (M_0 \cup M_1)| < \omega$ need not imply $M_0^+ \cup M_1^+ = \lambda\mathbb{N}$!). Using (*) and (**) it follows that $\mathscr{S}$ satisfies (iv). $\mathscr{S}$ also satisfies (v), because of Lemma 2.6 (recall that $P(\omega_1)$ is in ZFC and hence that we do not use CH or P($\mathfrak{c}$) here).

Finally $\mathscr{S}$ satisfies (vi). Let $A = \bigcup_{i\in\omega} (M_i^+ \cap \Sigma(\mathbb{N}))$ and $B = \bigcup_{i\in\omega} (L_i^+ \cap \Sigma(\mathbb{N}))$ such that $A \cap B = \varnothing$. Using the same technique as (*) it follows that $\mathrm{cl}_{\mathscr{S}}(A) \subset (\bigcup_{i\in\omega} M_i)^+$

$\cap \Sigma(\mathbb{N})$ and $\mathrm{cl}_{\mathscr{S}}(B) \subset (\bigcup_{i\in\omega} L_i)^+ \cap \Sigma(\mathbb{N})$. Now suppose that $\mathrm{cl}_{\mathscr{S}}(A) \cap \mathrm{cl}_{\mathscr{S}}(B) \neq \varnothing$. It then follows that $|(\bigcup_{i\in\omega} M_i)^* \cap (\bigcup_{i\in\omega} L_i)^*| \neq 0$ and hence there are $i, j \in \omega$ such that $|M_i \cap L_j| = \omega$. Contradiction.

Finally of course $\Sigma(\mathbb{N})$ is a compact Hausdorff space of weight $\mathfrak{c}$ and possesses no isolated points because of Theorem 2.9 (see also Verbeek [19]).

"$\Leftarrow$" Define operations $\wedge$, $\vee$, $'$ on $\mathscr{S}$ by

$$A \vee B = \mathrm{cl}_{\mathscr{S}}(A \cup B),$$
$$A \wedge B = \mathrm{int}_{\mathscr{S}}(A \cap B),$$
$$A' = X \setminus A.$$

We will show that $\langle \mathscr{S}, \wedge, \vee, ', 0, 1 \rangle$ is a Boolean algebra, where $0 = \varnothing$ and $1 = X$. Notice that for all $A, B \in \mathscr{S}$ we have that $A \wedge B \subset A \cap B$ and $A \cup B \subset A \vee B$. Because of (ii) $A \vee B \in \mathscr{S}$ for all $A, B \in \mathscr{S}$. Also $A \wedge B \in \mathscr{S}$ because of the equality

$$A \wedge B = (A' \vee B')'.$$

To prove this notice that $A \wedge B = \cup \{X \setminus S \mid S \in \mathscr{S} \text{ and } X \setminus S \subset A \cap B\} = \cup \{S \mid S \in \mathscr{S}$ and $S \subset A \cap B\}$ by (i). Now take $S \in \mathscr{S}$ such that $S \subset A \cap B$. Then $A' \cup B' \subset (X \setminus S)$ and consequently $\mathrm{cl}_{\mathscr{S}}(A' \cup B') \subset (X \setminus S)$. Therefore $S \subset X \setminus \mathrm{cl}_{\mathscr{S}}(A' \cup B') = (A' \vee B')$. Since $(A' \vee B')' \in \mathscr{S}$, by (i) and (ii) it follows that $A \wedge B = (A' \vee B')'$.

Define a relation $\leqq$ on $\mathscr{S}$ by putting $A \leqq B$ iff $A \wedge B = A$.

