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Convexity preserving mappings in subbase convexity theory

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INTRODUCTION

The present paper is a continuation of investigations on subbase convexity theory, started in [7] and in [8]. We are now concerned with so-called *convexity preserving* (cp) *mappings*, a notion comparable to affine mappings in vector space theory.

A first result is a characterization of cp maps in terms of subbasic line segments, from which it can be deduced that normal binary subbases on a given space are incomparable. It is also proved that a cp map commutes with the fundamental operations on spaces with normal binary subbases. This leads to a uniqueness theorem of induced Jensen mappings on superextensions, and to a new order theoretic classification of the superextensions of a space. We finally prove the existence of metrics which are intimately related to normal binary subbases of metrizable compacta.

1. CONVEXITY PRESERVING MAPPINGS

1.1 SUBBASIC CONVEX SETS. Let $\mathscr{S}$ be a closed subbase of a space X. A nonempty set $C \subset X$ is called *$\mathscr{S}$-closed* (or *$\mathscr{S}$-convex*) if there is a family $\mathscr{C} \subset \mathscr{S}$ such that $C = \cap \mathscr{C}$. We let $H(X, \mathscr{S})$ denote the set of all $\mathscr{S}$-closed subsets of X, equipped with the subspace topology of $H(X)$, the hyperspace of X. These notions originated from investigations on normal binary subbases and superextension theory: see [6], [7] and [9].

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1.2 DEFINITION. Let X and Y be spaces, and let $\mathscr{S}$ and $\mathscr{T}$ be closed subbases of, respectively, X and Y. A function $f: X \to Y$ is called a *convexity preserving map* (briefly: a *cp map*) relative to $\mathscr{S}$ and $\mathscr{T}$ if for each $T \in H(Y, \mathscr{T})$ it is true that $f^{-1}(T) \in H(X, \mathscr{S}) \cup \{\emptyset\}$. In this case we shall write

$$f: (X, \mathscr{S}) \to (Y, \mathscr{T}).$$

Notice that a cp map is automatically continuous.

A definition of cp maps has already occured in van Mill and Wattel [8] in the case of binary subbases. Using theorem 1.5 below, it can be seen that the latter definition coincides with ours in the case of normal binary subbases. The main advantage of the present definition is that the composition of cp maps is again a cp map. It easily follows that there is a *category*, whose objects are the pairs $(X, \mathscr{S})$, X a space and $\mathscr{S}$ a closed subbase of X, and whose morphisms are the cp maps.

1.3 EXAMPLES AND BASICAL CONCEPTS

(a) For each $i \in I$, let X_i be a space and let $\mathscr{S}_i$ be a closed subbase of X_i. The product space, $X = \prod_{i\in I} X_i$, is given the following *product subbase*:

$$\mathscr{S} = \prod_{i\in I} \mathscr{S}_i = \{\pi_i^{-1}(S_i) | S_i \in \mathscr{S}_i \quad i \in I\},$$

where $\pi_i: X \to X_i$ denotes the projection. Each map π_i is then a cp map with respect to $\mathscr{S}$ and $\mathscr{S}_i$.

(b) Let $\mathscr{S}$ be a closed subbase of X. A *linked system* in $\mathscr{S}$ is a family $\mathscr{L} \subset \mathscr{S}$ such that each two members of $\mathscr{L}$ meet. A maximal linked system in $\mathscr{S}$ is briefly called an *mls*. Let $\lambda(X, \mathscr{S})$ be the set of all mls's in $\mathscr{S}$, topologized by the following closed (Wallman-type) subbase:

$$\mathscr{S}^+ = \{S^+ | S \in \mathscr{S}\},$$

where $S^+ = \{\mathscr{L} \in \lambda(X, \mathscr{S}) | S \in \mathscr{L}\}$.

This subbase has the obvious property that each linked system $\mathscr{L} \subset \mathscr{S}^+$ has a nonempty intersection. Such a subbase is called *binary*. The resulting topological space is called a *superextension of* X.

If X is a T_1-space, and if $\mathscr{S}$ is a T_1-*subbase* (i.e. for each $x \in X$ and $S \in \mathscr{S}$ such that $x \notin S$, there is an $S' \in \mathscr{S}$ with $x \in S' \subset X - S$), then there is a *canonical embedding* $i: X \to \lambda(X, \mathscr{S})$, defined by

$$i(x) = \{S | x \in S \in \mathscr{S}\}.$$

It is easy to see that i is a cp map, relative to $\mathscr{S}$ and $\mathscr{S}^+$. For details, see Verbeek [10].

(c) Let $\mathscr{S}$ be a closed T_1-subbase of the T_1-space X. As usual, $H(X, \mathscr{S})$ denotes the space of all $\mathscr{S}$-closed subsets of X. At certain occassions, the collection $H(\mathscr{S})$, consisting of all sets of type

$$\langle C, X\rangle = \{D \in H(X, \mathscr{S}) | D \cap C \neq \emptyset\}, \ C \in H(X, \mathscr{S})$$
$$\langle C\rangle \quad = \{D \in H(X, \mathscr{S}) | D \subset C\}, \qquad C \in H(X, \mathscr{S}),$$

forms a closed subbase of $H(X, \mathscr{S})$. This is the case if $\mathscr{S}$ satisfies the following conditions: $\mathscr{S}$ is closed under intersections, $H(X, \mathscr{S})$ is compact, and $\mathscr{S}$ is *normal* (i.e. for each pair of disjoint sets $S_1, S_2 \in \mathscr{S}$ there exist $S_1', S_2' \in \mathscr{S}$ such that $S_1 \subset S_1' - S_2'$, $S_2 \subset S_2' - S_1'$, and $S_1' \cup S_2' = X$): cf. van Mill and van de Vel [7].

