AN EXTERNAL CHARACTERIZATION OF SPACES WHICH ADMIT BINARY NORMAL SUBBASES.

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Abstract. We give an external characterization of spaces which admit binary normal subbases. This is done by constructing sufficiently many nice Urysohn mappings to separate points in the space under consideration.

1. Introduction. In this note we give a characterization of those spaces which admit a closed subbase which is both binary (each of its linked[1] subsystems has a nonvoid intersection) and normal (two disjoint subbase elements are separated by two disjoint complements of subbase sets). A space with a binary closed subbase usually is called *supercompact* (cf. de Groot [5]). Many spaces are supercompact (for example all compact metric spaces, cf. Strok and Szymański [10]) but many compact spaces are not supercompact (for example βN: cf. Bell [1]; or, more generally, all infinite F-spaces: cf. van Douwen and van Mill [3]).

As noted above, supercompact spaces were introduced by de Groot [5]. A space X with a normal closed subbase $\mathbb{S}$, which in addition is T_1 (for all $x \in X$ and $S \in \mathbb{S}$ with $x \notin S$ there is an $S_0 \in \mathbb{S}$ with $x \in S_0$ and $S_0 \cap S = \emptyset$) is completely regular (this was proved by Frink [4] under the additional assumption that $\mathbb{S}$ is closed under finite intersections and finite unions; however, Frink's technique easily can be adapted to obtain the above result). For a nontrivial generalization of this result, see de Groot and Aarts [6]. Spaces which have a closed subbase which is both binary and normal behave surprisingly nicely. They have very rich "geometric" structure. If, in addition, such a space is connected, then the following assertions are true:

i. It is locally connected (cf. Verbeek [12]).
ii. It is generalized arcwise connected (cf. van Mill [8]).
iii. It can be partially ordered in a natural way by means of an order dense partial ordering (cf. van Mill [8]).

Manuscript received February 9, 1977.

[1] A linked subsystem is a subsystem in which each two members meet.

American Journal of Mathematics, Vol. 100, No. 5, pp. 987–994
0002-9327/78/1005-0987 $01.50

iv. It has the fixed point property for continuous maps (cf. van de Vel [11]).
v. It is an absolute retract if it is metrizable (cf. van Mill [7]).

We will give an "external" characterization of spaces possessing a binary normal subbase, by showing that such a space can be embedded in a product of closed unit segments in a prescribed nice way. We use an adaption of Urysohn's lemma to separate the points of the space under consideration by real valued continuous functions, which respect the convexity structure induced by the binary subbase (cf. Schrijver [2, 9]). Then we embed the space into the product of the images under those functions.

On the way we re-prove Frink's [4] result. All topological spaces under discussion are assumed to be T_1.

2. An Adaptation of Urysohn's Lemma. A closed subbase $\mathcal{S}$ for a T_1 space X is defined to be a

i. *T_1-subbase* if for all $x \in X$ and $S \in \mathcal{S}$ with $x \notin S$ there is an $S_0 \in \mathcal{S}$ with $x \in S_0$ and $S_0 \cap S = \varnothing$;
ii. *normal subbase* if for all $S_0, S_1 \in \mathcal{S}$ with $S_0 \cap S_1 = \varnothing$ there are $T_0, T_1 \in \mathcal{S}$ with $S_0 \cap T_1 = \varnothing = T_0 \cap S_1$ and $T_0 \cup T_1 = X$;
iii. *binary subbase* if for all $\mathcal{M} \subset \mathcal{S}$ with $\bigcap \mathcal{M} = \varnothing$ there are $M_0, M_1 \in \mathcal{M}$ with $M_0 \cap M_1 = \varnothing$;
iv. *binary normal subbase* if it is both binary and normal.

It is easy to see that a binary subbase is a T_1-subbase (cf. van Mill and Schrijver [9]) and that each supercompact space is compact.

If $\mathcal{S}$ is a closed subbase, then the collection of all countable intersections of members of $\mathcal{S}$ is denoted by $\mathcal{S}_\delta$. For a, b real the symbol $[a, b]$ denotes the closed real line segment between a and b, no matter whether $a < b$ or $a > b$.

