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COMPACTIFICATIONS OF LOCALLY COMPACT SPACES WITH ZERO-DIMENSIONAL REMAINDER

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For a locally compact space X we give a necessary and sufficient condition for every compactification αX of X with zero-dimensional remainder to be regular Wallman. As an application it follows that the Freudenthal compactification of a locally compact metrizable space is regular Wallman.

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1. Introduction

Every Tychonoff space X admits Hausdorff compactifications obtainable as the ultra-filter space of some normal base on X. Such compactifications are called *Wallman compactifications*. Until now the question, raised in [2] and [4], whether all Hausdorff compactifications are Wallman compactifications remains unanswered,. although many well known compactifications turned out to be Wallman compactifications [1, 6, 7, 9, 10, 11].

In this paper we will consider Wallman compactifications of locally compact spaces only; in particular compactifications of locally compact spaces with zero-dimensional remainder (that such a compactification is a Wallman compactification follows from a theorem of Njåstad [9]). Our main result is that a locally compact space X admits a separating ring of regular closed sets if and only if every compactification of X with zero-dimensional remainder is regular Wallman (in the sense of Steiner [11]; such a space is a Wallman compactification of each dense subspace). As an application of this theorem it follows that the Freudenthal compactification of a locally compact metrizable space is regular Wallman.

2. s-rings

All topological spaces under discussion are assumed to be Tychonoff.

Let X be a topological space and let $\mathscr{S}$ be a collection of subsets of X. We will write $\vee.\mathcal{S}$ for the family of finite unions of elements of $\mathcal{S}$ and $\wedge.\mathcal{S}$ for the family of finite intersections of elements of $\mathcal{S}$. The family $\wedge.\vee.\mathcal{S} = \vee.\wedge.\mathcal{S}$ is closed both under finite unions and intersections; it is called the *ring* generated by $\mathcal{S}$. We say that $\mathcal{S}$ is *separating* if for each closed subset $F \subset X$ and each $x \in X \backslash F$ there exist S_0, $S_1 \in \mathcal{S}$, such that $x \in S_0$, $F \subset S_1$ and $S_0 \cap S_1 = \emptyset$. In addition $\mathcal{S}$ is called a *separating ring* if $\mathcal{S} = \vee.\wedge.\mathcal{S}$ and $\mathcal{S}$ is separating. A *regular Wallman* space is a compact space which possesses a separating ring of regular closed sets. It is known that *each regular Wallman space is a Wallman compactification of each dense subspace* [11]. For shortness, from now on a separating ring of regular closed subsets of X will be called an *s-ring*.

Proposition 2.1. *Any open subspace of a regular Wallman space possesses an s-ring.*

Proof. Let U be an open subspace of the compact space X and let $\mathcal{F}$ be an s-ring in X. Then it is easy to see that $\mathcal{S} = \{F \cap U \mid F \in \mathcal{F}\}$ is an s-ring in U. □

When A and B are open subsets of X and $A \cap B = \emptyset$, we will write $A + B$ instead of $A \cup B$. If X is a locally compact space and $\mathcal{F}$ is an s-ring in X then we will write $\mathcal{F}^* = \{F \in \mathcal{F} \mid F$ is compact or $(X \backslash F)$ is relatively compact$\}$. Clearly $\mathcal{F}^*$ is an s-ring. In addition, if αX is any compactification of X, we define a collection $\alpha\mathcal{F}$ of X in the following manner:

$S \in \alpha\mathcal{F}$: $\Leftrightarrow$ there are $F \in \mathcal{F}^*$, compact $K \subset X$ and open subsets V_1, V_2 of αX such that:

(i) $F \cap K = \emptyset$,

(ii) $\alpha X \backslash K = V_1 + V_2$ and $S = F \cap V_1$.