In this case, the (canonical) embedding of X in $H(X, \mathscr{S})$ is a cp map again.

(d) The most interesting examples of such "compact" subbases are the *normal binary subbases*, for which a rich "geometric" theory can be built up with as a main tool the so-called *nearest point map*

$$p: X \times H(X, \mathscr{S}) \to X.$$

This map is constructed as follows. For each subset A of X, the *$\mathscr{S}$-convex closure* of A is the set

$$I_{\mathscr{S}}(A) = \cap \{S \in \mathscr{S} | S \supset A\}.$$

(with the intersection of the empty family equal to X). If A is a two-point set $\{x_1, x_2\}$, then $I_{\mathscr{S}}(A)$ is also called the *$\mathscr{S}$-interval joining x_1 and x_2*. Then for each $x \in X$ and for each $C \in H(X, \mathscr{S})$, $p(x, C)$ is the unique point contained in the set

$$\bigcap_{c \in C} I_{\mathscr{S}}(x, c) \cap C.$$

See [7], where it has also been proved that p is continuous. We shall prove that the map p is a cp map in each variable separately. For convenience, the mappings

$$p(-, C) \text{ with } C \in H(X, \mathscr{S}) \text{ fixed,}$$
$$p(x, -) \text{ with } x \in X \text{ fixed}$$

will also be called "nearest point maps". Notice that $p(-, C)$ is a retraction of X onto C for each $C \in H(X, \mathscr{S})$.

(e) A particular type of cp maps has already been used by Jensen (cf. [10]) in extending mappings to superextensions. This goes as follows. Let $\mathscr{S}$ be a closed subbase of X, and let $\mathscr{T}$ be a closed normal subbase of Y. If $f: X \to Y$ is such that $f^{-1}(T) \in \mathscr{S}$ for each $T \in \mathscr{T}$, then there is a continuous map

$$\lambda(f) = \lambda(f; \mathscr{S}, \mathscr{T}): \lambda(X, \mathscr{S}) \to \lambda(Y, \mathscr{T}),$$

sending $\mathscr{L} \in \lambda(X, \mathscr{S})$ onto the unique mls in $\mathscr{T}$ containing the linked system

$$\{T | f^{-1}(T) \in \mathscr{L}\}$$

(a linked system, which is contained in a unique maximal linked system is often called a *pre-mls*). $\lambda(f)$ is called the *induced Jensen map of* f. If $\mathscr{S}$ and $\mathscr{T}$ are T_1-subbases, and if X and Y are T_1-spaces, then $\lambda(f)$ extends f relative to the canonical embeddings

$$X \subset \lambda(X, \mathscr{S}); \;\; Y \subset \lambda(Y, \mathscr{T}).$$

We shall prove below that $\lambda(f)$ is a cp map, relative to the induced subbases $\mathscr{S}^+$ and $\mathscr{T}^+$. We shall also prove a uniqueness property for $\lambda(f)$.

It is assumed throughout that *all topological spaces are T_1-spaces.*

1.4 THEOREM. *Let $\mathscr{S}$ and $\mathscr{T}$ be normal binary subbases of the spaces X and Y, respectively, and let $f: X \to Y$ be a continuous map. Then the following assertions are equivalent:*
(a) *f is a cp map relative to $\mathscr{S}$ and $\mathscr{T}$.*
(b) *for each pair of points $x_1, x_2 \in X$,*

$$f(I_{\mathscr{S}}(x_1, x_2)) \subset I_{\mathscr{T}}(f(x_1), f(x_2)).$$

PROOF OF (a) $\Rightarrow$ (b). Let $x_1, x_2 \in X$. Then

$$x_1, x_2 \in f^{-1}(I_{\mathscr{T}}(f(x_1), f(x_2)),$$

and using the convexity of the latter set,

$$I_{\mathscr{S}}(x_1, x_2) \subset f^{-1}(I_{\mathscr{T}}(f(x_1), f(x_2)).$$

PROOF OF (b) $\Rightarrow$ (a). Let $T \in H(Y, \mathscr{T})$, and let $x_1, x_2 \in f^{-1}(T)$. Then $f(x_1), f(x_2) \in T$ and hence $I_{\mathscr{T}}(f(x_1), f(x_2)) \subset T$. Using (b), it follows that

$$I_{\mathscr{S}}(x_1, x_2) \subset f^{-1}(T).$$

f being continuous, $f^{-1}(T)$ is closed. It has been proved in [7] that a non-empty closed set C of X is $\mathscr{S}$-closed if (and only if) for each $x_1, x_2 \in C$, $I_{\mathscr{S}}(x_1, x_2) \subset C$. Hence, $f^{-1}(T)$ is $\mathscr{S}$-closed or empty. □

1.5 THEOREM. *Let $f: (X, \mathscr{S}) \to (Y, \mathscr{T})$ be a cp map, where $\mathscr{S}$ and $\mathscr{T}$ are normal binary subbases of X and Y respectively. Then for each $S \in H(X, \mathscr{S})$, the set $f(S)$ is the trace on* im (f) *of some $\mathscr{T}$-closed set of Y. In particular, we have for each $x_1, x_2 \in X$ that*

$$f(I_{\mathscr{S}}(x_1, x_2)) = I_{\mathscr{T}}(f(x_1), f(x_2)) \cap \operatorname{im}(f).$$