THEOREM 2.1. *Let $\mathcal{S}$ be a normal T_1-subbase for the space X. If p and q are distinct points of X (if P and Q are disjoint sets of $\mathcal{S}$), then there is a function $f: X \to [0,1]$ such that $f(p) = 0$ and $f(q) = 1$ ($f[P] = 0$, $f[Q] = 1$), while for every $t \in [0,1]$ the sets $f^{-1}([0,t])$ and $f^{-1}([t,1])$ are members of $\mathcal{S}_\delta$.*

Proof. In the case of the two points p and q we can define two sets P and Q in $\mathcal{S}$ such that $p \in P$ and $q \in Q$ and $P \cap Q = \varnothing$, because $\mathcal{S}$ is a T_1-subbase for the T_1-space X. Thus we use the same proof for both assertions.

We construct two nests of subbase members, indexed by the dyadic rationals, such that P is in all members of the first, and Q is in all members of the second nest. We proceed by induction.

Start: Choose $H, K \in \mathfrak{S}$ such that $H \cap Q = \varnothing$, $K \cap P = \varnothing$, and $H \cup K = X$. We define $H(\frac{1}{2}) := H$, $K(\frac{1}{2}) := K$, $H(0) := P$, $K(1) := Q$ and finally $H(1) := K(0) := X$. Note that for every two dyadic rationals $d < r$ we have

$$H(d) \subset H(r), \qquad K(r) \subset K(d) \quad \text{and} \quad H(d) \cap K(r) = \varnothing, \qquad (*)$$

$$H(r) \cup K(r) = X \qquad (**)$$

whenever those sets are defined.

Step: Assume that all $H(d)$ and $K(d)$ are defined for $0 \leqslant d \leqslant 1$, where d is a dyadic rational with denominator less than or equal to 2^n, satisfying (*). Moreover we assume that no $H(d)$ or $K(d)$ is as yet defined for dyadic rationals with a larger denominator.

Let $r < s$ be any two consecutive fractions for which $H(r)$ and $K(s)$ are defined. Then precisely one of them has (in simplest terms) denominator 2^n. Choose $S_0, S_1 \in \mathfrak{S}$ such that $H(r) \cap S_1 = \varnothing = S_0 \cap K(s)$ and $S_0 \cup S_1 = X$. Then define $H(\frac{1}{2}(r+s)) := S_0$ and $K(\frac{1}{2}(r+s)) := S_1$. We proceed in this way simultaneously for every two consecutive fractions with denominator less than or equal to 2^n and thus obtain all $H(d)$ and $K(d)$ for dyadic d with denominator 2^{n+1}. The now defined sets, together with the earlier defined ones, still satisfy the induction requirements (*), (**).

This completes the inductive construction.

Now define $f: X \to I$ by

$$f(x) := \inf\{r \mid x \in H(r)\}.$$

Observe that if s is a dyadic rational smaller than $f(x)$, then $x \in K(s)$, and also if s is a dyadic rational larger than $f(x)$, then there is an $r < s$ such that $x \in H(r)$ and hence $x \notin K(s)$. Therefore

$$f(x) = \sup\{s \mid x \in K(s)\}.$$

Moreover for all $t \in [0,1]$ we have that

$$f^{-1}[[0,t]] = \{x \mid \inf\{r \mid x \in H(r)\} \leqslant t\} = \bigcap_{r>t} H(r)$$

and

$$f^{-1}[[t,1]] = \{x \mid \sup\{s \mid x \in K(s)\} \geqslant t\} = \bigcap_{s>t} K(s)$$

respectively. Both inverse images are the intersection of a countable collection of closed subbase members. Therefore f is continuous, and since clearly $f(P) = 0$ and $f(Q) = 1$, this proves the theorem. $\square$

COROLLARY 2.2 (Frink [4]). *A space X is completely regular if and only if it admits a normal closed T_1-subbase.*

Proof. If X is completely regular, then the ring of zero-sets $Z(X)$ is a normal closed T_1-subbase (cf. Frink [4]).

