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SOUSLIN DENDRONS

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ABSTRACT. A dendron is a continuum in which every two distinct points have a separation point. We call a dendron X a Souslin dendron provided that X satisfies the countable chain condition, is not separable and has the additional property that every countable subset of X is contained in a metrizable subcontinuum of X. We prove that the existence of a Souslin line is equivalent to the existence of a Souslin dendron. In addition, each Souslin dendron is a continuous image of some Souslin continuum.

1. Introduction. A *dendron* or compact tree-like space is a connected compact Hausdorff space (or briefly "continuum") in which every two distinct points have a separation point. Clearly every orderable continuum is a dendron; however the class of dendrons is much bigger (see e.g. Kok [7]).

We call a dendron X a *Souslin dendron* provided that it satisfies the following three conditions

(i) X satisfies the countable chain condition;

(ii) X is not separable;

(iii) each countable subset of X is contained in a metrizable subcontinuum of X.

(Notice that the condition (iii) implies that a Souslin line never is a Souslin dendron.)

In this paper we will prove that the existence of a Souslin line is equivalent to the existence of a Souslin dendron. Hence $\diamond$ implies that there is a Souslin dendron (cf. Jensen [5]) and MA + $\neg$CH implies that there is no Souslin dendron (cf. Rudin [13]). We will also prove that each Souslin dendron is the continuous image of some Souslin line.

2. Dendrons. Let X be a dendron. For all distinct $a, b \in X$ let $S(a, b) \subset X$ be defined by

$$S(a, b) := \{x \in X \mid x \text{ separates } a \text{ from } b\} \cup \{a, b\}.$$

It is well known that $S(a, b)$ is an orderable continuum (cf. Proizvolov [10], Kok [7]). In fact $S(a, b)$ is ordered by the usual cut point ordering. In [7], Kok has proved that $S(a, b)$ can also be represented as the intersection of all closed connected subsets of X containing $\{a, b\}$. This implies that if X is a Souslin dendron then $S(a, b)$ is metrizable and moreover $S(a, b)$ is order isomorphic to [0, 1], the closed unit interval (cf. Ward [14]). This observation leads to the following:

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2.1 LEMMA. *Let X be a dendron which is ccc and not separable. Then the following statements are equivalent*:

(i) *X is a Souslin dendron*;

(ii) *for all distinct $a, b \in X$: $S(a, b)$ is metrizable*;

(iii) *for all distinct $a, b \in X$: $S(a, b)$ is homeomorphic to* $[0, 1]$;

(iv) *for all distinct $a, b \in X$: $S(a, b)$ is separable.*

PROOF. (iv) $\Rightarrow$ (i) The union of all $S(a_1, a_2)$ with $a_1, a_2 \in A$, in which A is a countable subset of X, is separable. Its closure is contained in a separable dendron, and (i) now follows from Proizvolov [11]. The other implications are clear. □

Let X be a dendron. For all $B \subset X$ the intersection of all subcontinua of X containing B is denoted by $S(B)$. By the above cited result of Kok [7] it follows that $S(B)$ is a subcontinuum of X.

The following proposition follows from earlier results (cf. van Mill and Schrijver [8], van Mill and van de Vel [9]).

2.2 PROPOSITION. *Let X be a dendron and let $A \subset X$ be a subcontinuum. Then the mapping $r: X \to A$ defined by*

$$\{r(x)\} = \bigcap_{a \in A} S(\{x, a\}) \cap A$$

is a retraction.

3. ccc Dendrons. In this section we investigate some special properties of ccc dendrons and prove an important lemma which is used in the proof of the main result in the present paper.

3.1 THEOREM. *Let X be a dendron which satisfies the countable chain condition. Then X is hereditarily ccc, hereditarily Lindelöf and consequently is perfectly normal.*

PROOF. By a result of Cornette [2] there is an ordered continuum L and a continuous surjection $f: L \to X$. Let $A \subset L$ be a closed set such that $f \restriction A$ is irreducible, that is, if B is a closed subset of A with $f[B] = X$ then $B = A$ (the existence of A is an easy consequence of Zorn's lemma). For all nonvoid open $U \subset A$ define $U^\# \subset X$ by $U^\# := X \setminus f[A \setminus U]$. Since f is closed and irreducible the set $U^\#$ is open and nonvoid. In addition, for all nonvoid open $U, V \subset A$ we have that $U \cap V = \emptyset$ implies $U^\# \cap V^\# = \emptyset$. This implies that A is ccc. An ordered space which satisfies the countable chain condition is hereditarily ccc and hereditarily Lindelöf (this is well known, see for instance Faber [3]). We conclude that X is hereditarily ccc and hereditarily Lindelöf. □