Let us prove that $A \leqq B$ iff $A \subset B$ for all $A, B \in \mathscr{S}$. Indeed assume that $A \subset B$ Then $A \wedge B = (A' \vee B')' = (A')' = A$ and therefore $A \leqq B$. Next, suppose that $A \leqq B$ and that there exists an $x \in A \setminus B$. Then $x \notin A \wedge B$ and consequently $A \wedge B \neq A$. Contradiction. It now follows that the relation $\leqq$ is a partial ordering. Also it is clear that for all $A, B \in \mathscr{S}$ the set $A \wedge B$ is the greatest lower bound of A and B with respect to this ordering and in the same way $A \vee B$ is the least upper bound for A and B. Hence $(\mathscr{S}, \leqq)$ is a lattice. Also $(\mathscr{S}, \leqq)$ is distributive because of (iv) and clearly it is complemented. Hence $\langle \mathscr{S}, \vee, \wedge, ', 0, 1 \rangle$ is a Boolean algebra. Let us show that this Boolean algebra is Cantor separable. Take $A_n \in \mathscr{S}$ $(n \in \omega)$ and $B \in \mathscr{S}$ such that $A_0 < \cdots < A_n < \cdots < B$. Define $S_n = B \wedge A_n'$ $(n \in \omega)$. We will show that $S_n \neq 0$ $(n \in \omega)$. For suppose to the contrary that for some $n_0 \in \omega$ we have $S_{n_0} = 0$. Then $1 = S_{n_0}' = (B \wedge A_{n_0}')' = B' \vee A_{n_0}$ and hence, by (iii), $B' \cup A_{n_0} = X$. This is a contradiction, since $A_{n_0} < B$ (notice that in fact we have shown that for all $A, B \in \mathscr{S}$ we have $A \wedge B \neq 0$ iff $A \cap B \neq \varnothing$). Also $A_n < A_{n+1}$ implies $B \cap A_{n+1}' \subset B \cap A_n'$ and consequently $B \wedge A_{n+1}' \subset B \wedge A_n'$ $(n \in \omega)$. By (v) there is a nonvoid C such that $C \subset \bigcap_{n\in\omega} S_n$. Then $A_0 < A_1 < \cdots < A_n < \cdots < C' < B$.

Let us prove that $\langle S, \vee, \wedge, ', 0, 1 \rangle$ is dense in itself.

Take $A, C \in \mathscr{S}$ such that $A < C$. If $A = 0$, then $C \neq \varnothing$ implies that there are two distinct points x and y in C since X contains no isolated points. By the fact that $\mathscr{S}$ is binary there is an $S \in \mathscr{S}$ and $y \notin S$. Then define $B = C \wedge S$. Notice that $B \neq 0$. Then $A < B < C$. If $A \neq 0$ define $D = C \wedge A'$. Then $D \neq 0$, since $C \cap A' \neq \varnothing$ and define $B = D' \wedge C$. Then clearly $A < B < C$. Let us prove that $\langle \mathscr{S}, \wedge, \vee, ', 0, 1 \rangle$ is Dubois-Reymond separable. Suppose that $A_0 < \cdots < A_n < \cdots < B_n < \cdots < B_0$ for some $A_n, B_n \in \mathscr{S} (n \in \omega)$. Then $\bigcup_{n\in\omega} A_n$ and $\bigcup_{n\in\omega} B'_n$ are disjoint countable unions of elements of $\mathscr{S}$ and hence have, by (vi), disjoint $\mathscr{S}$-closures. Let $C_0 \supset \mathrm{cl}_{\mathscr{S}}(\bigcup_{n\in\omega} A_n)$ and $C_1 \supset \mathrm{cl}_{\mathscr{S}}(\bigcup_{n\in\omega} B'_n)$ such that $C_0, C_1 \in \mathscr{S}$ and $C_0 \cap C_1 = \varnothing$. Then clearly $A_n < C_0$ and $B'_n < C_1$ for all $n \in \omega$. It now follows that

$$A_0 < \cdots < A_n < \cdots < C_0 < \cdots < B_n < \cdots < B_0.$$

Also, since the weight of X is $\mathfrak{c}$ the cardinality of $\mathscr{S}$ equals $\mathfrak{c}$ since $\mathscr{S}$ is a subbase. Now by Parovičenko's result the Boolean algebra $\langle \mathscr{S}, \wedge, \vee, ', 0, 1\rangle$ is isomorphic to the Boolean algebra of clopen subsets $\mathrm{CO}(\beta\mathbb{N} \setminus \mathbb{N})$ of $\beta\mathbb{N} \setminus \mathbb{N}$.

Let $\sigma: \mathscr{S} \to \mathrm{CO}(\beta\mathbb{N} \setminus \mathbb{N})$ be an isomorphism. Define a function $\varphi: X \to \Sigma(\mathbb{N})$ by

$$\varphi(x) := \{M \subset N \mid M^* \in \{\sigma(S) \mid x \in S\}\}.$$

Recall that $M^* = \mathrm{cl}_{\beta\mathbb{N}}(M) \setminus M$ for all $M \subset \mathbb{N}$. We will show that φ is a homeomorphism.

Take $x \in X$; then $\mathscr{S}_x = \{S \in \mathscr{S} \mid x \in S\}$ is a maximal linked system in the Boolean algebra $\langle \mathscr{S}, \wedge, \vee, ', 0, 1\rangle$.