PROOF. Let $S \in H(X, \mathscr{S})$. Then obviously

$$f(S) \subset I_{\mathscr{T}}(f(S)) \cap \operatorname{im}(f).$$

Let $y \in I_{\mathscr{T}}(f(S)) \cap \operatorname{im}(f)$. Then ([7] lemma 2.1)

$$\{y\} = \bigcap_{x \in S} I_{\mathscr{T}}(f(x), y),$$

and hence

$$\emptyset \neq f^{-1}(y) = \bigcap_{x \in S} f^{-1}(I_{\mathscr{T}}(f(x), y)),$$

where each member of the right hand intersecting family is $\mathscr{S}$-closed. Hence

$$\mathscr{L} = \{S\} \cup \{f^{-1}(I_{\mathscr{T}}(f(x), y)) | x \in S\}$$

is a linked system within $H(X, \mathscr{S})$, and $\mathscr{S}$ being binary, $\cap \mathscr{L} \neq \emptyset$. But

$$\cap \mathscr{L} = f^{-1}(y) \cap S,$$

proving that $y \in f(S)$. This argument yields that

$$I_{\mathscr{T}}(f(S)) \cap \text{im}\,(f) \subset f(S).$$

Applying this result on $S = I_{\mathscr{S}}(x_1, x_2)$, we find that

$$f(I_{\mathscr{S}}(x_1, x_2)) = I_{\mathscr{T}}(f(I_{\mathscr{S}}(x_1, x_2)) \cap \text{im}\,(f).$$

Now,

$$I_{\mathscr{T}}(f(x_1), f(x_2)) \subset I_{\mathscr{T}}(f(I_{\mathscr{S}}(x_1, x_2))$$

since $f(x_1), f(x_2) \in I_{\mathscr{T}}(f(I_{\mathscr{S}}(x_1, x_2))$ and since the latter set is $\mathscr{T}$-closed, whereas

$$f(I_{\mathscr{S}}(x_1, x_2)) \subset I_{\mathscr{T}}(f(x_1), f(x_2))$$

since f is a cp map, and hence

$$I_{\mathscr{T}}(f(I_{\mathscr{S}}(x_1, x_2))) \subset I_{\mathscr{T}}(f(x_1), f(x_2)).$$

This proves that

$$I_{\mathscr{T}}(f(I_{\mathscr{S}}(x_1, x_2))) = I_{\mathscr{T}}(f(x_1), f(x_2)),$$

and hence that

$$f(I_{\mathscr{S}}(x_1, x_2)) = I_{\mathscr{T}}(f(x_1), f(x_2) \cap \text{im}\,(f)). \qquad \square$$

1.6 COROLLARY. *Incomparability of normal binary subbases. If $\mathscr{S}_1$ and $\mathscr{S}_2$ are normal binary subbases of the space X such that $H(X, \mathscr{S}_1) \subset H(X, \mathscr{S}_2)$, then $H(X, \mathscr{S}_1) = H(X, \mathscr{S}_2)$.*

PROOF. By assumption, the identity map

$$\text{id}: (X, \mathscr{S}_2) \to (X, \mathscr{S}_1)$$

is a cp map. Applying theorem 1.5 then yields that

$$H(X, \mathscr{S}_2) \subset H(X, \mathscr{S}_1). \qquad \square$$

The above corollary motivates the following definition: two closed subbases $\mathscr{S}_1$ and $\mathscr{S}_2$ of a space X are called *equivalent* if $H(X, \mathscr{S}_1) = H(X, \mathscr{S}_2)$.

1.7 COROLLARY. *Let X be a compact tree-like space. Then up to equivalence, X admits a unique normal binary subbase.*

PROOF. There is a normal binary subbase $\mathscr{S}_0 = H(X, \mathscr{S}_0)$ of X, consisting of all nonempty subcontinua of X (cf. van Mill and Schrijver [6]). Let $\mathscr{S}_1$ be another normal binary subbase of X. Using the associated nearest point map (cf. example 1.2(d)), each member of $H(X, \mathscr{S}_1)$ is a retract of X, and hence a subcontinuum. This proves that $H(X, \mathscr{S}_1) \subset \subset H(X, \mathscr{S}_0)$, and by corollary 1.6, $\mathscr{S}_0$ and $\mathscr{S}_1$ are equivalent. $\square$

If X is a metrisable continuum which admits a normal binary subbase, then X is an AR (cf. van Mill [5]). Hence, if X is 1-dimensional moreover, then X is a dendron, i.e. a metric compact tree-like space (cf. Borsuk [2]). This leads to the following problem. If X is a metric continuum of dimension >1 which admits a normal binary subbase, does X then admit at least two non-equivalent normal binary subbases? Does X admit two normal binary subbases which are not even isomorphic in the category of cp maps?

2. MORE PROPERTIES AND EXAMPLES OF CP MAPS

There are three fundamental operations on spaces with normal binary subbases: *the nearest point map, the convex closure operator* and *the intersection operator*. It turns out that each of them is preserved by a cp map. Moreover, we prove that the nearest point map is cp in each variable separately and that cp extensions on superextensions are unique.

2.1 THEOREM. *Let $\mathscr{S}$ be a normal binary subbase of X and let $C \in H(X, \mathscr{S})$. Then the nearest point mapping $p: X \to C$ is a cp map.*

PROOF. C is given the canonical trace subbase, derived from $\mathscr{S}$. Let $D \in H(X, \mathscr{S}) \cap \langle C \rangle$. Let $x, y \in p^{-1}(D)$ and assume that there is a point $z \in I_{\mathscr{S}}(x, y) - p^{-1}(D)$. Then $p(z) \notin D$ and consequently, by normality of $\mathscr{S}$, there are $S_0, S_1 \in \mathscr{S}$ such that $p(z) \in S_0 - S_1$, $D \subset S_1 - S_0$ and $S_0 \cup S_1 = X$. We claim that $z \in S_0 - S_1$. For assume to the contrary that $z \in S_1$. Fix a point $d \in D \subset C$. Then

$$\{p(z)\} = \bigcap_{c \in C} I_{\mathscr{S}}(z, c) \cap C \subset I_{\mathscr{S}}(z, d) \subset S_1,$$

which is a contradiction. Hence $z \in S_0 - S_1$. However, this implies that $|\{x, y\} \cap S_0| \geqslant 1$, for, suppose that x and y are both contained in S_1. Then so is $I_{\mathscr{S}}(x, y)$, which contradicts the fact that $z \notin S_1$. Therefore we may assume that $x \in S_0$. This is again a contradiction however, since

$$\{p(x)\} = \bigcap_{c \in C} I_{\mathscr{S}}(x, c) \cap C \subset I_{\mathscr{S}}(x, p(z)) \subset S_0.$$