3.2 COROLLARY (MA + $\neg$CH). *Let X be a dendron. Then the following statements are equivalent*:

(i) *X is metrizable*;

(ii) *X is ccc.*

PROOF. (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (i). Follows from 3.1; 2.1 and Juhász [6].□

3.3 LEMMA. *Let X be a dendron which satisfies the countable chain condition. Let X_α and X_β be subcontinua of X such that $X_\alpha \subset X_\beta$. Let r_α (resp. r_β) be the retractions of X onto X_α (resp. X_β) described in Proposition 2.2. For $\delta \in \{\alpha, \beta\}$ let $M_\delta := \{x \in X_\delta \mid |r_\delta^{-1}(x)| \geq 2\}$. Then*
(i) *for all $x \in X$: $r_\alpha r_\beta(x) = r_\alpha(x)$;*
(ii) *if $\delta \in \{\alpha, \beta\}$ then $r_\delta^{-1}(x) \setminus \{x\}$ is open for all $x \in X_\delta$;*
(iii) *if $\delta \in \{\alpha, \beta\}$ then $|M_\delta| \leq \omega$.*

PROOF. (i) is trivial using the precise definition of r_α and r_β and (iii) follows from (ii) since X is ccc. To prove (ii), take $x \in M_\alpha$ and $y \in r_\alpha^{-1}(x) \setminus \{x\}$ and let p be a separation point of x and y. Since $S(y, x) \cap M_\alpha = \{x\}$, as can easily be seen, it follows that $S(p, x) \cap M_\alpha$ also equals $\{x\}$ (notice that $S(p, x) \subset S(y, x)$). Consequently $r_\alpha(p) = x$, i.e. $p \in r_\alpha^{-1}(x)$. Let U be the component of $X \setminus \{p\}$ containing y. Then U is open since X is locally connected (cf. Gurin [4]).

We claim that $U \subset r_\alpha^{-1}(x)$. Indeed, take $q \in U$. Then

$$S(q, p) \subset U \cup \{p\},$$

since $U \cup \{p\}$ is connected and closed. In addition, $U \cup \{p\}$ does not intersect X_α, since p separates x from y and $x \in X_\alpha$ and X_α is connected. Therefore

$$\begin{aligned} S(q, x) \cap X_\alpha &\subset (S(q, p) \cup S(p, x) \cap X_\alpha) \\ &= (S(q, p) \cap X_\alpha) \cup (S(p, x) \cap X_\alpha) \\ &\subset ((U \cup \{p\}) \cap X_\alpha) \cup \{x\} = \{x\}, \end{aligned}$$

which implies that $r_\alpha(q) = x$. Hence $U \subset r_\alpha^{-1}\{x\}$ and consequently $r_\alpha^{-1}(x) \setminus \{x\}$ is open. □

4. The existence of Souslin dendrons. In this section we prove that the existence of a Souslin line implies the existence of a Souslin dendron. We use inverse limit techniques. For more information concerning inverse limits we refer to Capel [1].

$(X_\alpha, f_{\alpha\beta}, \kappa)$, where κ is an ordinal number, means that for all $\alpha < \kappa$, X_α is a topological space and that for all $\beta < \alpha < \kappa$, $f_{\alpha\beta}: X_\alpha \to X_\beta$ is continuous such that $\gamma < \beta < \alpha$ implies that $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$.

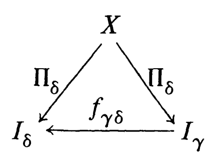
4.1 LEMMA. *Let X be a compact connected ordered space of weight at most ω_1. Then there is an inverse system $(I_\alpha, f_{\alpha\beta}, \omega_1)$ where each I_α is a copy of the closed unit interval and where each $f_{\alpha\beta}$ is monotone and onto such that inv lim $(I_\alpha, f_{\alpha\beta}, \omega_1)$ is homeomorphic to X. In addition, the mappings $f_{\alpha\beta}$ can be chosen in such a way that they have at most one nondegenerate point inverse if $\alpha = \beta + 1$.*

PROOF. Let D be a dense subset of X of cardinality at most ω_1. Let

$$E := \{(d_0, d_1)|d_0, d_1 \in D \text{ and } d_0 \neq d_1\}.$$

In addition, let Γ be the collection of nonlimit ordinals in ω_1. List E as $\{(d_0^\gamma, d_1^\gamma)|\gamma \in \Gamma\}$. We will now inductively construct the inverse system approximating X. We will do it in such a way that for each $\alpha < \omega_1$ there is a monotone surjection Π_α: $X \to I_\alpha$ such that for all $\alpha < \beta < \omega_1$ we have that $\Pi_\alpha = f_{\beta\alpha} \circ \Pi_\beta$ while in addition for each $\gamma \in \Gamma$ the points $\Pi_\gamma(d_0^\gamma)$ and $\Pi_\gamma(d_1^\gamma)$ are distinct.

