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Path Connectedness, Contractibility, and LC-Properties of Superextensions

by

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Summary. A number of recent results on the contractibility or on LC-type properties of superextensions are considerably strengthened by means of a technique involving the nearest point map and the convex closure operator on a superextension.

The following results on superextensions have recently been proved:

1. If X is compact, and either contractible or suspended, then its superextension $\lambda(X)$ is contractible (Verbeek [15]). By Theorem 3.1. below, $\lambda(X)$ is even LC*.
2. If X is a metric continuum, then $\lambda(X)$ is an AR (compact metric) (Van Mill [8] or Van de Vel [15]). In particular, $\lambda(X)$ is contractible and LC*.
3. If X is a connected normal T_1 space, then $\lambda(X)$ is acyclic and lc (Van de Vel [15]). (For a definition of lc, see Begle [2]).

In this paper we make a first attempt to fill up the gaps which obviously exist among the above results. We shall concentrate on superextensions of completely regular T_1 spaces.

1. Some definition and preliminary results. A closed subbase $\mathcal{S}$ of a T_1 space X is a called a T_1 *subbase* if for each $S \in \mathcal{S}$ and for each $x \in X-S$, there is an $S' \in \mathcal{S}$ with $x \in S' \subset X-S$. $\mathcal{S}$ is called a *normal subbase* if for each pair $S_1, S_2 \in \mathcal{S}$ of disjoint sets there exist $S_1', S_2' \in \mathcal{S}$ such that

$$S_1 \subset S_1' - S_2';\ S_2 \subset S_2' - S_1';\ S_1' \cup S_2' = X.$$

A *linked system in* $\mathcal{S}$ is a subfamily $\mathcal{S}'$ of $\mathcal{S}$ such that each two members of $\mathcal{S}'$ intersect. The *superextension* $\lambda(X, \mathcal{S})$ of X *relative to a closed subbase* $\mathcal{S}$ is the T_1 space defined on the set of all maximal linked systems (*mls's*) in $\mathcal{S}$, topologized by means of a Wallman-type closed subbase

$$\mathcal{S}^+ = \{S^+ \mid S \in \mathcal{S}\},$$

where S^+ denotes the set of all $\mathcal{L} \in \lambda(X, \mathcal{S})$ with $S \in \mathcal{L}$. If $\mathcal{S} = H(X)$, the set (space) of all nonempty closed subsets of X, then $\lambda(X) = \lambda(X, H(X))$ is called *the* superextension of X. If $\mathcal{S}$ is a T_1-subbase, then there is an obvious embedding of X in $\lambda(X, \mathcal{S})$.

The closed subbase $\mathcal{S}^+$ of $\lambda(X, \mathcal{S})$ has the property that each linked system in $\mathcal{S}^+$ has a nonempty intersection. Such a subbase is called *binary*. By Alexander's lemma, a space carrying a binary subbase is compact (it is called a *supercompact space*). If $\mathcal{S}$ is a binary subbase of X, then obviously $\lambda(X, \mathcal{S}) \approx X$. These notions were introduced by De Groot in [3].

The usual topology for the *hyperspace* $H(X)$ of X is generated by the open base, consisting of all sets of type

$$\langle 0_1, \ldots, 0_n \rangle = \{A \mid A \subset \bigcup_{i=1}^{n} 0_i \text{ and } A \cap 0_i \neq \varnothing \text{ for all } i\},$$

where $0_1, \ldots, 0_n \subset X$ are open (see e.g. Michael [6]).

A closed subset C of $\lambda(X, \mathcal{S})$ is called *convex* (relative ot $\mathcal{S}^+$) if it equals an intersection of subbasic closed sets. The subspace of $H(\lambda(X, \mathcal{S}))$, consisting of all nonempty convex sets in $\lambda(X, \mathcal{S})$, will be denoted by $K(\lambda(X, \mathcal{S}))$. The notion of subbase convexity was introduced in [12].

An important class of convex sets in $\lambda(X, \mathcal{S})$ can be described as follows: Let $\mathcal{M}, \mathcal{N} \in \lambda(X, \mathcal{S})$. The *interval joining* $\mathcal{M}$ *and* $\mathcal{N}$ is the (convex) set

$$I(\mathcal{M}, \mathcal{N}) = \bigcap \{P^+ \mid P \in \mathcal{M} \cap \mathcal{N}\}.$$

(Van Mill and Schrijver [11]). Notice that $I(\mathcal{M}, \mathcal{N})$ is the smallest convex set containing $\mathcal{M}$ and $\mathcal{N}$. More generally, the *convex closure* of a set $A \subset \lambda(X, \mathcal{S})$ is defined to be the set

$$I(A) = \bigcap \{S^+ \mid S \in \mathcal{S} \text{ and } A \subset S^+\}.$$

1.1. THEOREM. *Let X be a T_1 space and let $\mathcal{S}$ be a normal T_1 subbase for X. Then the convex closure map*

$$I\colon H(\lambda(X, \mathcal{S})) \to K(\lambda(X, \mathcal{S}))$$

is a continuous retraction.