The result then follows from the continuity of p (cf. [7]) and from theorem 1.4. $\square$

2.2 THEOREM. *Let $\mathscr{S}$ be a binary normal subbase of X, and let $x_0 \in X$. Then the nearest point map $p: H(X, \mathscr{S}) \to X$, sending $A \in H(X, \mathscr{S})$ to $p(x_0, A)$, is a cp retraction.*

PROOF. Let $H(\mathscr{S})$ denote the canonical subbase of $H(X, \mathscr{S})$, as described in example 1.3. It has been shown in [7] that this subbase is normal and binary if $\mathscr{S}$ is. Then:

$$(\star) \qquad I_{H(\mathscr{S})}(A, B) = \langle I_{\mathscr{S}}(A \cup B)\rangle \cap \bigcap \{\langle I_{\mathscr{S}}(a, b), X\rangle | a \in A, b \in B\}$$

i.e. $C \in I_{H(\mathscr{S})}(A, B)$ iff $C \subset I_{\mathscr{S}}(A \cup B)$ and for each $a \in A, b \in B: C \cap \cap I_{\mathscr{S}}(a, b) \neq \emptyset$. In fact,

$$I_{H(\mathscr{S})}(A, B) = \bigcap \{\langle D\rangle | A, B \in \langle D\rangle\} \cap \bigcap \{\langle E, X\rangle | A, B \in \langle E, X\rangle\},$$

by definition, and formula $(\star)$ easily follows.

Let $A, B \in H(X, \mathscr{S})$ and let $C \in I_{H(\mathscr{S})}(A, B)$. For simplicity of notation, we write $x_D = p(x_0, D)$ for each $D \in H(X, \mathscr{S})$. Assume that $x_C \notin I_{\mathscr{S}}(x_A, x_B)$. By normality of $\mathscr{S}$ there exist $S_0, S_1 \in \mathscr{S}$ such that $x_C \in S_0 - S_1$, $I_{\mathscr{S}}(x_A, x_B) \subset S_1 - S_0$, and $S_0 \cup S_1 = X$. Then A, or B, meets S_0, for otherwise $A \cup B \subset I_{\mathscr{S}}(A \cup B) \subset S_1$ and consequently $C \subset S_1$, contradicting that $x_C \in C - S_1$. Assume e.g. that A meets S_0. Then $x_0 \notin S_0$, for otherwise, $x_A \in S_0$. Hence $x_0 \in S_1$; also $I_{\mathscr{S}}(x_A, x_B) \subset S_1$ and $C \cap I_{\mathscr{S}}(x_A, x_B) \neq \emptyset$ whence C meets S_1. But this implies that $x_C \in S_1$, which is a contradiction. Using the fact that p is continuous (cf. van Mill and van de Vel [7]), theorem 1.4 implies that p is a cp map. Clearly, p is a retraction. □

REMARK. The above theorems suggest the question whether the nearest point mapping $p: X \times H(X, \mathscr{S}) \to X$ is a cp map. The following example answers this question negatively. Let $X = I$ and $\mathscr{S} = \{[0, x] | 0 \leqslant x \leqslant 1\} \cup \cup \{[x, 1] | 0 \leqslant x \leqslant 1\}$. There is an obvious embedding $\phi: H(I, \mathscr{S}) \to I^2$ defined by

$$\phi(A) := (\min A, \ \max A).$$

Identify $H(I, \mathscr{S})$ and $\phi(H(I, \mathscr{S})) = \{(x, y) \in I^2 | x \leqslant y\}$. Then it is easily seen that

$$p^{-1}(\tfrac{1}{2}) = ([0, \tfrac{1}{2}] \times \{(\tfrac{1}{2}, y) | \tfrac{1}{2} \leqslant y\}) \cup ([\tfrac{1}{2}, 1] \times \{(x, \tfrac{1}{2}) | x \leqslant \tfrac{1}{2}\}) \cup$$
$$\cup (\{\tfrac{1}{2}\} \times \{(x, y) | x \leqslant \tfrac{1}{2}, y \geqslant \tfrac{1}{2}\}).$$

It can be verified that the canonical product subbase of $X \times H(X, \mathscr{S})$ coincides with the trace of canonical product subbase of I^3 on the subspace $I \times \phi(H(X, \mathscr{S}))$. It easily follows that $p^{-1}(\tfrac{1}{2})$ is not subbasic closed.