Suppose that we have completed the construction for all $\alpha < \beta$. First suppose that $\beta = 0$. Without loss of generality assume that $d_0^0 < d_1^0$. It now is easy to construct a monotone Urysohn mapping f: $X \to I = [0, 1]$ such that $f(d_0^0) = 0$ and $f(d_1^0) = 1$. Define $I_0 := I$ and $\Pi_0 := f$. If $\beta \neq 0$ we consider the case that β is a limit ordinal first. Consider the inverse system $(I_\alpha, f_{\alpha\gamma}, \beta)$. Since for each $\gamma < \beta$ there is, by induction hypothesis, a monotone surjection Π_γ: $X \to I_\gamma$ such that for each $\delta < \gamma < \beta$ the diagram



commutes, the mapping e: $X \to \text{inv} \lim(I_\alpha, f_{\alpha\gamma}, \beta)$ defined by

$$e(x)_\gamma := \Pi_\gamma(x) \qquad (\gamma < \beta)$$

is a continuous surjection. It is easily seen that e is monotone. Let $J = \text{inv} \lim(I_\alpha, f_{\alpha\gamma}, \beta)$. By a result of Capel [**1**], J is an ordered compactum, while in addition J is metrizable since β is a countable ordinal. Hence J is homeomorphic to [0, 1]. Define $I_\beta := J$ and for all $\alpha < \beta$ let $f_{\beta\alpha}$ be the projection of $J = \text{inv} \lim(I_\alpha, f_{\alpha\gamma}, \beta)$ onto I_α; in addition define $\Pi_\beta := e$. It is easy to show that our inductive assumptions are satisfied. If β is a nonlimit, say $\beta = \alpha + 1$, then there are two cases; if $\Pi_\alpha(d_0^\beta) \neq \Pi_\alpha(d_1^\beta)$ then we do nothing, i.e. we define $I_\beta := I_\alpha$, $\Pi_\beta := \Pi_\alpha$ and for all $\gamma < \beta$ define $f_{\beta\gamma} := f_{\alpha\gamma}$ if $\gamma \neq \alpha$ and $f_{\beta\gamma} = \text{id}_{X_\alpha}$ if $\gamma = \alpha$. Now suppose that $d_0^\beta < d_1^\beta$ and that $\Pi_\alpha(d_0^\beta) = \Pi_\alpha(d_1^\beta)$. (W.l.o.g. we may assume that Π_α is order preserving.)

Let g: $[d_0^\beta, d_1^\beta] \to I = [0, 1]$ be a continuous monotone surjection such that $g(d_0^\beta) = 0$ and $g(d_1^\beta) = 1$. In I_α split the point $\Pi_\alpha(d_0^\beta)$ in two points, say a and b; let $a < b$ and identify a with 0 and b with 1. The resulting set J is ordered in the natural way, hence is homeomorphic to the closed unit interval. Define a mapping f: $X \to J$ by

$$\begin{cases} f(x) = \Pi_\alpha(x) & \text{if } x < a, \\ f(x) = \Pi_\alpha(x) & \text{if } b < x, \\ f(x) = g(x) & \text{if } a \leqslant b \leqslant x. \end{cases}$$

Then f is monotone and hence continuous. Let h: $J \to I_\alpha$ be the mapping which collapses $[a, b]$ to $\Pi_\alpha(d_0^\beta)$. Define $I_\beta := J$, $\Pi_\beta := f$ and $f_{\beta\alpha} := h$. Finally define $f_{\beta\gamma}(\gamma < \alpha)$ as the composition of $f_{\beta\alpha}$ and $f_{\alpha\gamma}$. It is easily seen that our inductive hypotheses are satisfied. This completes the transfinite construction.

Now define a mapping e: $X \to \operatorname{inv}\lim(I_\alpha, f_{\alpha\beta}, \omega_1)$ by $e(x)_\alpha := \Pi_\alpha(x)$. Then e is well defined and consequently is a continuous surjection (X is compact!). It suffices to prove that e is one-to-one. Indeed, take $x, y \in X$ such that $x < y$; choose distinct $d_0, d_1 \in D$ such that $x < d_0 < d_1 < y$. Then $(d_0, d_1) \in E$ and hence there is a nonlimit ordinal number β such that $\Pi_\beta(d_0) \neq \Pi_\beta(d_1)$. By the fact that Π_β is monotone it now follows that $\Pi_\beta(x) \neq \Pi_\beta(y)$; consequently $e(x) \neq e(y)$. We conclude that e is a homeomorphism. □