Lemma 2.2. *Let X be a locally compact space, αX a compactification of X, and $\mathcal{F}$ an s-ring in X. Then $\alpha\mathcal{F}$ is closed under finite intersections, and $\vee.\alpha\mathcal{F}$ is again an s-ring.*

Proof. First notice that $\alpha\mathcal{F}$ consists of regular closed sets. Secondly we show that $\alpha\mathcal{F}$ is closed under finite intersections. Take S_0, $S_1 \in \alpha\mathcal{F}$. Then for $i \in \{0, 1\}$ there exist $F_i \in \mathcal{F}^*$, compact $K_i \subset X$ and open U_i, $V_i \subset \alpha X$ such that $\alpha X \backslash K_i = U_i + V_i$ and $F_i \cap K_i = \emptyset$ and $S_i = F_i \cap U_i$. Then $S_0 \cap S_1 = (F_0 \cap F_1) \cap (U_0 \cap U_1)$. Since $K_0 \cup K_1$ is compact, $(F_0 \cap F_1) \cap (K_0 \cup K_1) = \emptyset$, and

$$\begin{aligned} \alpha X \backslash (K_0 \cup K_1) &= (\alpha X \backslash K_0) \cap (\alpha X \backslash K_1) \\ &= (U_0 + V_0) \cap (U_1 + V_1) \\ &= (U_0 \cap U_1) + \{(U_0 \cap V_1) \cup (V_0 \cap U_1) \cup (V_0 \cap V_1)\} \end{aligned}$$

it follows that $S_0 \cap S_1 \in \alpha\mathcal{F}$.

Trivially $\mathcal{F}^* \subset \alpha\mathcal{F}$ and hence $\alpha\mathcal{F}$ is separating if $\mathcal{F}^*$ is. To prove the latter, let $x \in X$ and let G be a closed set in X such that $x \notin G$. Take an open U in X such that

$x \in U \subset \mathrm{cl}_X(U)$ and $\mathrm{cl}_X(U) \cap G = \emptyset$, while moreover $\mathrm{cl}_X(U)$ is compact. This is possible, since X is locally compact. Now, $\mathscr{F}$ is separating and therefore there exist $F_0, F_1 \in \mathscr{F}$ such that $x \in F_0$, $X \backslash U \subset F_1$ and $F_0 \cap F_1 = \emptyset$. Evidently $F_0, F_1 \in \mathscr{F}^*$ and hence $\mathscr{F}^*$ is separating. Since the union of finitely many regular closed sets is again regular closed it now follows that $\vee . \alpha\mathscr{F} = \vee . \wedge . \alpha\mathscr{F}$ is an s-ring. □

Theorem 2.3. *Let X be a locally compact space. Then the following properties are equivalent*:

(*i*) *X possesses an s-ring.*

(*ii*) *Any compactification αX of X with zero-dimensional remainder $\rho X = \alpha X \backslash X$ is regular Wallman.*

Proof. (ii) ⇒ (i) Follows from Proposition 2.1.

(i) ⇒ (ii) Let $\mathscr{F}$ be an s-ring in X and let $\mathscr{S} = \{\mathrm{cl}_{\alpha X}(S) \mid S \in \alpha\mathscr{F}\}$. We will show that $\vee . \mathscr{S}$ is an s-ring in αX. Hence αX is regular Wallman.

(a) Let $F \in \mathscr{F}^*$ and let K be a compact subset of X such that $\alpha X \backslash K = V_0 + V_1$ and $F \cap K = \emptyset$; we put $S_i = F \cap V_i$. Then

$$\mathrm{cl}_{\alpha X}(S_i) = S_i \quad \text{or} \quad \mathrm{cl}_{\alpha X}(S_i) = S_i \cup (V_i \cap \rho X).$$

Indeed, if F is compact, then also S_i is compact; consequently $\mathrm{cl}_{\alpha X}(S_i) = S_i$. If $X \backslash F$ is relatively compact, then $\mathrm{cl}_{\alpha X}(F) = F \cup \rho X$ and consequently

$$\begin{aligned}\mathrm{cl}_{\alpha X}(S_i) &= \mathrm{cl}_{\alpha X}(F \cap V_i) \subset (F \cup \rho X) \cap \mathrm{cl}_{\alpha X}(V_i)\\ &\subset (F \cup \rho X) \cap (V_i \cup K) = (F \cap V_i) \cup (\rho X \cap V_i)\\ &= S_i \cup (\rho X \cap V_i).\end{aligned}$$