2.3 THEOREM. *Let $f: (X, \mathscr{S}) \to (Y, \mathscr{T})$, where $\mathscr{S}$ and $\mathscr{T}$ are normal binary subbases. Then f commutes with the nearest point map, i.e. if $T \in H(Y, \mathscr{T})$ and if*

$$p: X \to f^{-1}(T); \ p: Y \to I_{\mathscr{T}}(ff^{-1}(T))$$

denote the corresponding nearest point mappings, then the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] & f^{-1}(T) \arrow[d, "f|f^{-1}(T)"] \\ Y \arrow[r, "p"] & I_{\mathscr{T}}(ff^{-1}(T)) \end{tikzcd}$$

PROOF. Choose $x \in X$. Then

$$\{p(x)\} = \bigcap_{v \in f^{-1}(T)} I_{\mathscr{S}}(x, v) \cap f^{-1}(T).$$

Consequently, by theorem 1.4,

$$\begin{aligned} \{fp(x)\} &\subset \bigcap_{v \in f^{-1}(T)} f(I_{\mathscr{S}}(x, v) \cap f^{-1}(T)) \\ &\subset \bigcap_{v \in f^{-1}(T)} I_{\mathscr{T}}(f(x), f(v)) \cap ff^{-1}(T) \\ &\subset \bigcap_{t \in ff^{-1}(T)} I_{\mathscr{T}}(f(x), t) \cap I_{\mathscr{T}}(ff^{-1}(T)) \\ &= \{pf(x)\}. \end{aligned}$$

(for the last equality see [7], lemma 1.2). □

2.4 THEOREM. *Let f: $(X, \mathscr{S}) \to (Y, \mathscr{T})$, where $\mathscr{S}$ and $\mathscr{T}$ are normal binary subbases. Then f commutes with the convex closure operators $I_{\mathscr{S}}$ and $I_{\mathscr{T}}$, i.e. for each closed set $A \subset X$,*

$$f(I_{\mathscr{S}}(A)) = I_{\mathscr{T}}(f(A)) \cap \text{im}\,(f).$$

PROOF. Since $f(A) \subset f(I_{\mathscr{S}}(A))$, we have that $I_{\mathscr{T}}(f(A)) \cap \text{im}\,(f) \subset f(I_{\mathscr{S}}(A))$, by theorem 1.5. On the other hand, $A \subset f^{-1}I_{\mathscr{T}}(f(A))$ and since f is a cp map we conclude that $I_{\mathscr{S}}(A) \subset f^{-1}I_{\mathscr{T}}(f(A))$. Therefore,

$$fI_{\mathscr{S}}(A) \subset I_{\mathscr{T}}(f(A)) \cap \text{im}\,(f).$$ □

2.5 THEOREM. *Let f: $(X, \mathscr{S}) \to (Y, \mathscr{T})$, where $\mathscr{S}$ and $\mathscr{T}$ are normal binary subbases. Then f commutes with the intersection operator, i.e. for each linked system $\mathscr{L} \subset H(X, \mathscr{S})$ we have that $f(\cap \mathscr{L}) = \bigcap_{L \in \mathscr{L}} f(L)$.*

PROOF. Let $y \in \bigcap_{L \in \mathscr{L}} f(L)$. Since f is a cp map, the fiber $f^{-1}(y)$ is in $H(X, \mathscr{S})$. Then $\mathscr{L} \cup \{f^{-1}(y)\}$ is a linked system of $\mathscr{S}$-closed sets. Consequently, by the binarity of $\mathscr{S}$, we have that $\cap \mathscr{L} \cap f^{-1}(y) \neq \emptyset$, proving that $y \in f(\cap \mathscr{L})$. □

2.6 THEOREM. *Let $\mathscr{S}$ and $\mathscr{T}$ be normal T_1-subbases for the spaces X and Y, respectively, and let f: $X \to Y$ be a mapping such that $f^{-1}(T) \in \mathscr{S}$*

for each $T \in \mathscr{T}$. *Then the induced Jensen mapping*

$$\lambda(f) = \lambda(f; \mathscr{S}, \mathscr{T}) : \lambda(X, \mathscr{S}) \to \lambda(Y, \mathscr{T})$$

is a cp mapping extending f. *Moreover,* $\lambda(f)$ *is the unique cp mapping which extends* f.

Due to the fact that a space X is usually not dense in $\lambda(X, \mathscr{S})$ (e.g. if X is compact and if $\mathscr{S}$ is not binary), there may as well exist more than one *continuous* extension of the map f. Restricting to the category of cp mappings, the extension is unique. Hence, superextension theory can be regarded as "ordinary compactification theory" within the appropriate category.

PROOF. First notice that the Jensen mapping $\lambda(f) : \lambda(X, \mathscr{S}) \to \lambda(Y, \mathscr{T})$ is continuous. Let $T \in \mathscr{T}$. We will prove that $\lambda(f)^{-1}[T^+] \in H(\lambda(X, \mathscr{S}), \mathscr{S}^+)$ or is empty, which suffices to prove that $\lambda(f)$ is a cp map. Take $\mathscr{M}, \mathscr{N} \in \lambda(f)^{-1}[T^+]$ and $\mathscr{P} \in I_{\mathscr{S}^+}(\mathscr{M}, \mathscr{N})$ such that $\mathscr{P} \notin \lambda(f)^{-1}[T^+]$. As $\{T \in \mathscr{T} \mid f^{-1}(T) \in \mathscr{P}\}$ is a pre-mls for $\lambda(f)(\mathscr{P})$ there is a $T_0 \in \mathscr{T}$ such that $f^{-1}(T_0) \in \mathscr{P}$ and $T_0 \cap T = \emptyset$. Take $T_0', T' \in \mathscr{T}$ such that $T_0 \subset T_0' - T'$, $T \subset T' - T_0'$ and $T_0' \cup T' = Y$. Then $f^{-1}(T_0') \cup f^{-1}(T') = X$ and consequently $f^{-1}(T_0')^+ \cup \cup f^{-1}(T')^+ = \lambda(X, \mathscr{S})$. Now if $\mathscr{M}$ and $\mathscr{N}$ both belong to $f^{-1}(T')^+$ we conclude that

$$\mathscr{P} \in I_{\mathscr{S}^+}(\mathscr{M}, \mathscr{N}) \subset f^{-1}(T')^+,$$

which is a contradiction since $\mathscr{P} \in f^{-1}(T_0)^+$ and $f^{-1}(T_0)^+ \cap f^{-1}(T')^+ = \emptyset$. Hence, without loss of generality we may assume that $\mathscr{M} \in f^{-1}(T_0')^+$. Then $T_0' \in \lambda(f)(\mathscr{M})$ and as $T_0' \cap T = \emptyset$, this is a contradiction. Now, from the continuity of $\lambda(f)$ and from the characterization of $\mathscr{S}^+$-closed sets in $\lambda(X, \mathscr{S})$ mentioned in the proof of theorem 1.4, we conclude that $\lambda(f)$ is a cp map.