See [12].

1.2. NEAREST POINT MAPPING THEOREM. *Let X be a T_1 space and let $\mathcal{S}$ be a normal T_1 subbase for X. If $\mathcal{M} \in \lambda(X, \mathcal{S})$ and if $C \subset \lambda(X, \mathcal{S})$ is nonempty and convex, then there is a unique point $p(\mathcal{M}, C)$ in $\lambda(X, \mathcal{S})$ with the property that*

$$I(\mathcal{M}, p(\mathcal{M}, C)) \cap C = \{p(\mathcal{M}, C)\},$$

and the mapping

$$p\colon \lambda(X, \mathcal{S}) \times K(\lambda(X, \mathcal{S})) \to \lambda(X, \mathcal{S})$$

so obtained, is continuous.

See [12]. p is called the nearest point map of $\lambda(X, \mathcal{S})$ in view of certain metric and order-theoretic considerations: see [13] and [15]. Techniques involving this mapping have already got a variety of applications. See Van Mill [9] and [10], Van de Vel [15]. New applications are given below.

2. Contractibility of certain superextensions. The following general result will be our main tool in deriving contractibility results on $\lambda(X, S)$:

2.1. Lemma. *Let X be a T_1-space and let $\mathscr{S}$ be a normal T_1 subbase for X. Assume that there exists a continuous mapping*

$$\varphi\colon [0, 1] \to H(X)$$

such that $\varphi(0)$ is a singleton, and $\varphi(1) = X$. Then there is a contraction of $\lambda(X, \mathscr{S})$ onto $\varphi(0)$ keeping $\varphi(0)$ fixed.

Proof. Regarding X as a subspace of $\lambda(X, \mathscr{S})$, there is a mapping

$$\psi\colon H(X) \to H(\lambda(X, \mathscr{S}))$$

sending $D \in H(X)$ onto its closure $\bar{D}$ in $\lambda(X, \mathscr{S})$. This map is continuous since $\lambda(X, \mathscr{S})$ is normal, being compact and Hausdorff (Verbeek [16]). Define

$$\varphi'\colon [0, 1] \to H(\lambda(X, \mathscr{S}))$$

as follows:

$$\varphi'(t) = \bigcup\{\psi\varphi(u) \mid u \leqslant t\}.$$

$\varphi'(t)$ is compact, being the union of a compact family of compact sets, and φ' is obviously continuous again. Notice that $\varphi'(0) = \varphi(0)$ and that φ' is increasing.

We now use the convex closure map

$$I\colon H(\lambda(X, \mathscr{S})) \to K(\lambda(X, \mathscr{S})) \quad \text{(cf. section 1)}.$$

It is easy to verify that I preserves singletons, and that $I(D) = D^+$ for each $D \in \mathscr{S}$. Moreover, $\varphi'(1) = \bar{X} \subset \lambda(X, \mathscr{S})$, whence $I\varphi'(1) = \lambda(X, \mathscr{S})$ ($\lambda(X, \mathscr{S})$ is the only convex set containing X).

Let x_0 be the unique point in $I\varphi'(0)$, and define a map

$$F\colon \lambda(X, \mathscr{S}) \times [0, 1] \to \lambda(X, \mathscr{S})$$

by $F(\mathscr{M}, t) = p(\mathscr{M}, I\varphi'(t))$, where p is the nearest point map (see Section 1). Then by the construction of the map,

$$F(\mathscr{M}, 0) = p(\mathscr{M}, \{x_0\}) = x_0;$$

$$F(\mathscr{M}, 1) = p(\mathscr{M}, \lambda(X, \mathscr{S})) = \mathscr{M}.$$

Moreover, $x_0 \in I\varphi'(t)$ for each t, whence

$$F(x_0, t) = p(x_0, I\varphi'(t)) = x_0,$$

proving that F is a contraction of $\lambda(X, \mathscr{S})$ onto x_0 keeping x_0 fixed.