Since $\mathrm{cl}_{\alpha X}(S_0 \cup S_1) \cap \rho X = \rho X$ and $\mathrm{cl}_{\alpha X}(S_0) \cap \mathrm{cl}_{\alpha X}(S_1) = \emptyset$ it follows that

$$\mathrm{cl}_{\alpha X}(S_i) = S_i \cup (\rho X \cap V_i) \qquad (i \in \{0, 1\}).$$

(b) For all $S_0, S_1 \in \alpha\mathscr{F}$ we have $\mathrm{cl}_{\alpha X}(S_0) \cap \mathrm{cl}_{\alpha X}(S_1) = \mathrm{cl}_{\alpha X}(S_0 \cap S_1)$. If S_0 or S_1 is compact, then this is a triviality. Therefore, suppose neither is compact. For $i \in \{0, 1\}$ let K_i be a compact subset of X, $F_i \in \mathscr{F}^*$ and U_i, V_i open subsets of αX such that $S_i = F_i \cap V_i$, while $\alpha X \backslash K_i = V_i + U_i$ and $F_i \cap K_i = \emptyset$. Then

$$\begin{aligned}\mathrm{cl}_{\alpha X}(S_0) \cap \mathrm{cl}_{\alpha X}(S_1) &= (S_0 \cup (V_0 \cap \rho X)) \cap (S_1 \cup (V_1 \cup \rho X))\\ &= (S_0 \cap S_1) \cup (\rho X \cap V_0 \cap V_1).\end{aligned}$$

Suppose there exists an $x \in (\mathrm{cl}_{\alpha X}(S_0) \cap \mathrm{cl}_{\alpha X}(S_1)) \backslash \mathrm{cl}_{\alpha X}(S_0 \cap S_1)$. Then $x \in V_0 \cap V_1$. Now, as $\mathrm{cl}_{\alpha X}(F_0 \cap F_1) \cap \rho X = \rho X$, it follows that (cf. the proof of Lemma 2.2)

$$\begin{aligned}x \in V_0 \cap V_1 \cap \mathrm{cl}_{\alpha X}(F_0 \cap F_1) &\subset \mathrm{cl}_{\alpha X}((V_0 \cap V_1) \cap (F_0 \cap F_1))\\ &= \mathrm{cl}_{\alpha X}(S_0 \cap S_1),\end{aligned}$$

which is a contradiction.

It follows that $\mathscr{T} = \vee . \wedge . \mathscr{S}$ is a ring consisting of regular closed sets.

(c) $\mathcal{T}$ is separating.

Let $x_0 \in \alpha X$ and let G be a closed set of αX such that $x_0 \notin G$.

If $x_0 \in X$, then the existence of $T_0, T_1 \in \mathcal{T}$ such that $x_0 \in T_0 = \text{cl}_{\alpha X}(T_0)$ and $G \subset \text{cl}_{\alpha X}(T_1)$ and $\text{cl}_{\alpha X}(T_0) \cap \text{cl}_{\alpha X}(T_1) = \emptyset$ is evident. So, we may assume that $x_0 \in \rho X$. Since ρX is zero-dimensional, it possesses a base of open and closed sets. Let C be an open and closed subset of ρX such that $x_0 \in C$ and $C \cap G = \emptyset$. Define $C_0 = \rho X \backslash C$. Then C and C_0 are disjoint closed subsets in αX such that $C_0 \cup C = \rho X$. As αX is normal, there exist open $U_0, U_1 \subset \alpha X$ such that $C_0 \cup G \subset U_0$, $C \subset U_1$ and $U_0 \cap U_1 = \emptyset$. Then $K = \alpha K \backslash (U_0 \cup U_1)$ is a compact subset of X such that $K \cap G = \emptyset$. Choose a relatively compact, open O in X such that $K \subset O \subset \text{cl}_X(O)$ and $\text{cl}_X(O) \cap (G \cap X) = \emptyset$. As $\mathcal{F}^*$ is separating,

$$X \backslash O = \cap\{F \in \mathcal{F}^* \mid X \backslash O \subset F\}$$

and consequently, by the compactness of K, there exists an $F \in \mathcal{F}^*$ such that $X \backslash O \subset F$ and $F \cap K = \emptyset$. Define $S_0 = F \cap U_0$ and $S_1 = F \cap U_1$. From (a) it now follows that $x_0 \in \text{cl}_{\alpha X}(S_1)$ and $G \subset \text{cl}_{\alpha X}(S_0)$, while $\text{cl}_{\alpha X}(S_0) \cap \text{cl}_{\alpha X}(S_1) = \emptyset$.