We next prove that $\lambda(f)$ is unique. Suppose that $g : \lambda(X, \mathscr{S}) \to \lambda(Y, \mathscr{T})$ is another cp map which extends f. Let $\mathscr{M} \in \lambda(X, \mathscr{S})$. Then

$$\begin{aligned}
\{g(\mathscr{M})\} &= g\Big(\bigcap_{M \in \mathscr{M}} M^+\Big) \\
&\subset \bigcap_{M \in \mathscr{M}} g(M^+) \\
&= \bigcap_{M \in \mathscr{M}} g(I_{\mathscr{S}^+}(M)) && \text{(since } M^+ = I_{\mathscr{S}^+}(M)\text{)} \\
&= \bigcap_{M \in \mathscr{M}} I_{\mathscr{T}^+}(g(M)) \cap \operatorname{im}(g) && \text{(theorem 2.4)} \\
&\subset \bigcap_{M \in \mathscr{M}} I_{\mathscr{T}^+}(f(M)) && \text{(since } f \text{ equals } g \text{ on } X\text{)} \\
&= \bigcap_{T \in \mathscr{T}} \{T^+ \mid T \in \mathscr{T} \text{ and } \exists M \in \mathscr{M} : f(M) \subset T\} \\
&= \{\lambda(f)(\mathscr{M})\},
\end{aligned}$$

since $\{T \in \mathscr{T} \mid \exists M \in \mathscr{M} : f(M) \subset T\}$ is a pre-mls for $\lambda(f)(\mathscr{M})$. □

The above theorem can now be used to construct a natural partial ordering on the set of superextensions with respect to normal subbases of a fixed space. To this end, let X be a topological space and define

$$\Lambda(X) := \{\lambda(X, \mathscr{S}) | \mathscr{S} \text{ is a normal } T_1\text{-subbase for } X\}.$$

2.7 DEFINITION Two elements $\lambda(X, \mathscr{S})$ and $\lambda(X, \mathscr{T})$ of $\Lambda(X)$ are called *equivalent* when there is a cp homeomorphism $\phi: \lambda(X, \mathscr{S}) \to \lambda(X, \mathscr{T})$ which extends id_X

Now define an order $\leqslant$ on $\Lambda(X)$ by putting

$$\lambda(X, \mathscr{S}) \leqslant \lambda(X, \mathscr{T}) \text{ iff there exists a cp surjection}$$
$$f: \lambda(X, \mathscr{T}) \to \lambda(X, \mathscr{S}) \text{ which extends } \mathrm{id}_X.$$

We then have:

2.8 THEOREM. *Up to equivalence, $\leqslant$ is a partial ordering.*

PROOF. We only need to prove that $\leqslant$ is antisymmetric. Take $\lambda(X, \mathscr{S})$, $\lambda(X, \mathscr{T}) \in \Lambda(X)$ and assume that there exist cp surjections $f: \lambda(X, \mathscr{S}) \to$ $\to \lambda(X, \mathscr{T})$ and $g: \lambda(X, \mathscr{T}) \to \lambda(X, \mathscr{S})$ extending id_X. Then

$$f \circ g: \lambda(X, \mathscr{T}) \to \lambda(X, \mathscr{T})$$

is a cp surjection which extends id_X. By theorem 2.6, we have that $f \circ g = \mathrm{id}_{\lambda(X,\mathscr{T})}$. In the same way, $g \circ f = \mathrm{id}_{\lambda(X,\mathscr{S})}$. Hence $\lambda(X, \mathscr{S})$ and $\lambda(X, \mathscr{T})$ are equivalent. □

3. APPLICATION: THE EXISTENCE OF SUBBASE-CONVEX METRICS

3.1 DEFINITION. Let $\mathscr{S}$ be a closed subbase of the space X, and let d be a metric on X. Then d is called *$\mathscr{S}$-convex* provided that for each $S \in H(X, \mathscr{S})$ and for each $r \geqslant 0$,

$$B_r(S) = \{x | d(x, S) \leqslant r\}$$

is $\mathscr{S}$-closed.

We shall present examples showing that the above defined notion is independent of the classical notion of a convex metric, even in the case of normal binary subbases of connected spaces (a connected space, which admits a normal binary subbase, is also locally connected: cf. Verbeek [10]. If the space is metrizable, then it is a peano continuum – even an AR – and hence it admits a convex metric, cf. Bing [1]).

3.2 EXAMPLES

(a) ω denotes the cardinal number of the natural number system. Let $Q = [0, 1]^\omega$ be the Hilbert cube, and let $\mathscr{T}_\omega$ be the canonical (product) subbase of Q, which is normal and binary. Then the metric d on Q, defined by

$$d((x_n)_{n\in\omega}, (y_n)_{n\in\omega}) = \max \{2^{-n}\cdot|x_n - y_n| \mid n \in \omega\}$$

is a $\mathscr{T}_\omega$-convex metric on Q.

(b) Let X be a compact space, and let d be a metric on X. Then the formula

$$\bar{d}(\mathscr{M}, \mathscr{N}) = \inf \{r \mid \forall M \in \mathscr{M} : B_r(M) \in \mathscr{N}\}$$

defines a metric $\bar{d}$ on $\lambda(X, H(X))$ extending d (cf. Verbeek [10]). We prove that $\bar{d}$ is convex relative to the induced subbase $H(X)^+$ of $\lambda(X, H(X))$ (it is customary to write $\lambda(X)$ instead of $\lambda(X, H(X))$).