4.2 Theorem. *The existence of a Souslin line implies the existence of a Souslin dendron.*

Proof. It is well known that the existence of a Souslin line implies the existence of a Souslin continuum of weight ω_1, cf. Rudin [**13**]. Let L be a Souslin continuum of weight (or, equivalently, density) ω_1. By Lemma 4.1, $L \approx \operatorname{inv}\lim(I_\alpha, f_{\alpha\beta}, \omega_1)$ where each $f_{\alpha\beta}$ is monotone. For each $\alpha < \omega_1$ we construct a metric dendron T_α and a mapping ξ_α: $I_\alpha \overset{\text{onto}}{\to} T_\alpha$ such that for each $\alpha < \beta < \omega_1$ there is a monotone *retraction* $r_{\beta\alpha}$: $T_\beta \to T_\alpha$ with the property that the diagram

$$\begin{array}{ccc} I_\alpha & \xleftarrow{\;f_{\beta\alpha}\;} & I_\beta \\ \xi_\alpha \downarrow & & \downarrow \xi_\beta \\ T_\alpha & \xleftarrow{\;r_{\beta\alpha}\;} & T_\beta \end{array}$$

commutes.

Suppose that the construction is completed for all $\alpha < \beta < \omega_1$. If $\beta = 0$ then define $T_0 := I_0$ and $\xi_0 := \mathrm{id}_{I_0}$. If β is a limit ordinal set $T_\beta = \operatorname{inv}\lim(T_\alpha, r_{\alpha\gamma}, \beta)$ and define all mappings in the obvious way. It is easy to see that T_β indeed is a dendron since the inverse limit of dendrons with monotone surjective bonding maps is a dendron (this result is not stated explicitly in Capel [**1**], but it can be proved using the same technique). If β is a successor, say $\beta = \alpha + 1$, consider the point x for which $f_{\beta\alpha}^{-1}(x)$ is nondegenerate. (If it exists; otherwise we do nothing.)

Let $Z := T_\alpha \times I$ and consider the subspace $Y = (T_\alpha \times \{0\}) \cup (\{\xi_\alpha(x)\} \times I)$. This space clearly is a metric dendron and in addition the projection Π_1: $Y \to T_\alpha$ onto the first coordinate is (equivalent to) a retraction. Let a and b be the endpoints of the interval $f_{\beta\alpha}^{-1}(x)$. Let ϕ: $f_{\beta\alpha}^{-1}(x) \to I$ be a continuous surjection such that $\phi(a) = \phi(b) = 0$. Now define $T_\beta := Y$, $r_{\beta\alpha} := \Pi_1$ and ξ_β by the following rules

$$\begin{cases} \xi_\beta(y) = (\xi_\alpha f_{\beta\alpha}(y), 0) & \text{if } y \notin [a, b], \\ \xi_\beta(y) = (\xi_\alpha(x), \phi(y)) & \text{if } y \in [a, b]. \end{cases}$$

It then is easily seen that our inductive hypotheses are satisfied.

Now put $T = \text{inv lim}(T_\alpha, r_{\alpha\beta}, \omega_1)$. Then there is a continuous surjection f: $L \to T$ which implies that T is perfectly normal and ccc. That T is a dendron is obvious (cf. the above remark). Since each $r_{\beta\alpha}$ is a retraction it is easy to identify each T_α in a canonical way with a subspace of T. We will do so. Then since $T = \text{inv lim}(T_\alpha, r_{\alpha\beta}, \omega_1)$ we see that $\bigcup_{\alpha<\omega_1} T_\alpha$ is dense in T and also that $\alpha < \beta$ implies that $T_\alpha \subset T_\beta$. Since T is first countable (cf. 3.1) and since there are precisely ω_1 distinct T_α's we conclude that $\bigcup_{\alpha<\omega_1} T_\alpha$ is closed in T. Consequently $\bigcup_{\alpha<\omega_1} T_\alpha = T$, since $\bigcup_{\alpha<\omega_1} T_\alpha$ is dense in T. This implies that T is not separable and also that each countable subset of T is contained in some T_α. The T_α's being metrizable we conclude that T is a Souslin dendron. □

5. The existence of Souslin lines. In this section we prove that each Souslin dendron is a continuous image of some Souslin line. As a corollary it follows that the existence of a Souslin line is equivalent to the existence of a Souslin dendron.