This completes the proof of the theorem. □

3. Some applications

If X is a topological space, then each compactification αX of X is a quotient of the Čech-Stone compactification βX of X, by a quotient map f which on X is the identity. A point $p \in \alpha X \backslash X$ is called a *multiple point* of αX if $f^{-1}(p)$ consists of more than one point.

Corollary 3.1. *Let X be a topological space and let αX be a compactification of X such that the set M of multiple points is compact and zero-dimensional. If βX is regular Wallman, then also αX is regular Wallman.*

Proof. According to Proposition 2.1, $\alpha X \backslash M$ possesses an s-ring and hence, as αX is a compactification of $\alpha X \backslash M$, αX is regular Wallman (Theorem 2.3). □

Note that in the above corollary X need not be locally compact.

In [7], Misra showed that βX is regular Wallman if X is separable metric and that $\beta(\sum_{i \in I} X_i)$ is regular Wallman if βX_i is regular Wallman for each $i \in I$. It is well known that a locally compact metrizable space is a topological sum of locally compact separable metric spaces and hence βX is regular Wallman in case X is locally compact and metrizable.

Corollary 3.2. *Let X be a locally compact metrizable space. Then each bounding system compactification of Gould, all finite and countable compactifications, all finite multiple point compactifications and the Freudenthal compactification are regular Wallman.*

Proof. Bounding system compactifications of Gould have only 1 multiple point [8] and the Freudenthal compactification has zero-dimensional remainder. □

In [7], Misra also showed that βX is regular Wallman in case X is normal and homeomorphic to a finite product of locally compact ordered spaces. Thus the above corollaries also hold for these spaces.

Added in proof. Recently R. C. Solomon has constructed an example of a compact Hausdorff space which is not regular Wallman (General Topology Appl. 6 (1976)).

More recent yet is V.I. Ul'janov's result that not every Hausdorff compactification is Wallman (Dokl. Akad. Nauk SSR 233 (1977)).

References.

[1] J.M. Aarts, Every metric compactification is a Wallman-type compactification, Proc. Intern. Symp. on Topology and its Applications, Herceg-Novi (Yugoslavia) (1968).

[2] B. Banachschewski, On Wallman's method of compactification, Math. Nachr. 27 (1963) 105–114.

[3] H. Freudenthal, Kompaktisierungen und Bikompaktisierungen, Indag. Math. 13 (1951) 184–192.

[4] O. Frink, Compactifications and semi-normal spaces, Amer. J. Math. 86 (1964) 602–607.

[5] J. van Mill, A note on Wallman compactifications, Nw. Arch. Wisk. 24 (1976) 168–172.

[6] J. van Mill, Compactifications in which the collection of multiple points is Lindelöf semi-stratifiable, Indag. Math. 37 (1976) 349–356.

[7] A.K. Misra, Some regular Wallman βX, Indag. Math. 35 (1973) 237–242.

[8] O. Njåstad, A note on compactifications by bounding systems, J. London Math. Soc. 40 (1965) 526–532.

[9] O. Njåstad, On Wallman-type compactifications, Math. Zeitschrift 91 (1966) 267–277.

[10] A.K. Steiner and E.F. Steiner, Products of compact metric spaces are regular Wallman, Indag. Math. 30 (1968) 428–430.

[11] E.F. Steiner, Wallman spaces and compactifications, Fund. Math. 61 (1968) 295–304.

[12] H. Wallman, Lattices and topological spaces, Annals of Math. 39 (1938) 112–126.