Let $T \in H(\lambda(X), H(X)^+)$. Then there is a (linked) collection $\mathscr{C} \subset H(X)$ such that

$$T = \cap \{C^+ \mid C \in \mathscr{C}\}.$$

We prove that for each $r \geqslant 0$,

$$(\star) \qquad B_r(T) = \cap \{B_r(C)^+ \mid C \in \mathscr{C}\}.$$

Let $\mathscr{M} \in B_r(T)$. By the compactness of $\lambda(X)$, there is an $\mathscr{N} \in T$ with $\bar{d}(\mathscr{M}, \mathscr{N}) \leqslant r$. Hence $B_r(N) \in \mathscr{M}$ for each $N \in \mathscr{N}$, and since $\mathscr{C} \subset \mathscr{N}$, we find that $\mathscr{M}$ is in the right hand set of the equation $(\star)$.

If the latter is true, then by the symmetry of d,

$$B_r(M) \cap C \neq \emptyset, \qquad M \in \mathscr{M},\ C \in \mathscr{C}.$$

By Zorn's lemma, there is an mls $\mathscr{N}$ in $H(X)$ such that

$$\mathscr{N} \supset \mathscr{C} \cup \{B_r(M) \mid M \in \mathscr{M}\}.$$

Hence $\mathscr{N} \in T$ and $\bar{d}(\mathscr{M}, \mathscr{N}) \leqslant r$ by construction.

(c) The usual metric d on $[0, 1]^2$,

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{\frac{1}{2}},$$

is convex, but not $\mathscr{T}_2$-convex, where $\mathscr{T}_2$ is the canonical (product) subbase of the square.

(d) Let the metric d on the square $[0, 1]^2$ be defined by,

$$d((x_1, x_2), (y_1, y_2)) = \max \{|x_1 - y_1|, |x_2 - y_2|\}$$

and consider the metric subspace

$$X = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1]).$$

Then the trace of the canonical subbase $\mathscr{T}_2$ of the square on X yields a normal binary subbase of X, the metric d on X is subbase convex, but it is not convex on X. Notice that $X \approx [0, 1]$, and hence the subbase of X in consideration is equivalent to the usual one (theorem 1.6): *the unit interval admits a nonconvex, subbase convex metric.*

The main objective of this final section is to prove the existence, on metrizable spaces with a given normal binary subbase, of an associated subbase convex metric. Such a metric has certain desirable features, as is shown in our next result:

3.3 THEOREM. *Let $\mathscr{S}$ be a normal binary subbase of X, let d be an $\mathscr{S}$-convex metric on X, and let $S \in H(X, \mathscr{S})$. Then the corresponding nearest point map*

$$p_S: X \to S$$

has the following properties:

(a) *p_S is a metric nearest point map, i.e. for each $x \in X$,*

$$d(x, S) = d(x, p_S(x))$$

(b) *p_S is a metric contraction, i.e. for each $x_1, x_2 \in X$,*

$$d(p_S(x_1), p_S(x_2)) \leqslant d(x_1, x_2).$$

Recall, moreover, that p_S is a retraction of X onto S.

PROOF OF (a). Let $r = d(x, S)$. Since $\{x\} \in H(X, \mathscr{S})$, we find that

$$B_r(x) \in H(X, \mathscr{S}).$$

X being compact,

$$S \cap B_r(x) \neq \emptyset.$$

Hence, by the construction of p_S, $p_S(x) \in B_r(x)$, i.e. $d(x, p_S(x)) \leqslant r$, and (a) easily follows from this.

PROOF OF (b). Let $d(x_1, x_2) = r$ and $d(p_S(x_1), p_S(x_2)) = s$. Since

$$I_{\mathscr{S}}(x_1, p_S(x_1)) \cap S = \{p_S(x_1)\},$$

we find that

$$I_{\mathscr{S}}(x_1, p_S(x_1)) \cap I_{\mathscr{S}}(p_S(x_1), p_S(x_2)) = \{p_S(x_1)\}.$$

Hence, putting $T = I_{\mathscr{S}}(x_1, p_S(x_1))$, we obtain

$$p_T(p_S(x_2)) = p_S(x_1).$$

Applying (a) on p_T,

$$d(p_S(x_2), I_{\mathscr{S}}(x_1, p_S(x_1))) = d(p_S(x_2), p_S(x_1)) = s.$$

Since d is $\mathscr{S}$-convex, the set $B_r(I_{\mathscr{S}}(x_1, p_S(x_1)))$ is in $H(X, \mathscr{S})$.

Now, $x_2 \in B_r(I_{\mathscr{S}}(x_1, p_S(x_1)))$, and $B_r(I_{\mathscr{S}}(x_1, p_S(x_1))) \cap S \neq \emptyset$, whence by the construction of p_S,

$$p_S(x_2) \in B_r(I_{\mathscr{S}}(x_1, p_S(x_1))).$$

It follows that

$$s = d(p_S(x_2), I_{\mathscr{S}}(x_1, p_S(x_1)) \leqslant r. \qquad \square$$

The first property, (a), has been known for some time on the superextension $\lambda(X)$ of a compact metric space X, using the metric $\bar{d}$ on $\lambda(X)$ which is described in example 3.2(b).

The result (a) also comes close to an old result of Kuratowski (cf. [4]) which states that a subspace A of a metrizable space X is a retract of X iff there is a metric ϱ on X such that for each $x \in X$ there is a unique $f(x) \in A$ such that $\varrho(x, f(x)) = \varrho(x, A)$. In the present case, the desired metric is a subbase convex metric, which does the job with respect to a large number of retractions at the time. Only, the nearest point need not be unique with respect to the metric.