5.1 THEOREM. *Each Souslin dendron is a continuous image of some Souslin continuum.*

PROOF. Let T be a Souslin dendron. We construct a Souslin continuum L which can be mapped continuously onto T by means of transfinite induction using a suitable inverse limit system $(T_\alpha, r_{\alpha\beta}, \omega_1)$ of metrizable dendrons approximating T. The T_α's are comparable subdendrons of T and the $r_{\alpha\beta}$'s are the canonical retractions between them (cf. Proposition 2.2).

For every $\alpha < \omega_1$ we now construct a metrizable subcontinuum $T_\alpha \subset T$ such that

(i) $\beta < \alpha < \omega_1 \to T_\beta \subsetneq T_\alpha$;

(ii) $|r_{\alpha+1}^{-1}(x)| = 1$ for each $x \in T_\alpha$, where $r_{\alpha+1}$: $T \to T_{\alpha+1}$ is the canonical retraction of T onto $T_{\alpha+1}$.

Assume that we have completed the construction for all $\beta < \alpha < \omega_1$. If $\alpha = 0$ then choose two distinct endpoints a_0 and b_0 of T and define $T_0 := S(a_0, b_0)$. Now suppose that α is a nonlimit ordinal, say $\alpha = \delta + 1$. Define M_δ as in 3.3. Then $|M_\delta| \leqslant \omega$. Let $\{a_{\delta i} | i \in \omega\}$ be an enumeration of M_δ. Then for each $i \in \omega$ the set $T \setminus \{a_{\delta i}\}$ has at most countably many components and consequently has at most countably many components not intersecting T_δ. Choose an endpoint $b_{\delta ij}$ from the jth component disjoint from T_δ of $T \setminus \{a_{\delta i}\}$. We put

$$T_\alpha := T_\delta \cup \bigcup_{\substack{i \in \omega \\ j \in \omega}} S(a_{\delta i}, b_{\delta ij}).$$

First we claim that T_α is closed in T (it is obvious that T_α is connected). Take $x \notin T_\alpha$. Then $r_\delta(x) = a_{\delta i}$ for some $i \in \omega$; in addition $x \in C_{ij}$ for some component C_{ij} of $T \setminus \{a_{\delta i}\}$. Let $b_{\delta ij}$ be the endpoint chosen from this component. Since $r_\delta\colon T \to T_\delta$ maps C_{ij} onto $a_{\delta i}$ we see that

$$U := C_{ij} \setminus S(a_{\delta i}, b_{\delta ij})$$

is an open neighbourhood of x disjoint from T_α. We conclude that T_α is a metrizable subcontinuum of T since clearly T is separable. In addition the point x cannot be an element of $r_\alpha^{-1}(y)$ for some $y \in T_\delta \subset T_\alpha$, where r_α is the canonical retraction of T onto T_α. For suppose to the contrary that there were such a y. Since $r_\alpha[\{a_{\delta i}\} \cup C_{ij}] = S(a_{\delta i}, b_{\delta ij})$ the point y equals $a_{\delta i}$. Then $S(x, a_{\delta i}) \cap S(a_{\delta i}, b_{\delta ij}) = \{a_{\delta i}\}$ which implies that $a_{\delta i} \in S(x, b_{\delta ij})$; in other words $a_{\delta i}$ separates x from $b_{\delta ij}$. Then x and $b_{\delta ij}$ are not in the same component of $T \setminus \{a_{\delta i}\}$, which is a contradiction.

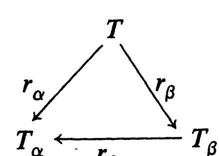
If α is a limit ordinal then we put $T_\alpha := \mathrm{cl}_T(\bigcup_{\beta<\alpha} T_\beta)$. Then T_α is metrizable since α is a countable ordinal. Since T is not separable for all $\beta < \alpha$ we have that $T_\beta \subsetneq T_\alpha$. This completes the transfinite construction.

We claim that $\bigcup_{\alpha<\omega_1} T_\alpha = T$. Indeed, assume to the contrary there exists an $x \in T \setminus \bigcup_{\alpha<\omega_1} T_\alpha$. First of all, notice that since T is first countable (cf. Theorem 3.1) the set $Z := \bigcup_{\alpha<\omega_1} T_\alpha$ is closed in T and hence is a proper subcontinuum of T. Let $r\colon T \to Z$ be the canonical retraction. Suppose that $r(x) \in T_\alpha (\alpha < \omega_1)$. By construction of the retractions of Proposition 2.2 we see that

$$r(x) = r_\alpha(x) = r_{\alpha+1}(x),$$

which contradicts (ii) since $r_\alpha(x) \in T_\alpha$ and $x \in r_{\alpha+1}^{-1}(r_\alpha(x)) \setminus \{r_\alpha(x)\}$.

