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PAROVIČENKO'S CHARACTERIZATION OF $\beta\omega - \omega$ IMPLIES CH

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ABSTRACT. Parovičenko characterized $\beta\omega - \omega$ (dually: the field of subsets of ω modulo the finite sets) under CH. We show that his characterization implies CH.

What we do: It will be convenient to call a space X a Parovičenko space if

(α) X is a zero-dimensional compact space without isolated points with weight $\mathfrak{c}$.

(β) every two disjoint open F_σ's in X have disjoint closures, and

(γ) every nonempty G_δ in X has nonempty interior.

We complete the proof of the following theorem, begun by Parovičenko.

THEOREM. CH *is equivalent to the statement that every Parovičenko space is homeomorphic to* $\beta\omega - \omega$.

[We leave the translation of this theorem in Boolean algebraic language to the reader.]

Parovičenko proved the implication from CH. We prove the converse implication by constructing two real examples of Parovičenko spaces which are not homeomorphic to each other under $\neg$CH.

In [**vD**] it is shown that several other results about spaces satisfying (β), which were proved from CH in the literature, also are in fact equivalent to CH.

How we do it: Recall that if X is a space and $p \in X$, then $\chi(p, X)$, the *character* of p in X, is the minimum cardinality of a neighborhood base for p. We identify cardinals with initial ordinals.

EXAMPLE 1. A Parovičenko space S having a point p such that $\chi(p, S) = \omega_1$.

Let X be any *Parovičenko* space, e.g. $\beta\omega - \omega$. There is an ω_1-sequence $\langle U_\alpha : \alpha < \omega_1 \rangle$ of clopen sets in X with $U_\alpha \subset U_\beta$ if $\beta < \alpha < \omega_1$ ($\subset$ denotes proper inclusion). Let $P = \bigcap_{\alpha<\omega_1} U_\alpha$, and let $S = S/P$, the quotient space obtained from X by collapsing P to one point.

One can easily check that S and $p = \{P\}$ are as required.

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EXAMPLE 2. A Parovičenko space T such that $\chi(x, T) = \mathfrak{c}$ for all $x \in T$.

We define $T = \beta(\omega \times {}^{\mathfrak{c}}2) - \omega \times {}^{\mathfrak{c}}2$, where ${}^{\mathfrak{c}}2$ denotes the product of $\mathfrak{c}$ copies of 2, the two-point discrete space. Clearly T is compact.

$\omega \times {}^{\mathfrak{c}}2$ is (strongly) zero-dimensional, hence so is $\beta(\omega \times {}^{\mathfrak{c}}2)$, **[GJ**, 16.11]. Also, $\omega \times {}^{\mathfrak{c}}2$ is a Lindelöf space with weight $\mathfrak{c}$, hence $\omega \times {}^{\mathfrak{c}}2$ has $\mathfrak{c}^{\omega} = \mathfrak{c}$ clopen subsets, hence $\beta(\omega \times {}^{\mathfrak{c}}2)$ has weight $\mathfrak{c}$. It follows that T is a zero-dimensional space with weight $\leqslant \mathfrak{c}$. There are several reasons that T has weight $\geqslant \mathfrak{c}$ and has no isolated points; one is given below.

T satisfies (β), i.e. T is an F-space, since $\omega \times {}^{\mathfrak{c}}2$ is σ-compact and locally compact, **[GJ**, 14.27].

T satisfies (γ) since $\omega \times {}^{\mathfrak{c}}2$ is real compact and locally compact, **[FG**, 3.1].

For $\alpha < \mathfrak{c}$ denote the αth projection ${}^{\mathfrak{c}}2 \to 2$ by π_α. For $\alpha < \mathfrak{c}$ and $i = 0$ or 1 define

$$K(\alpha, i) = T \cap \mathrm{cl}(\omega \times \pi_\alpha^{\leftarrow}\{i\}).$$

Note that each $K(\alpha, i)$ is a nonempty clopen subset of T and that $K(\alpha, i) = K(\alpha', i')$ iff $\alpha = \alpha'$ and $i = i'$. Define

$$\mathcal{K} = \{K(\alpha, i) \colon \alpha < \mathfrak{c}, i = 0 \text{ or } 1\}.$$

CLAIM. Any intersection of ω_1 distinct members of $\mathcal{K}$ has empty interior.