2.2. Corollary. *Let X be a T_1 space, let $\mathscr{S}$ be a normal T_1 subbase of X, and assume that $\lambda(X, \mathscr{S})$ is contractible. Then for each $\mathscr{M}_0 \in \lambda(X, \mathscr{S})$ there is a contraction of $\lambda(X, \mathscr{S})$ onto $\mathscr{M}_0$ keeping $\mathscr{M}_0$ fixed.*

Proof. Let $F: \lambda(X, \mathscr{S}) \times [0,1] \to \lambda(X, \mathscr{S})$ be a contraction of $\lambda(X, \mathscr{S})$. Then there is an associated continuous map

$$\varphi: [0,1] \to H(\lambda(X, \mathscr{S}))$$

defined by $\varphi(t) = F(\lambda(X, \mathscr{S}) \times \{t\})$ (cf. Van de Vel [15]). In particular, $\varphi(0)$ is some singleton and $\varphi(1) = \lambda(X, \mathscr{S})$.

Let $\mathscr{M}_0 \in \lambda(X, \mathscr{S})$ be arbitrary. Then there is a path

$$\alpha: [0, 1] \to \lambda(X, \mathscr{S})$$

with $\alpha(0) = \mathscr{M}_0$ and $\{\alpha(1)\} = \varphi(0)$. Define

$$\psi: [0, 1] \to H(\lambda(X, \mathscr{S}))$$

by

$$\psi(t) = \{\alpha(2t)\} \quad \text{if} \quad 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$\psi(t) = \{\alpha(1)\} \cup \varphi(2t-1) \quad \text{if} \quad \tfrac{1}{2} \leqslant t \leqslant 1.$$

Then ψ is a well-defined continuous path in $H(\lambda(X, \mathscr{S}))$ joining $\{\mathscr{M}_0\}$ with $\lambda(X, \mathscr{S})$. Since $\mathscr{S}^+$ is a normal subbase of $\lambda(X, \mathscr{S})$ and since $\lambda(\lambda(X, \mathscr{S}), \mathscr{S}^+) \approx \approx \lambda(X, \mathscr{S})$ ($\mathscr{S}^+$ is binary), Lemma 2.1 yields that $\lambda(X, \mathscr{S})$ is contractible to $\mathscr{M}_0$ with a contraction keeping $\mathscr{M}_0$ fixed.

As a second application, we now show that, as far as *normal* spaces are concerned, one has to look after *the superextension*:

2.3. COROLLARY. *Let X be a normal T_1 space such that $\lambda(X)$ is contractible. Then for each normal T_1 subbase $\mathscr{S}$ of X, the superextension $\lambda(X, \mathscr{S})$ is also contractible.*

Proof. First, notice that if $f: Y \to Z$ is a continuous map of a T_1 space Y to a normal T_1 space Z, then the mapping

$$H(f): H(Y) \to H(Z),$$

sending $A \subset Y$ to $\mathrm{Cl}_Z f(A)$, is also continuous. If φ is a path in $H(Y)$ joining some singleton with Y, then $H(f) \circ \varphi$ joins some singleton of Z with Z in $H(Z)$, provided that f is onto (or, at least, that $f(Y)$ is dense in Z).

Assume now, that $\mathscr{S}$ is a normal T_1 subbase for X. Then there is a continuous surjection

$$f: \lambda(X) \to \lambda(X, \mathscr{S})$$

(the so-called *Jensen-map*, cf. Verbeek [16]). Notice that $\lambda(X, \mathscr{S})$ is normal, being compact, and being Hausdorff by the normality of the subbase $\mathscr{S}$. Also, each superextension is T_1. If $\lambda(X)$ is contractible, then there is a path in $H(\lambda(X))$ joining some singleton of $\lambda(X)$ with $\lambda(X)$ (see the proof of Corollary 2.2). Combining the above remark with Lemma 2.1 then yields the desired result.

We now come to our main results.

2.4. THEOREM. *Let X be a separable T_1 space such that each finite subset of X is contained in a metric subcontinuum of X. Let $\mathscr{S}$ be a normal T_1 subbase for X. Then $\lambda(X, \mathscr{S})$ is contractible.*

Proof. We need two auxiliary results.

(2.5; 1) *There is an increasing sequence* $(K_n)_{n=0}^{\infty}$ *of metrizable subcontinua of* X, *such that* K_0 *is a singleton, and* $(K_n)_{n=0}^{\infty}$ *converges* X *in* $H(X)$.