For every $\alpha < \beta < \omega_1$ define $r_{\beta\alpha}\colon T_\beta \to T_\alpha$ by $r_{\beta\alpha} := r_\alpha \restriction T_\beta$. Then, by Lemma 3.3(i) for each $\alpha < \beta$ the diagram

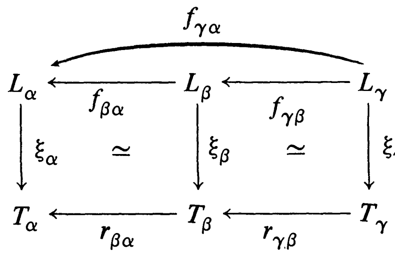


commutes. It follows that $(T_\alpha, r_{\alpha\beta}, \omega_1)$ is an inverse system such that

$$T \approx \text{inv}\lim(T_\alpha, r_{\alpha\beta}, \omega_1);$$

for take distinct $x, y \in T$. Then since $T = \bigcup_{\alpha<\omega_1} T_\alpha$ there is an $\alpha < \omega_1$ such that x and y both belong to T_α, hence $r_\alpha(x) \neq r_\alpha(y)$ and consequently the mapping $e\colon\ T \to \text{inv}\lim(T_\alpha, r_{\alpha\beta}, \omega_1)$ defined by $e(x)_\alpha = r_\alpha(x)$ is a homeomorphism.

Next we construct for every $\alpha < \omega_1$ a continuous surjection $\xi_\alpha\colon I \to T_\alpha$ and for each $\alpha < \beta < \omega_1$ a monotone surjection $f_{\alpha\beta}\colon I \to I$ such that the following diagram commutes



For the sake of convenience we consider a collection of ω_1 different copies of I called $\{L_\alpha | \alpha < \omega_1\}$. In addition, we construct the mappings ξ_α in such a way that ξ_α is two-to-one in all but countably many points for every nonlimit ordinal $\alpha < \omega_1$.

Suppose that the construction is completed for all $\alpha < \gamma$. If $\gamma = 0$ then let $L_0 := [0, 1]$. Let ϕ: $[0, \frac{1}{2}] \to S(a_0, b_0) = T_0$ be an (order) isomorphism. Then define ξ_0: $L_0 \to T_0$ by $\xi_0(x) := \phi(\min\{x, 1 - x\})$. Next suppose that $\gamma = \delta + 1$. If M_δ is dense in itself (cf. Lemma 3.3) then we divide the M_δ into two subsets M_δ^+ and M_δ^- which are dense in M_δ (notice that M_δ is homeomorphic to the space of the rationals). Let $a_{\delta i} \in M_\delta^+$; now let v_i be the first member of L_δ such that $\xi_\delta(v_i) = a_{\delta i}$. If $a_{\delta i} \in M_\delta^-$ or if M_δ is not dense in itself then we choose v_i to be the last member of L_δ such that $\xi_\delta(v_i) = a_{\delta i}$. For each segment $S(a_{\delta i}, b_{\delta ij})$ there exists a mapping $\xi_{\delta ij}$ from a copy $L_{\delta ij}$ of I onto $S(a_{\delta i}, b_{\delta ij})$ such that $\xi_{\delta ij}(0) = \xi_{\delta ij}(1) = a_{\delta i}$ and which is two-to-one except on $b_{\delta ij}$ and which in addition is an isomorphism with respect to the cutset ordering on both $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

We define

$$L_\gamma := L_\delta \cup \bigcup \{L_{\delta ij} \setminus \{1_{\delta ij}\} | i, j \in \omega\}$$

and we obtain an ordering on it by defining:

$$\begin{cases} \text{if } x, y \in L_\delta \text{ and } x < y \text{ in } L_\delta \text{ then also in } L_\gamma; \\ \text{if } x \in L_\delta \text{ and } y \in L_{\delta ij} \text{ and } x < v_i \text{ in } L_\delta \text{ then } x < y \text{ in } L_\gamma; \\ \qquad\qquad\qquad\qquad\qquad \text{if } x \geqslant v_i \text{ in } L_\delta \text{ then } x > y \text{ in } L_\gamma; \\ \text{if } x \in L_{\delta ij} \text{ and } y \in L_{\delta mn} \text{ and } v_i < v_m \text{ in } L_\delta \text{ then } x < y \text{ in } L_\gamma; \\ \text{if } x \in L_{\delta ij} \text{ and } y \in L_{\delta ik} \text{ and } j < k \text{ then } x < y \text{ in } L_\gamma; \\ \text{if } x, y \in L_{\delta ij} \text{ and } x < y \text{ in } L_{\delta ij} \text{ then } x < y \text{ in } L_\gamma. \end{cases}$$

The mapping ξ_γ is defined to be ξ_δ on L_δ and $\xi_{\delta ij}$ on $L_{\delta ij} \setminus \{1_{\delta ij}\}$; in addition $f_{\gamma\delta}$ is the identity on L_δ and $\{v_i\}$ on $L_{\delta ij} \setminus \{1_{\delta ij}\}$. For all $\alpha < \delta$ the mapping $f_{\gamma\alpha}$ is defined to be the composition $f_{\delta\alpha} \circ f_{\gamma\delta}$.