PROOF OF CLAIM. For symmetry reasons it suffices to prove that $I = \bigcap_{\alpha<\omega_1} K(\alpha, 0)$ has empty interior. Suppose I does not have empty interior. Then there is a clopen U in $\beta(\omega \times {}^{\mathfrak{c}}2)$ such that $\varnothing \neq U \cap T \subset I$. For every $\alpha < \omega_1$ the set $U - (\omega \times \pi_\alpha^{\leftarrow}\{0\})$ is a compact subset of $\omega \times {}^{\mathfrak{c}}2$, and since $U \cap (\omega \times {}^{\mathfrak{c}}2)$ is not compact because $U \cap T \neq \varnothing$, there is an integer n_α such that $\varnothing \neq U \cap (\{n_\alpha\} \times {}^{\mathfrak{c}}2) \subset \{n_\alpha\} \times \pi_\alpha^{\leftarrow}\{0\}$. There is an integer n such that $A = \{\alpha < \omega_1 \colon n_\alpha = n\}$ is infinite. But then $\{n\} \times \bigcap_{\alpha \in A} \pi_\alpha^{\leftarrow}\{0\}$ is a subset of $\{n\} \times {}^{\mathfrak{c}}2$ with nonempty interior, which is impossible.

Let $x \in T$ be arbitrary, and let $\mathcal{U}$ be a neighborhood base for x. The family $\mathcal{F} = \{K \in \mathcal{K} \colon x \in K\}$ has cardinality $\mathfrak{c}$. For each $K \in \mathcal{F}$ there is a $U(K) \in \mathcal{U}$ with $U(K) \subseteq K$, hence $|\mathcal{U}| \geqslant |\mathcal{F}| = \mathfrak{c}$ since the claim implies that $|\{K \in \mathcal{K} \colon U(K) = U\}| \leqslant \omega$ for all $U \in \mathcal{U}$. It follows that $\chi(x, T) = \mathfrak{c}$ since we know already that T has weight $\leqslant \mathfrak{c}$. It also follows that x is not isolated.

REMARKS. (A) If S is constructed from T, then S is homeomorphic to $\beta\omega - \omega$ iff CH holds. Indeed, every nonempty clopen subspace of $\beta\omega - \omega$ is homeomorphic to $\beta\omega - \omega$, but under $\neg$CH no clopen subspace of S which does not contain p is homeomorphic to S.

Note that the fact that $\chi(p, S) = \omega_1$ does not by itself imply that S and $\beta\omega - \omega$ are nonhomeomorphic, since it is consistent with $\neg$CH that $\chi(q, \beta\omega - \omega) = \omega_1$ for some point q of $\beta\omega - \omega$, **[K]**.

(B) We do not know if T can be homeomorphic to $\beta\omega - \omega$ under $\neg$CH. However, it is easy to see that T and $\beta\omega - \omega$ are not homeomorphic under MA + $\neg$CH. For it is well known that MA implies that $(*)$ any nonempty intersection of $< \mathfrak{c}$ open sets in $\beta\omega - \omega$ has nonempty interior, e.g. adapt [**B**, 4.7]. But the claim shows that $\bigcap_{\alpha<\omega_1} K(\alpha, 0)$ is a nonempty intersection of ω_1 open sets with empty interior. Alternatively,

$$\left\{ \bigcap_{\alpha<\omega_1} K(\alpha, i(\alpha)) \colon i(\alpha) = 0 \text{ or } 1 \text{ for } \alpha < \omega_1 \right\}$$

is a cover of T consisting of 2^{ω_1} nowhere dense sets. But (*) implies that $2^{\omega_1} = \mathfrak{c}$, [**R**, p. 43], and (*) clearly implies that $\beta\omega - \omega$ is not the union of $\mathfrak{c}$ nowhere dense sets.

(C) It is well known that CH implies that $\beta\omega - \omega$ has $2^{\mathfrak{c}}$ autohomeomorphisms, [**Ru**, 4.7], but it is now known if this can be true under $\neg$ CH. But clearly T has $2^{\mathfrak{c}}$ autohomeomorphisms.

(D) The proof that $\chi(x, T) = \mathfrak{c}$ for all $x \in T$ is similar to the proof that $\chi(x, \beta\omega - \omega) = \mathfrak{c}$ for some $x \in \beta\omega - \omega$, [**Po**], see e.g. [**C**, 2.7]. Our use of two spaces is similar to the use of two spaces in Weiss' solution of the Blumberg problem, [**W**].

(E) Ryszard Frankiewicz has informed us, without giving a proof, that he has shown that Parovičenko's characterization implies $2^{\omega_1} > \mathfrak{c}$.

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