Let $\{x_n \mid n \in \mathbf{N}\}$ be a countable (counted) dense subspace of X. For each $n \geqslant 0$ we let L_n be a metric continuum containing $\{x_0, \ldots, x_n\}$. In particular, we choose $L_0 = \{x_0\}$. Then $K_n = \bigcup_{n=0}^{n} L_i$ is a metric continuum, $(K_n)_{n=0}^{\infty}$ is an increasing sequence, and $\bigcup_{n=0}^{\infty} K_n$ is dense in X. Let $\langle 0_1, \ldots, 0_p \rangle$ be a basic open set in $H(X)$ containing X as a member. Then $0_i \neq \emptyset$ for each i, and $\bigcup_{i=1}^{p} 0_i = X$. For each $i = 1, \ldots, p$ there is an $n_i \in \mathbf{N}$ such that $K_{n_i} \cap 0_i \neq \emptyset$, and hence $K_n \cap 0_i \neq \emptyset$ for all $n \geqslant n_i$. If n_0 denotes the maximum of $\{n_1, \ldots, n_p\}$, then $K_n \in \langle 0_1, \ldots, 0_p \rangle$ for each $n \geqslant n_0$, proving that $(K_n)_{n=0}^{\infty}$ converges to X.

(2.5; 2) *If* $K \subset L$ *are metric subcontinua of* X, *then there is a continuous increasing mapping*

$$\varphi: [0, 1] \to H(X)$$

with $\varphi(0) = K$ *and* $\varphi(1) = L$.

Using the fact that $H(L) \subset H(X)$, this statement is a direct consequence of a result of Borsuk and Mazurkiewicz, which can be found e.g. in Kuratowski [5] p. 127.

We now combine the two statements. For each $n > 0$ we have a continuous increasing map (with rearranged domain)

$$\varphi_n: \left[1 - \frac{1}{n},\ 1 - \frac{1}{n+1}\right] \to H(X)$$

such that $\varphi_n\left(1 - \frac{1}{n}\right) = K_{n-1}$ and $\varphi_n\left(1 - \frac{1}{n+1}\right) = K_n$. Since each φ_n is monotonic and since $(K_n)_{n=0}^{\infty}$ converges to X, the join

$$\varphi: [0, 1] \to H(X)$$

of the maps φ_n, with $\varphi(1) = X$, is also continuous. Applying Lemma 2.1 then proves the Theorem.

Some classes of spaces satisfying the hypotheses of Theorem 2.4. are worth mentioning (all spaces are T_1 and completely regular):

(i) the class of all separable, path connected spaces, with, in particular, the class of separable topological vector spaces;

(ii) the class of all separable, compactly connected, compactly metrizable spaces. (If P is a topological property, then a space is called compactly P if each compact subspace is contained in a compact subspace satisfying P).

As a particular consequence of Theorem 2.4, it follows that $\lambda(\mathbf{R})$ is contractible, in contrast with the fact that the Cěch–Stone compactification $\beta(\mathbf{R}) \subset \lambda(\mathbf{R})$ is not contractible, not even path connected. The contractibility of $\lambda(\mathbf{R})$ was claimed previously by Verbeek ([16] p. 133). His proof is wrong, however, as it relies on the contrability of $\beta(\mathbf{R})$.

By (i) above, a countable product of real lines also has contractible superextensions. Notice that R^∞ is homeomorphic to the space l_2 of all square summable sequences in $\mathbf{R}$ by a result of Anderson [1].

Recall that a space X is said to be of *category* $\leqslant n$, where $n > 0$ is a natural number, if X equals the union of n closed subspaces, each deformable to a point in X. X is of *finite category* if it is of category $\leqslant n$ for some n.

2.5. THEOREM. *Let X be a connected T_1 space of finite category, containing a dense σ-compact subspace. If $\mathscr{S}$ is a normal T_1 subbase for X then $\lambda(X, \mathscr{S})$ is contractible.*

Proof. Let $C_1, \ldots, C_p$ be nonempty closed subspaces of X such that $X = \bigcup_{i=1}^{p} C_i$, and such that C_i is deformable to a point $x_i \in X$, i.e. there is a mapping

$$F_i \colon C_i \times [0, 1] \to X$$

with $F_i(-, 0)$ = constant map onto x_i, and $F_i(-, 1)$ = inclusion map of C_i in X. Let $(K_n)_{n=0}^{\infty}$ be a sequence of compact subspaces of X such that $\bigcup_{n=0}^{\infty} K_n$ is dense in X. We may assume that this sequence is increasing, and that $K_n \cap C_i \neq \emptyset$ for all n, i.