Let $\mathfrak{S}$ be a normal T_1-subbase for X. Given $x \in X$ and a closed set $G \subset X$ not containing x, choose a finite $\mathfrak{F} \subset \mathfrak{S}$ such that $G \subset \bigcup \mathfrak{F}$ and $x \notin \bigcup \mathfrak{F}$. By the proof of Theorem 2.1, for each $F \in \mathfrak{F}$ there is a continuous mapping $f_F : X \to [0,1]$ such that $f_F(x) = 0$ and $f_F[F] = 1$. Next we define

$$g : X \to [0,1]$$

by $g(x) := \max\{ f_F(x) | F \in \mathfrak{F}\}$. Since $\mathfrak{F}$ is finite, g is a continuous mapping. Moreover $g(x) = 0$ and $g[G] = 1$, as can easily be seen. $\square$

3. An External Characterization of Spaces Which Admit Binary Normal Closed Subbases. Let X be a set, and let $I : X \times X \to \mathfrak{P}(X)$. We write $I(x,y)$ for $I((x,y))$. Then I is called an *interval structure* (Schrijver [2, 9]) on X if

i $x, y \in I(x,y)$ $(x, y \in X)$;
ii. $I(x,y) = I(y,x)$ $(x, y \in X)$;
iii. if $u, v \in I(x,y)$, then $I(u,v) \subset I(x,y)$ $(u,v,x,y \in X)$;
iv. $I(x,y) \cap I(x,z) \cap I(y,z) \neq \varnothing$ $(x,y,z \in X)$.

A subset B of X is called *I-convex* iff for all $x, y \in B$ we have $I(x,y) \subset B$.

If X is a supercompact space with binary subbase $\mathfrak{S}$, then the function $I_{\mathfrak{S}} : X \times X \to \mathfrak{P}(X)$ defined by

$$I_{\mathfrak{S}}(x,y) := \cap \{ S \in \mathfrak{S} \,|\, x, y \in S \}$$

is easily seen to be an interval structure. Clearly each $S \in \mathfrak{S}$ is $I_{\mathfrak{S}}$-convex. The converse also is true. *If X is a compact space which admits an interval structure I and a closed subbase $\mathfrak{S}$ consisting of I-convex sets, then $\mathfrak{S}$ is binary* (Schrijver [2, 9]); *in particular X is supercompact.*

Definition 3.1. Let X and Y be supercompact spaces with binary subbases $\mathfrak{S}$ and $\mathfrak{T}$ respectively. We say that a continuous mapping $f : X \to Y$ is *convexity preserving* provided that $f[I_{\mathfrak{S}}(x,y)] = I_{\mathfrak{T}}(f(x), f(y)) \cap f[X]$ for all $x, y \in X$.

THEOREM 3.2. *Let X and Y be supercompact spaces with binary subbases $\mathfrak{S}$ and $\mathfrak{T}$. Let $f : X \to Y$ be a convexity preserving surjection. Then for all $S \in \mathfrak{S}$ we have that $f[S]$ is $I_{\mathfrak{T}}$-convex in Y. Moreover, for all $T \in \mathfrak{T}$ the set $f^{-1}[T]$ is also $I_{\mathfrak{S}}$-convex in X.*

Proof. Given $S \in \mathcal{S}$, let y_0 and y_1 be in $f[S]$. Choose x_0 and x_1 in S such that $f(x_i) = y_i$ $(i \in \{0,1\})$. Then $I_{\mathcal{S}}(x_0, x_1) \subset S$, by definition of $I_{\mathcal{S}}(x_0, x_1)$. Hence $I_{\mathcal{T}}(y_0, y_1) = f[I_{\mathcal{S}}(x_0, x_1)] \subset f[S]$; hence $f[S]$ is $I_{\mathcal{T}}$-convex.

On the other hand, let $T \in \mathcal{T}$ and assume that $f^{-1}[T]$ is not $I_{\mathcal{S}}$-convex. There are $x_0, x_1 \in f^{-1}[T]$ such that $I_{\mathcal{S}}(x_0, x_1) \cap (X \setminus f^{-1}[T]) \neq \varnothing$. But then $\varnothing \neq f[I_{\mathcal{S}}(x_0, x_1)] \cap (Y \setminus T) = I_{\mathcal{T}}(f(x_0), f(x_1)) \cap (Y \setminus T)$, which is a contradiction, since each $T \in \mathcal{T}$ is $I_{\mathcal{T}}$-convex. □

In the remainder of this section we are interested in spaces which admit a closed subbase which is both binary and normal. As noted in the introduction, the existence of such a subbase implies may interesting topological properties of the space under consideration.