In this way we can consider L_γ to be constructed from L_δ by replacing countably many times a point by a closed unit interval, and in this way it is easy to show that the L_γ is homeomorphic to the closed unit interval and $f_{\gamma\delta}$ is a monotone mapping (cf. Lemma 4.1). The continuity of ξ_γ can be checked directly if we consider the inverse images of components of points in T_γ (the components of points in T_γ form an open subbase for T_γ, as can easily be seen).

If γ is a limit ordinal then we take $L_\gamma := \text{inv}\lim(L_\alpha, f_{\alpha\beta}, \gamma)$. Then since γ is a countable ordinal, L_γ is again homeomorphic to the closed unit interval and the mappings $f_{\gamma\alpha}$ and ξ_γ can be defined in the natural way (because $T_\gamma \approx \text{inv}\lim(T_\alpha, r_{\alpha\beta}, \gamma)$ which can be proved in practically the same way as $T \approx \text{inv}\lim(T_\alpha, r_{\alpha\beta}, \omega_1)$ above). This completes the transfinite construction.

Let L be $\text{inv}\lim(L_\alpha, f_{\alpha\beta}, \omega_1)$ with projections $\{f_\alpha\colon L \to L_\alpha | \alpha < \omega_1\}$. Let $\xi\colon L \to \text{inv}\lim(T_\alpha, r_{\alpha\beta}, \omega_1) \approx T$ be defined by $\xi(x)_\alpha = \xi_\alpha(f_\alpha(x))$. Then ξ is a continuous surjection. Note that L is an ordered compactum (cf. Capel [1]) since the mappings $f_{\alpha\beta}$ are monotone ($\beta < \alpha < \omega_1$). Since T is not separable, the space L cannot be separable either, and the only thing left to prove is that L satisfies the countable chain condition.

The space $T_{\delta+1} \setminus T_\delta$ is a countable union of half open intervals. If $M_{\delta+1}$ is dense in $T_{\delta+1} \setminus T_\delta$ (notice that by construction $M_{\delta+1} \subset T_{\delta+1} \setminus T_\delta$!) then $M_{\delta+1}$ is dense in itself and hence can be divided into two dense subsets. If $M_{\delta+1}$ is not dense then there is a nonvoid open set $O_{\delta+1}$ (open in $T_{\delta+1} \setminus T_\delta$) in $T_{\delta+1} \setminus T_\delta$ disjoint from $M_{\delta+1}$. Clearly $O_{\delta+1}$ is also open in $T_{\delta+1}$ since T_δ is closed in $T_{\delta+1}$. Moreover, since $r_{\delta+1}$ maps $T \setminus T_{\delta+1}$ onto $M_{\delta+1}$ we find that $r_{\delta+1}^{-1}[O_{\delta+1}] = O_{\delta+1}$ and consequently $O_{\delta+1}$ is open in T too. From this observation it follows that there exist only countably many δ's for which $M_{\delta+1}$ is not dense in $T_{\delta+1} \setminus T_\delta$ and consequently we can find a $\theta < \omega_1$ such that $M_{\delta+1}$ is dense in $T_{\delta+1} \setminus T_\delta$ for all $\theta < \delta < \omega_1$.

Let $\mathcal{C}$ be a collection of pairwise disjoint nonvoid connected open subsets of L. We will prove that $|\mathcal{C}| \leqslant \omega$. For each $C \in \mathcal{C}$ we can define an ordinal number $\phi(C) < \omega_1$ as the least number $\alpha < \omega_1$ such that $\text{int}_{L_\alpha}(f_\alpha[C]) \neq \varnothing$. Since f_α is monotone (cf. Capel [1]) we have that for all $\alpha \geqslant \phi(C)$ the set

$$C^* = f_\alpha^{-1}\left[\text{int}_{L_\alpha}(f_\alpha[C])\right]$$

is an open interval in C. The number $\phi(C) = \alpha$ can never be a limit ordinal since for limit ordinals β we have that $L_\beta = \text{inv}\lim(L_\gamma, f_{\gamma\delta}, \beta)$ and

$$\left\{f_{\beta\gamma}^{-1}[O] | O \text{ is open in } L_\gamma (\gamma < \beta)\right\}$$