Fix $i \in \{1, \ldots, p\}$ for a while. For each $n \geqslant 0$ there is an associated continuous map

$$\psi_{n,i} \colon [0, 1] \to H(X)$$

with $\psi_{n,i}(t) = F_i(K_{n,i} \times \{t\})$, where $K_{n,i} = K_n \cap C_i$ (Van de Vel [15]). Notice that $\psi_{n,i}(0) = \{x_i\}$ and that $\psi_{n,i}(1) = K_{n,i}$. We put $\varphi'_{0,i} = \psi_{0,i}$, and for each $n \geqslant 0$ we let

$$\varphi'_{n+1,i} \colon [0, 1] \to H(X)$$

be a path joining $K_{n,i}$ with $K_{n+1,i}$ (using the paths $\psi_{n,i}$ and $\psi_{n+1,i}$). These mappings are now made monotonic and compatible as follows. Define $\varphi_{0,i}$ by

$$\varphi_{0,i}(t) = \bigcup_{t' \leqslant t} \varphi'_{0,i}(t').$$

Notice that $\varphi_{0,i}$ has compact values, and $K_{0,i} \subset \varphi_{0,i}(1)$. If $\varphi_{0,i}, \ldots, \varphi_{n,i}$ have been constructed such that each $\varphi_{m,i}$ is continuous, monotonic, with compact values, and such that $\varphi_{m,i}(1) = \varphi_{m+1,i}(0)$ if $m < n$, and $K_{n,i} \subset \varphi_{n,i}(1)$, then we put

$$\varphi_{n+1,i}(t) = \bigcup_{t' \leqslant t} \varphi'_{n+1,i}(t') \cup \varphi_{n,i}(1).$$

Hence, $\varphi_{n+1,i}$ is continuous, monotonic, and with compact values again. Moreover,

$$\varphi_{n+1,i}(0) = \varphi'_{n+1,i}(0) \cup \varphi_{n,i}(1) = \varphi_{n,i}(1),$$

since $\varphi'_{n+1,i}(0) = K_{n,i} \subset \varphi_{n,i}(1)$, and

$$K_{n+1,i} = \varphi'_{n+1,i}(1) \subset \bigcup_{t \leqslant 1} \varphi'_{n+1,i}(t) \cup \varphi_{n,i}(1) = \varphi_{n+1,i}(1),$$

completing the inductive construction.

Let D_i be the closure of $\bigcup_{n=0}^{\infty} \varphi_{n,i}(1)$. We now proceed as in the proof of Theorem 2.4: the sequence $(\varphi_{n,i}(1))_{n=0}^{\infty}$ converges to D_i in $H(X)$, and the monotonic maps $\varphi_{n,i}$ can be joined such as to yield a mapping

$$\varphi_i: [0, 1] \to H(X)$$

with $\varphi_i(1) = D_i$. In particular, $\varphi_i(0) = \{x_i\}$.

Proceeding as above for each $i \in \{1, \ldots, p\}$ we obtain mappings $\varphi_i: [0, 1] \to$ $\to H(X)$ with

$$\varphi_i(1) = D_i = \mathrm{Cl}_X\Big(\bigcup_{n=0}^{\infty} \varphi_{n,i}(1)\Big),$$

where $\varphi_{n,i}(1) \supset K_{n,i} = K_n \cap C_i$. Hence

$$\bigcup_{i=1}^{p} D_i = Cl_X\Big(\bigcup_{i=1}^{p} \bigcup_{n=0}^{\infty} \varphi_{n,i}(1)\Big) \supset Cl_X\Big(\bigcup_{n=0}^{\infty} \bigcup_{i=1}^{p} K_n \cap C_i\Big) = Cl_X\Big(\bigcup_{n=0}^{\infty} K_n\Big) = X.$$

A connected space of finite category is easily seen to be path connected. Fix $x_0 \in X$ and for each $i = 1, \ldots, p$, fix a path

$$\alpha_i: [0, 1] \to X$$

joining $x_0 = \alpha_i(0)$ with $x_i = \alpha_i(1)$. A path

$$\varphi: [0, 1] \to H(X)$$

joining $\{x_0\}$ with X can now be constructed as follows:

$$\varphi(t) = \{\alpha_i(2t) \mid i = 1, \ldots, p\} \qquad \text{if} \quad 0 \leqslant t \leqslant \tfrac{1}{2};$$

$$\varphi(t) = \varphi(\tfrac{1}{2}) \cup \bigcup_{i=1}^{p} \varphi_i(2t-1) \qquad \text{if} \quad \tfrac{1}{2} \leqslant t \leqslant 1.$$

This map is continuous, $\varphi(0) = \{x_0\}$, and $\varphi(1) \supset \bigcup_{i=1}^{p} D_i = X$ as we computed above. Lemma 2.1 can now be applied.