Theorem 2.1 gives us enough continuous function to separate the points in a space with a binary normal subbase in a nice way. This gives rise to special embeddings of such a space in a product of unit segments.

Recall that the unit segment $I = [0,1]$ is supercompact. For example, the collection $\mathcal{T}$ of all closed intervals is a binary subbase for I. The interval structure $I_{\mathcal{T}}$ defined by this subbase is very simple to describe. It is easy to see that $I_{\mathcal{T}}(x, y) = [x, y]$ for all $x, y \in I$. In the sequel we use this interval structure on I without further reference.

THEOREM 3.3. *Let X be a space which admits a binary normal closed subbase $\mathcal{S}$. Then for any two distinct $p, q \in X$ there is a continuous convexity preserving mapping $f: X \to I$ such that either $f[X] = \{0,1\}$ or $f[X] = I$ and which sends p to 0 and q to 1.*

Proof. Let $g: X \to I$ be a continuous mapping such that $g(p) = 0$ and $g(p) = 1$, while $g^{-1}[0,t]$ and $g^{-1}[t,1]$ are in $\mathcal{S}_\delta$ for each $t \in I$ (cf. Theorem 2.1).

Case 1: $g[X] \neq I$. Choose $s \in I \setminus g[X]$, and define a mapping $f: X \to I$ by

$$f(x) = \begin{cases} 0 & \text{if } g(x) < s, \\ 1 & \text{if } g(x) > s. \end{cases}$$

Then, in view of Theorem 2.1, f satisfies our requirements, since if for some $u, v \in X$ we have $g(u) \leqslant g(v) < s$, then $I_{\mathcal{S}}(u, v) \subset \bigcap \{S \mid u, v \in S\} \subset \subset \bigcap \{S \mid g^{-1}[0,s] \subset S \text{ and } S \in \mathcal{S}\} = g^{-1}[0,s]$ and $f[I_{\mathcal{S}}(u,v)] = 0$. Similarly, if $g(u) \geqslant g(v) > s$, we obtain $f[I_{\mathcal{S}}(u,v)] = 1$.

Case 2: $g[X] = I$. We will show that g is convexity preserving. For this, let $x, y \in X$ and assume that $g(x) \leqslant g(y)$. First of all notice that $g^{-1}[[g(x), g(y)]]$ is a countable intersection of elements of $\mathcal{S}$ and consequently is $I_{\mathcal{S}}$-convex.

Therefore

$$I_{\mathbb{S}}(x,y) \subset g^{-1}[[g(x),g(y)]],$$

and consequently $g[I_{\mathbb{S}}(x,y)] \subset [g(x),g(y)]$. Now if $g(x)=g(y)$, then clearly $g[I_{\mathbb{S}}(x,y)]=[g(x),g(y)]$. Therefore assume that there is a $t \in [g(x),g(y)]$ such that $g(x)<t<g(y)$. Then, since g is surjective,

$$\begin{aligned}\mathcal{L} := &\{S \in \mathbb{S} \mid g^{-1}[[0,t]] \subset S\} \cup \{S \in \mathbb{S} \mid g^{-1}[[t,1]] \subset S\}\\ &\cup \{S \in \mathbb{S} \mid I_{\mathbb{S}}(x,y) \subset S\}\end{aligned}$$

is a linked subsystem of $\mathbb{S}$,[2] since members in the first and the second collection intersect in $g^{-1}(t)$ and members of the first and the third collection intersect in x etc. Consequently, by binarity of $\mathbb{S}$,

$$\cap \mathcal{L} = g^{-1}[[0,t]] \cap g^{-1}[[t,1]] \cap I_{\mathbb{S}}(x,y) = g^{-1}[\{t\}] \cap I_{\mathbb{S}}(x,y)$$

is nonvoid. We conclude that $g[I_{\mathbb{S}}(x,y)]=[g(x),g(y)]$. This completes the proof of the theorem. □

If $x,y,z \in I$, then let $m(x,y,z)$ be the unique point in $[x,y] \cap [y,z] \cap [z,x]$. We call a subset X in a product of unit segments I^A *triple-convex* provided that for all $x,y,z \in X$ the point p of I^A defined by

$$p_\alpha := m(x_\alpha, y_\alpha, z_\alpha) \qquad (\alpha \in A)$$

also belongs to X. Triple-convex sets need not be convex, and convexity does not imply triple-convexity. We get the following characterization of spaces possessing a binary normal subbase:

THEOREM 3.4. *A compact space X admits a binary normal subbase if and only if it can be embedded as a triple-convex set in a product of closed unit segments.*

Proof. Assume that X is compact and is a triple-convex subspace of I^A. Define an interval structure on X by putting

$$I(x,y) := \bigcap_{\alpha \in A} \pi_\alpha^{-1}[[x_\alpha, y_\alpha]] \cap X$$

[2] A subsystem any two members of which meet.