is a base for L_β. If $\alpha \leqslant \theta$, where θ is as defined above, then

$$f_\theta[C^*] = f_{\theta\alpha}^{-1}\left[\text{int}_{L\alpha}(f_\alpha[C])\right]$$

is an open set in L_θ and $C^* = f_\theta^{-1}[f_\theta[C^*]]$. If $\alpha_1 = \phi(C_1) \leqslant \theta$ and $\alpha_2 = \phi(C_2) \leqslant \theta$ then $f_\theta[C_1^*] \cap f_\theta[C_2^*] = \varnothing$ since $C_1 \cap C_2 = \varnothing$. Since L_θ is ccc it follows that $\phi(C) \leqslant \theta$ for at most countably many $C \in \mathcal{C}$ and hence we can restrict our attention to the collection

$$\mathcal{D} := \left\{f_\alpha^{-1}\left[\text{int}_{L_\alpha}(f_\alpha[C])\right] | C \in \mathcal{C} \text{ and } \alpha = \phi(C) > \theta\right\}.$$

Fix $D \in \mathfrak{D}$, say $D = f_\alpha^{-1}[\text{int}_{L_\alpha}(f_\alpha[C])]$ where $C \in \mathcal{C}$ and $\alpha = \phi(C)$. Let δ be the predecessor of α. Define $D_\alpha := f_\alpha[D]$. Then D_α is an open interval in L_α but $f_{\alpha\delta}[D_\alpha]$ is convex without interior in L_δ, therefore $f_{\alpha\delta}[D_\alpha]$ has to be some point v_i of L_δ with $\xi_\delta(v_i) = a_{\delta i}(i \in \omega)$. Therefore D_α must be contained in

$$\bigcup \{ L_{\delta ij} \setminus \{ 1_{\delta ij} \} | j \in \omega \}$$

and consequently there exists an open interval (l_0, l_1) in some $L_{\delta ij}$ which is entirely contained in D_α. Without loss of generality we may assume that either l_0 and l_1 are both smaller than $\frac{1}{2}$ or both larger than $\frac{1}{2}$. Assume that $l_0 < l_1 < \frac{1}{2}$. The set (l_0, l_1) is mapped by ξ_α onto the open interval (t_0, t_1) in the set $S(a_{\delta i}, b_{\delta ij})$ in T. Since (t_0, t_1) is open in $T_\alpha \setminus T_\delta$ and M_α^+ is dense in $T_\alpha \setminus T_\delta$ there is a point $a_{\alpha k} \in M_\alpha^+ \cap (t_0, t_1)$. In the construction of $L_{\alpha+1}$ we have assigned a point v_k to $a_{\alpha k}$ and since $a_{\alpha k} \in M_\alpha^+$ and $l_0 < l_1 < \frac{1}{2}$ the point v_k is in (l_0, l_1). Let $E_D \subset T$ be defined by

$$E_D := r_\alpha^{-1}[\{a_{\alpha k}\}] \setminus \{a_{\alpha k}\}.$$

By Lemma 3.3(ii) this set is open in T and

$$r_{\alpha+1}[E_D] = \bigcup_{j \in \omega} S(a_{\alpha k}, b_{\alpha kj}) \setminus \{a_{\alpha k}\}$$

and consequently

$$\xi_{\alpha+1}^{-1}[r_{\alpha+1}[E_D]] \subset \bigcup_{j \in \omega} L_{\alpha kj}$$

and therefore $f_{\alpha+1,\alpha}[\xi_{\alpha+1}^{-1}[r_{\alpha+1}[E_D]]] = \{v_k\}$. We obtain

$$\xi^{-1}[E_D] \subset f_{\alpha+1}^{-1}\left[\bigcup_{j \in \omega} L_{\alpha kj}\right] \subset f_\alpha^{-1}(v_k) \subset D.$$

If we assume that $\frac{1}{2} < l_0 < l_1$ we can follow the same procedure for some $a_{\alpha k}$ in $M_\alpha^- \cap \xi_\alpha[(l_0, l_1)]$ and also in this case we find an open subset E_D of T such that $\xi^{-1}[E_D] \subset D$. Since the collection $\mathfrak{D}$ is pairwise disjoint, the corresponding collection $\mathfrak{E} = \{E_D | D \in \mathfrak{D}\}$ is also pairwise disjoint. The space T satisfies the countable chain condition and therefore neither $\mathfrak{E}$ nor $\mathfrak{D}$ can be uncountable. This finishes the proof. □

5.2 COROLLARY. *The existence of a Souslin line is equivalent to the existence of a Souslin dendron.* □

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