3.4 THEOREM. *Let $\mathscr{S}$ and $\mathscr{T}$ be normal binary subbases of, respectively, X and Y, and let $e\colon (X, \mathscr{S}) \to (Y, \mathscr{T})$ be a cp embedding. Then each $\mathscr{T}$-convex metric on Y induces an $\mathscr{S}$-convex metric on X.*

PROOF. Let d be a $\mathscr{T}$-convex metric on Y. We use the same symbol d to denote the induced metric on X making the map e into an isometry. We let $B_r^X(A)$ denote the closed r-ball around $A \subset X$, and $B_r^Y(A)$ the closed r-ball around $A \subset Y$.

Let $S \in H(X, \mathscr{S})$ and $r \geqslant 0$. We prove that

$$(\star) \qquad e(B_r^X(S)) = B_r^Y(I_{\mathscr{T}}(e(S))) \cap e(X)$$

In fact, one inclusion ($\subset$) is obvious, since e is an isometry. Let $e(x)$ now be in the right hand set of ($\star$), and consider the two nearest point maps

$p^X\colon X \to S$ (derived from $\mathscr{S}$)
$p^Y\colon Y \to I_{\mathscr{T}}(e(S))$ (derived from $\mathscr{T}$).

By theorem 2.4,

$$e(S) = I_{\mathscr{T}}(e(S)) \cap e(X)$$

and e being injective, it follows that $S = e^{-1}(I_{\mathscr{T}}(e(S)))$. Applying theorem 2.3, we find that

$$ep^X(x) = p^Y e(x),$$

and using theorem 3.3(a),

$$r \geqslant d(e(x), I_{\mathscr{T}}(e(S))) = d(e(x), ep^X(x)) = d(x, p^X(x) \geqslant d(x, S),$$

whence $x \in B_r^X(S)$. This proves the other half of ($\star$), and it follows that

$$B_r^X(S) = e^{-1}\,(B_r^Y I_{\mathscr{T}} e(S)) \in H(X, \mathscr{S})$$

since e is an injective cp map. $\square$

3.5 THEOREM. *Let $\mathscr{S}$ be a closed normal subbase of a compact space X. Then there is a cp embedding of $(X, \mathscr{S})$ in some Tychonov cube I^α with its canonical product subbase $\mathscr{T}_\alpha$. If X is metrizable, moreover, then the cardinal number α can be taken equal to ω.*

This theorem can easily be derived from the following result in van Mill and Wattel [8]: if $p \neq q$ are in X, then there is a map $f: X \to [0, 1]$ such that $f(p)=0$, $f(q)=1$, and for each $t \in [0, 1]$, both $f^{-1}([0, t])$ and $f^{-1}([t, 1])$ are (countable) intersections of members of $\mathscr{S}$. In our terminology, f is a cp map (where $[0, 1]$ carries its canonical normal binary subbase).

Let $\mathscr{F}$ be a family of cp maps $(X, \mathscr{S}) \to [0, 1]$ which separates the points of X. If X is metrizable, then $\mathscr{F}$ may be assumed to be countable. Then, if α is the cardinality of $\mathscr{F}$, we obtain the desired cp embedding

$$e: (X, \mathscr{S}) \to ([0, 1]^\alpha, \mathscr{T}_\alpha)$$

by putting

$$e(x) = (f(x))_{f \in \mathscr{F}} \qquad \square$$

3.6 COROLLARY. *Let X be a metrizable space with a normal binary subbase $\mathscr{S}$. Then X admits an $\mathscr{S}$-convex metric.*

PROOF. Use the $\mathscr{T}_\omega$-convex metric of the Hilbert cube and theorems 3.4, 3.5. $\square$

As we noticed above, a metrizable continuum carrying a normal binary subbase is a Peano continuum, and hence it carries a convex metric. It is an unsolved problem whether such a space admits a metric which is both convex and subbase convex. Lots of spaces possess such a metric, e.g. the cubes I^α, $\alpha \leqslant \omega$. Also:

3.7 THEOREM. *Let X be a peano continuum. Then the superextension $\lambda(X)$ of X admits a convex and subbase convex metric.*

PROOF. Let d be a convex metric for X (Bing [1]). Then its canonical extension $\bar{d}$ on $\lambda(X)$ (cf. example 3.2(b)) is $H(X)^+$-convex. We now prove that $\bar{d}$ is also a convex metric.

In fact, let $\mathscr{M}, \mathscr{N} \in \lambda(X)$ and let $\bar{d}(\mathscr{M}, \mathscr{N}) = r$. For a fixed $t \in [0, r]$, we claim that

$$\mathscr{P}_t' = \{B_t(M) | M \in \mathscr{M}\} \cup \{B_{r-t}(N) | N \in \mathscr{N}\}$$

is a linked system. Indeed, let $M \in \mathscr{M}$ and $N \in \mathscr{N}$. Since $B_r(M) \cap N \neq \emptyset$, we can choose $x \in M$, $y \in N$, such that $d(x, y) \leqslant r$. If $d(x, y) \leqslant t$, then obviously $B_t(M) \cap B_{r-t}(N) \neq \emptyset$.

Hence, assume $d(x, y) > t$. d being a convex metric there is a $z \in X$

such that $d(x, z)=t$ and $d(z, y)=d(x, y)-t \leqslant r-t$. Therefore,

$$z \in B_t(M) \cap B_{r-t}(N).$$

Let $\mathscr{P}_t$ be an mls containing $\mathscr{P}'_t$. By construction,

$$\bar{d}(\mathscr{M}, \mathscr{P}_t) \leqslant t; \ \bar{d}(\mathscr{N}, \mathscr{P}_t) \leqslant r-t,$$

and hence the equalities must hold. $\square$

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