Indeed take $S_0, S_1 \in \mathscr{S}_x$, then $S_0 \cap S_1 \neq \varnothing$ implies that $S_0 \wedge S_1 \neq 0$ which shows that $\mathscr{S}_x$ is a linked system. Also $\mathscr{S}_x$ is maximally linked for suppose there is an $A \in \mathscr{S}$ such that $\mathscr{S}_x \cup \{A\}$ is linked but $A \notin \mathscr{S}_x$. Then $x \notin A$ and consequently $x \in A'$. But $A \cap A' = \varnothing$ implies that $A \wedge A' = 0$. Contradiction. The Boolean isomorphism σ maps $\mathscr{S}_x$ onto a maximal linked system in $\mathrm{CO}(\beta\mathbb{N} \setminus \mathbb{N})$. Now it is not hard to see that

$$\{M \subset \mathbb{N} \mid M^* \in \{\sigma(S) \mid x \in S\}\}$$

is a maximal linked system in $\mathscr{P}(\mathbb{N})$ and that it is an element of $\Sigma(\mathbb{N})$. Also the fact that σ is an isomorphism implies that φ is one to one and surjective. Moreover φ is continuous since it is easy to see that

$$\varphi^{-1}(M^+ \cap \Sigma(\mathbb{N})) = \sigma^{-1}(M^*)$$

for all $M \subset \mathbb{N}$. Therefore φ is a homeomorphism. □

Corollary 4.2 [CH]. *If X is a zero-dimensional noncompact σ-compact and locally compact space with $|C(X)| = \mathfrak{c}$, then $\Sigma(\mathbb{N})$ and $\Sigma(X)$ are homeomorphic.*

Proof. It is easy to see that $\{M^+ \cap \Sigma(X) \mid M$ is open and closed in $X\}$ satisfies all conditions of Theorem 4.1 (notice that X Lindelöf, being σ-compact, implies that for all closed sets $A, B \subset X$ with $A \cap B = \varnothing$ there is an open and closed $U \subset X$ such that $A \subset U$ and $B \subset X \setminus U$). □

Remark. Corollary 4.2 also follows directly from Parovičenko's result. For if X is a zero-dimensional noncompact σ-compact and locally compact space with $|C(X)| = \mathfrak{c}$ then there is a homeomorphism $\varphi: \beta X \setminus X \to \beta\mathbb{N} \setminus \mathbb{N}$ using Parovicenko's characterization of $\beta\mathbb{N} \setminus \mathbb{N}$. This homeomorphism easily can be extended to a homeomorphism $\bar{\varphi}: \lambda(\beta X \setminus X) \to \lambda(\beta\mathbb{N} \setminus \mathbb{N})$ (Verbeek [19]). Now theorem 2.4 shows $\Sigma(X) \simeq \Sigma(\mathbb{N})$. □

Example 4.3. A locally compact and σ-compact separable space M for which $\Sigma(M)$ and $\Sigma(\mathbb{N})$ are homeomorphic under CH but not under $P(\mathfrak{c})+\neg$CH.

As noted in the introduction of this section this example is based on an example of van Douwen [4].

Let $M=\mathbb{N}\times\{0,1\}^{\mathfrak{c}}$. Then clearly $\Sigma(M)$ and $\Sigma(\mathbb{N})$ are homeomorphic under CH. Assume that $\omega_1<\mathfrak{c}$ and let $K=\{0,1\}^{\mathfrak{c}}$. Let $\mathscr{K}=\{\Pi_\alpha^{-1}[\{i\}]\mid\alpha\in\omega_1, i\in\{0,1\}\}$. Then $\{\mathbb{N}\times K\mid K\in\mathscr{K}\}$ is a collection of ω_1 clopen subsets of M each infinite subcollection of which has a void interior. As for each $\alpha\in\omega_1$ we have $(\mathbb{N}\times\Pi_\alpha^{-1}[\{0\}])\cup(\mathbb{N}\times\Pi_\alpha^{-1}[\{1\}])=M$ for each $\mathscr{M}\in\lambda M$ there is an $i\in\{0,1\}$ such that $\mathbb{N}\times\Pi_\alpha^{-1}[\{i\}]\in\mathscr{M}$ (recall that $\mathscr{M}$ is a maximal linked system). For each $\mathscr{M}\in\Sigma(M)$ define $\mathscr{K}(\mathscr{M})=\{K\in\mathscr{K}\mid\mathbb{N}\times K\in\mathscr{M}\}$. It follows that $\mathscr{K}(\mathscr{M})$ is uncountable for each $\mathscr{M}\in\Sigma(M)$ and also that $\{\mathscr{K}(\mathscr{M})\mid\mathscr{M}\in\Sigma(\mathbb{N})\}$ has cardinality 2^{ω_1}. Also

$$A=\{\cap\{(\mathbb{N}\times K)^+\mid K\in\mathscr{K}(\mathscr{M})\}\cap\Sigma(M)\}\mid\mathscr{M}\in\Sigma(M)\}$$

covers $\Sigma(M)$. The collection $\mathscr{A}$ has cardinality 2^{ω_1} and consists of pairwise disjoint intersections of ω_1 clopen subsets of $\Sigma(M)$.