This indeed is an interval structure, since for all $x,y,z \in X$ we have

$$I(x,y) \cap I(x,z) \cap I(y,z)$$
$$= \bigcap_{\alpha \in A} \left(\pi_\alpha^{-1}[[x_\alpha, y_\alpha]] \cap \pi_\alpha^{-1}[[x_\alpha, z_\alpha]] \cap \pi_\alpha^{-1}[[y_\alpha, z_\alpha]]\right) \cap X$$
$$= \bigcap_{\alpha \in A} \pi_\alpha^{-1}[\{m(x_\alpha, y_\alpha, z_\alpha)\}],$$

which consists of one point and which belongs to X, by assumption. It now follows that

$$\mathbb{S} = \left\{\pi_\alpha^{-1}[0,t] \cap X \,|\, 0 \leqslant t \leqslant 1 \;\&\; \alpha \in A\right\} \cup \left\{\pi_\alpha^{-1}[t,1] \cap X \,|\, 0 \leqslant t \leqslant 1 \;\&\; \alpha \in A\right\}$$

is a binary subbase for $\mathbb{S}$ (cf. the remark preceding Definition 3.1). Hence we only need to show that $\mathbb{S}$ is normal. For this, take $S_0, S_1 \in \mathbb{S}$ such that $S_0 \cap S_1 = \varnothing$. W.l.o.g. $S_0 = \pi_\alpha^{-1}[0,t] \cap X$ for some $\alpha \in A$. Then $[0,t] \cap \pi_\alpha[S_1] = \varnothing$, since $S_0 \cap S_1 = \varnothing$. Take $s \in I$ such that $t < s$ and $\pi_\alpha[S_1] \subset (s,1]$. This is possible because S_1 is compact and hence $\pi_\alpha[S_1]$ is compact too. Then $\{\pi_\alpha^{-1}[0,s] \cap X, \pi_\alpha^{-1}[s,1] \cap X\}$ is the desired covering of X by elements of $\mathbb{S}$.

On the other hand, assume that $\mathbb{S}$ is a binary normal subbase for X. Let $\mathcal{C} := \{f : X \to I \,|\, f$ is convexity preserving and either $f[X] = I$ or $f[X] = \{0,1\}\}$. Then Theorem 3.3 implies that the evaluation mapping $e : X \to I^{\mathcal{C}}$ defined by

$$e(x)_f := f(x)$$

is an embedding. We claim that $e[X]$ is triple-convex in $I^{\mathcal{C}}$. In order to show this, assume $p,q,r \in e[X]$. Then for each $f \in \mathcal{C}$ the point

$$m(p_f, q_f, r_f)$$

belongs to $f[X]$, because either $f[X] = I$ or $f[X] = \{0,1\}$. If we consider the possible inequalities between p_f, q_f, and r_f belonging to some $f \in A$ and p_g, q_g, and r_g belonging to some $g \in A$, we obtain that all intersections of the type $f^{-1}[[0, m(p_f, q_f, r_f)]] \cap g^{-1}[[m(p_g, q_g, r_g), 1]]$ are nonempty. Consequently the subsystem

$$\mathcal{L} := \left\{S \in \mathbb{S} \,|\, \exists f \in \mathcal{C} : f^{-1}[[0, m(p_f, q_f, r_f)]] \subset S \text{ or } f^{-1}[[m(p_f, q_f, r_f), 1]] \subset S\right\}$$

of $\mathbb{S}$ is linked. Choose $x \in \bigcap \mathcal{L}$. Then $e(x)_f = m(p_f, q_f, r_f)$ for all $f \in \mathcal{C}$. This completes the proof of the theorem. $\square$

Acknowledgment. We are grateful to the referee for his valuable comments.

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