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SOME VERY SMALL CONTINUA

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0. INTRODUCTION

Given spaces X, Y, we shall say $X \leq Y$ if Y embeds in a product of copies of X. This gives a preorder on the class of topological spaces. In what follows we shall pretend that "$\leq$" is an order; it will be clear how to formalize what we say, but we feel that our informal approach is more perspicuous.

The properties of "$\leq$" have been extensively studied (see [H], [P],[HP] for references). Our interest here is in a smaller class, the class of continua, and in particular the question of the existence of $\leq$-minimal continua and related questions.

In 1970 the first author asked whether or not the pseudoarc $\mathbb{P}$ is minimal. We answer this question in the negative, but with a very nonmetrizable continuum. It is natural to ask whether $\mathbb{P}$ is minimal among metric continua; we answer this question in the negative also.

CONVENTION 0.1. All given spaces are compact Hausdorff and have more than one point. In particular, continua are assumed to be nondegenerate.

DEFINITION 0.2. $X \leq Y$ if $C(X,Y)$ separates points of X.

Observe that for compact X, 0.2 is equivalent to the definition of $\leq$ in the first paragraph above.

Clearly, $\{0,1\} = 2 \leq X \leq [0,1]$ for all X, $X \leq 2$ iff X is zero-dimensional, and $[0,1] \leq X$ iff X contains an arc.

1. THE MAIN RESULTS

Henceforth all given spaces are continua. Put $\mathbb{H} = [0,\infty) \subseteq \mathbb{R}$, $\mathbb{H}^* = \beta\mathbb{H} - \mathbb{H}$.

THEOREM 1.1. $\mathbb{H}^*$ *is strictly smaller than any metric continuum.*

This is the smallest of the continua promised in the title. The result is a corollary of

THEOREM 1.2. *If* $f: \mathbb{H}^* \to K$ *is nonconstant then* $\mathbb{H}^* \leq K$.

A more general, and in one sense sharper, result on lower bounds is:

THEOREM 1.3. *If* κ *is a cardinal, and* $\mathcal{A}$ *is a collection of at most* κ *continua, each of weight at most* κ*, then there is a continuum* K *of weight* κ *with* $K \leq H$ *for each* $H \in \mathcal{A}$.

COROLLARY 1.3.1. *Every set of continua is bounded below.*

COROLLARY 1.3.2. *If* K_0 *and* K_1 *are metric continua then there is* K *with* $K \leq K_0$ *and* $K \leq K_1$.

Since it is known [R] that there is a plane continuum incomparable with $\mathbb{P}$, 1.3.2 shows that $\mathbb{P}$ is not minimal, indeed, that any minimally metric continuum is a minimum (i.e. a least metric continuum). We do not know whether such a beast exists.

2. PROOFS

PROOF OF THEOREM 1.1. By [AvEB] every metric continuum is the remainder in a compactification of $\mathbb{H}$, and hence an image of $\mathbb{H}^*$. Since every continuum has more than one point, the result follows from Theorem 1.2. □

PROOF OF THEOREM 1.2. For U open in $\mathbb{H}$, define

$$\hat{U} = \mathbb{H}^* - c\ell_{\beta\mathbb{H}}(\mathbb{H} - U).$$

We shall say that $\langle U,V\rangle$ is an *alternating sequence of intervals* if

$$U = \bigcup_{n=1}^{\infty} \langle a_n, b_n\rangle, \quad V = \bigcup_{n=1}^{\infty} \langle c_n, d_n\rangle,$$

$a_n < b_n < c_n < d_n < a_{n+1}$ for each n, and $\sup_{n<\omega} a_n = \infty$.

The theorem follows at once from the following three observations:

(a) If $\mathcal{U}$, $\mathcal{V}$ are disjoint open sets of $\mathbb{H}^*$, then there is an alternating sequence $\langle U,V\rangle$ of intervals with $\hat{U} \subseteq \mathcal{U}$, $\hat{V} \subseteq \mathcal{V}$.

(b) If p, q are distinct points of $\mathbb{H}^*$ then there is an alternating sequence $\langle U,V\rangle$ of intervals with $p \in \hat{U}$, $q \in \hat{V}$.

(c) All alternating sequences of intervals are the same; that is, for any two there is an autohomeomorphism of $\mathbb{H}$ taking one to the other. □

The above may be neatly summarized by saying that $\mathbb{H}^*$ is nearly homogeneous.

<u>PROOF OF THEOREM 1.3</u>. We leave the elementary verification of the following fact to the reader:

<u>FACT</u>. If H and K are continua, U and V are open subsets of H with $\bar{U} \cap \bar{V} = \emptyset$, and U' and V' are open proper subsets of K with $U' \cup V' = K$, then

$$H \times K - (U \times U' \cup V \times V')$$

is a continuum.