Let us prove that each $A\in\mathscr{A}$ has a void interior.

Assume that there exists open and closed $C_1,\ldots,C_n\subset M$ such that

$$\varnothing\neq\bigcap_{i\leq n}C_i^+\cap\Sigma(M)\subset A_0$$

for some $A_0\in\mathscr{A}$. Let $A_0=\cap\{(\mathbb{N}\times K)^+\mid K\in\mathscr{K}(\mathscr{M}_0)\}\cap\Sigma(M)$.

Now the fact that $\bigcap_{i\leq n}C_i^+\cap\Sigma(M)\subset\cap\{(\mathbb{N}\times K)^+\mid K\in\mathscr{K}(\mathscr{M}_0)\}\cap\Sigma(M)$ implies that for all $K\in\mathscr{K}(\mathscr{M}_0)$ there is an $i_K\leq n$ such that $C_i\setminus(\mathbb{N}\times K)$ is compact; for otherwise $\bigcap_{i\leq n}C_i^+\cap\Sigma(M)\not\subset(\mathbb{N}\times K)^+\cap\Sigma(M)$.

Hence there is an $i_0\leq n$ such that $\mathscr{L}=\{K\in\mathscr{K}(\mathscr{M}_0)\mid i_K=i_0\}$ is uncountable. Also, clearly, C_{i_0} is not compact. Choose for every $L\in\mathscr{L}$ an integer $i(L)$ such that $\varnothing\neq C_{i_0}\cap(\{i(L)\}\times K)\subset\{i(L)\}\times L$ (this is possible since $C_{i_0}\setminus(\mathbb{N}\times L)$ is compact!). There is an integer i such that the collection $\mathscr{B}=\{L\in\mathscr{L}\mid i(L)=i\}$ is infinite, because $\mathscr{L}$ is uncountable. But then $\cap\mathscr{B}$ has a nonvoid interior in K since $\varnothing\neq C_{i_0}\cap(\{i\}\times K)\subset\{i\}\times\cap\mathscr{B}$, which is a contradiction.

Now suppose that there is a homeomorphism φ: $\Sigma(\mathbb{N})\to\Sigma(M)$. Take $\mathscr{F}\in\beta\mathbb{N}\setminus\mathbb{N}$ and take $A\in\mathscr{A}$ such that $\mathscr{F}\in\varphi^{-1}(A)$. As A is an intersection of ω_1 clopen sets, $\varphi^{-1}(A)$ is also an intersection of ω_1 clopen subsets of $\Sigma(\mathbb{N})$. However $P(\mathfrak{c})$ implies that this intersection has a nonvoid interior (Lemma 2.6). Contradiction. □

Example 4.4. Two compact metric spaces X and Y which are not homeomorphic while yet λX and λY are homeomorphic.

Let $X=I$, the closed unit interval and let

$$Y=\{(0,y)\mid -1\leq y\leq 1\}\cup\{(x,\sin 1/x)\mid 0\leq x\leq 1\}.$$

Then clearly X and Y are not homeomorphic. In [10] it was shown that $\lambda X\simeq Q$, the Hilbert cube. This also implies that λY is homeomorphic to the Hilbert cube, because of the following theorem ([11]): *If* $Y \simeq \varprojlim \{X_i\}_i$ *and if all the bonding maps in the inverse sequence* $\{X_i\}_i$ *are surjections, then* $\lambda X_i \simeq Q (i \in \omega)$ *implies that* $\lambda X \simeq Q$.

Example 4.5. A separable compact Hausdorff space with ω_1 points which is not the continuous image of $\Sigma(\mathbb{N})$.

Let $T = {}^{\underline{\omega}}2 \cup {}^{\omega}2$ be a Cantor tree (cf. Rudin [15]) and let $L \subset {}^{\omega}2$ be such that $|L| = \omega_1$. In [6] it was shown that no compactification of the subspace $S = {}^{\underline{\omega}}2 \cup L$ of T is supercompact. With the same technique however it also follows that no compactification of S can be the continuous image of a supercompact space. However, as S is locally compact, the one point compactification of S is not the continuous image of $\Sigma(\mathbb{N})$. □

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