This theorem includes the contractibility results of Verbeek, mentioned in the introduction. In fact, a contractible (compact) space is of category 1 and a (compact) suspension is of category $\leqslant 2$.

A new class of examples is provided by the topological vector spaces, which are densely σ-compact, e.g. (uncountable) products of real lines. Countable products of $\mathbf{R}$ are even separable, and Theorem 2.4 can be called on in this case.

3. LC$^\infty$ and LC* spaces. We now use the nearest point mapping on a superextension to obtain LC-type properties. The reader is referred to Hu [4] for a definition of LC$^\infty$ and of LC*.

3.1. THEOREM. *Let X be a T_1 space that admits a normal binary subbase. Then*
(i) *X is LC$^\infty$ if it is path connected,*
(ii) *X is LC* if it is contractible.*

Proof. The following results can be found in [15]:

(3.1; 1) Each mapping of an n-sphere, $n>0$, into $\lambda(X)$ is homotopic to a constant map;

(3.1; 2) Each point of $\lambda(X)$ has a neighbourhood base, consisting of convex sets;

(3.1; 3) Each convex subset of $\lambda(X)$ is a retract of $\lambda(X)$.

The latter result first appeared in [9]. Its short proof involves the nearest point map p: if $C\subset\lambda(X)$ is convex, then

$$p(-, C): \lambda(X)\to\lambda(X)$$

is a retraction of $\lambda(X)$ onto C.

Theorem 3.1 is a direct consequence of these results, using the fact that a space with a normal binary subbase is a retract of its superextension (Van Mill [7]). Actually, the above cited results can be proved directly on the original space X, using the same method as in the $\lambda(X)$-case.

3.2. Corollary. *Let $\mathcal{S}$ be a normal T_1 subbase for the T_1 space X. Then $\lambda(X, \mathcal{S})$ is a contractible LC* space in each of the following cases:*
(i) *X is a densely σ-compact, connected space of finite category;*
(ii) *X is separable, compactly connected, and compactly metrizable;*
(iii) *X is separable and path connected.*

Notice that (i) covers the case of contractible or suspended compacta, and that (ii) covers the case of metric continua.

4. Some remarks and problems. In addition to the contractibility results of Van Mill and Verbeek on the superextension of a compact space, we have now proved that $\lambda(X)$ is also contractible if X is separable, compact, and path connected, or if X is a continuum of finite category. However, the following problem remains open:

4.1. Question. *Find necessary and sufficient conditions on a continuum X for $\lambda(X)$ to be path connected/contractible. Are there path connected non-contractible superextensions of continua?*

Concerning the first part of the question, we found the following examples:

4.2. Examples. (i) *Let X be a compact tree which is not path connected. Then $\lambda(X)$ is not path connected.*
(ii) *Let $X=\beta(\mathbf{R})$, the Cěch–Stone compactification of the real line $\mathbf{R}$. Then X is not path connected, but $\lambda(X)$ is contractible.*

The proofs are simple:
(i) A compact tree admits a normal binary subbase (Van Mill and Schrijver [11], and hence it is a retract of its superextension (Van Mill [7]);
(ii) $\lambda(\beta(\mathbf{R}))$ is homeomorphic to $\lambda(\mathbf{R})$ (Verbeek [16]).

Concerning the second part of question 4.1, Theorem 2.4 implies that path connectedness and contractibility are equivalent on separable superextensions.

It is well-known that AR's in the category of compact spaces are contractible and locally contractible (LC*) (see e.g. Saalfrank [14]). The two properties are not equivalent in general. However, in view of Van Mill's result that $\lambda(X)$ is an AR (compact metric) if X is a metric continuum, and in view of the nice convexity structure of superextensions, one is led to the following

4.3. Problem. *Find conditions on a continuum X in order that $\lambda(X)$ be an AR (compact).*

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Й. ван Милл, М. Ван де Вель, **Связанность, сжимаемость и LC-свойство суперрасширений**

Содержание. Число последних результатов по сжимаемости или по свойствам LC-типа значительно увеличилось с помощью техники' включающей ближайшую точку отображения и оператор выпуклого замыкания на суперрасширении.