Fix $\lambda \le \kappa$ and $\mathcal{A} = \{K_\alpha : \alpha < \lambda\}$ as in the hypotheses of the theorem. Let $j: \kappa \to \kappa^2 \times \lambda$ be a bijection such that if $j(\alpha) = \langle\beta,\gamma,\delta\rangle$, then $\beta \le \alpha$; we write $j(\alpha) = \langle j_1(\alpha), j_2(\alpha), j_3(\alpha)\rangle$. Fix, for each $\alpha \le \lambda$, proper open subsets A_α, B_α of K_α with $A_\alpha \cup B_\alpha = K_\alpha$.

We define inductively an inverse system $\langle H_\alpha, f_{\alpha\beta}, \kappa\rangle$ of continua of weight at most κ. Given H_α, let $\{U_\alpha : \alpha < \kappa\}$ be an open basis for H_α and let $\{\langle V^\alpha_\beta, W^\alpha_\beta\rangle : \beta < \kappa\}$ enumerate the pairs of basic open sets of H_α with disjoint closures.

(a) $H_0 = K_0$;

(b) For λ a limit, $H_\lambda = \varprojlim \langle H_\alpha, f_{\alpha\beta}, \lambda\rangle$;

(c) $H_{\alpha+1} = K_{j_3(\alpha)} \times H_\alpha - (A_{j_3(\alpha)} \times f^{-1}_{\alpha j_1(\alpha)}(V^{j_1(\alpha)}_{j_2(\alpha)}) \cup$
$\cup\, B_{j_3(\alpha)} \times f^{-1}_{\alpha j_1(\alpha)}(W^{j_1(\alpha)}_{j_2(\alpha)}))$,

and the f's are defined as the restrictions of the appropriate projections. We claim that $K = \varprojlim \langle H_\alpha, f_{\alpha\beta}, \kappa\rangle$ satisfies the conclusion of the theorem. We need only show that there are point-separating maps from K into K_α for $\alpha < \lambda$; so fix $\alpha < \lambda$ and distinct $p,q \in X$. For some $\gamma < \kappa$, $f_{\kappa\gamma}(p) \ne f_{\kappa\gamma}(q)$, and so for some $\xi < \kappa$, $f_{\kappa\gamma}(p) \in V^\gamma_\xi$ and $f_{\kappa\gamma}(q) \in W^\gamma_\xi$. Let $\eta = j^{-1}\langle\gamma,\xi,\alpha\rangle$. Then

$$H_{\eta+1} = K_\alpha \times H_\eta - (A_\alpha \times f_{\eta\gamma}^{-1}(V_\xi^\gamma) \cup B_\alpha \times f_{\eta\gamma}^{-1}(W_\xi^\gamma)),$$

and one sees directly that projection onto K_α separates $f_{\kappa,\eta+1}(p)$ from $f_{\kappa,\eta+1}(q)$. □

3. QUESTIONS AND REMARKS

As we have already noted we do not know whether there is a minimally metric continuum; one feels strongly however that there is not. One can ask also whether there is a minimal planar continuum.

We should note that one can easily construct, given a continuum K, a continuum H with $H \not\geq K$; with Theorem 1.3 this shows that there are no minimal continua.

Let us also add the following information. If X is a hereditarily indecomposable (metric) continuum, then $X \leq \mathbb{P}$, [B_1]. There is also a hereditarily indecomposable continuum M_1 in $\mathbb{R}^3$ such that $X \not\leq M$, for every plane continuum X, [C], [R]. Finally, $\mathbb{H}^*$ is indecomposable, [B_2].

REFERENCES

[AvEB] AARTS, J.M. & P. van EMDE BOAS, *Continua as remainders in compact extensions*, Nw. Arch. Wisk. (3) 15 (1967) 34-37.

[B_1] BELLAMY, D.P., *Mapping hereditarily indecomposable continua onto a pseudoarc*, Topology Conf. Virginia 1973, Lecture Notes Math. 375 (1974) 6-14.

[B_2] BELLAMY, D.P., *A nonmetric indecomposable continuum*, Duke Math. J. 38 (1971) 15-20.

[C] COOK, H., *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. 60 (1967) 241-249.

[H] HERRLICH, H., *Topologische Reflexionen und Coreflexionen*, Lecture Notes Math. 78 (1968).

[HP] HUŠEK, M. & J. PELANT, *Note about atom-categories of topological spaces*, Comment. Math. Univ. Carolinae 15 (1974) 767-773.

[P] PELANT, J., *Lattices of E-compact spaces*, Comment. Math. Univ. Carolinae 14 (1973) 719-738.

[R] ROGERS, J.W. JR., *Continuous mappings on continua*, Proc. Auburn Top. Conf. 1969 (Auburn 1969) 94-97.