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SUBBASE CHARACTERIZATIONS OF COMPACT TOPOLOGICAL SPACES

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In this paper we give characterizations of some classes of compact topological spaces, such as (products of) compact lattice, tree-like and orderable spaces, by means of the existence of a closed subbase of a special kind.

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1. Introduction, conventions and some definitions

All topological spaces, under discussion, are assumed to be T_1, and "subbase" will always mean a subbase for the closed sets.

Often, an important class of topological spaces can be characterized by the fact that each element of the class possesses a subbase of a special kind. For example compact spaces (Alexander's subbase lemma), completely regular spaces (De Groot and Aarts [13]), second countable spaces (by definition), metrizable spaces (Bing, cf. [8]), (products of) orderable spaces (Van Dalen and Wattel [6]; Van Dalen [5]; De Groot and Schnare [14]). Such characterizations we shall call *subbase characterizations.*

A class of spaces defined by the existence of a subbase of a special type is the class of *supercompact spaces* (De Groot [10]); this class consists of all spaces possessing a so-called *binary* subbase, that is a subbase $\mathcal{S}$ such that if $\mathcal{S}_0 \subset \mathcal{S}$ with $\bigcap \mathcal{S}_0 = \emptyset$ then there exist $S_0, S_1 \in \mathcal{S}_0$ such that $S_0 \cap S_1 = \emptyset$. It is clear that by the lemma of Alexander every supercompact space is compact. There are many interesting subclasses of the class of supercompact spaces, such as all compact metric spaces (Strok and Szymański [16]; cf. Theorem 2.6 of the present paper), compact orderable spaces (De Groot and Schnare [14]; cf. Theorem 5.2), compact tree-like spaces (Theorem 4.3), compact lattice spaces (Theorem 3.2) and products of these spaces. Not all compact Hausdorff spaces are supercompact as was shown by Bell [2] (see also Van Douwen and Van Mill [7]).

In this paper we will give subbase characterizations of the above classes of topological spaces. The characterization of compact metric spaces and compact orderable spaces are due to De Groot [11] and De Groot and Schnare [14].

An idea of De Groot was to represent a supercompact space with binary subbase $\mathcal{S}$ by the graph with vertex set $\mathcal{S}$ and an edge between S_0 and S_1 in $\mathcal{S}$ if and only if $S_0 \cap S_1 \neq \emptyset$. De Groot [12] proved that the space is completely determined by this graph. In our approach we will represent a supercompact space with binary subbase $\mathcal{S}$ by the graph with vertex set $\mathcal{S}$ and an edge between S_0 and S_1 in $\mathcal{S}$ if and only if $S_0 \cap S_1 = \emptyset$. This not essentially different approach seems to have some advantages (e.g. connectedness and bipartiteness of this latter graph imply interesting properties of the space). This graph representation is often helpful to determine a subbase characterization.

This paper is organized as follows. In Section 2 we give a characterization of supercompactness by means of "interval structures" and show the relation between supercompact spaces and graphs. Sections 3, 4 and 5 deal with lattice spaces, tree-like spaces and orderable spaces, respectively. As an application of Section 2 we show that some of the results can be extended to products of these spaces.

2. Supercompact spaces and graphs

We shall first define the notion of interval structure and we characterize supercompactness by means of this concept. Second, a correspondence between graphs and supercompact spaces is demonstrated.

Definition. Let X be a set and let $I: X \times X \to \mathcal{P}(X)$. Write $I(x, y) = I((x, y))$. Then I is called an *interval structure* on X if:

(i) $x, y \in I(x, y)\ (x, y \in X)$,

(ii) $I(x, y) = I(y, x) \qquad (x, y \in X)$,

(iii) if $u, v \in I(x, y)$, then $I(u, v) \subset I(x, y) \qquad (u, v, x, y \in X)$,

(iv) $I(x, y) \cap I(x, z) \cap I(y, z) \neq \emptyset \qquad (x, y, z \in X)$.

Axioms (i), (ii) and (iii) together can be replaced by the following axiom:

$$u, v \in I(x, y) \quad \text{iff} \quad I(u, v) \subset I(x, y) \quad (u, v, x, y \in X).$$

A subset B of X is called *I-convex* if for all $x, y \in B$ we have $I(x, y) \subset B$. If $(X, \leq)$ is a lattice, then $I(x, y) = \{z \in X \mid x \wedge y \leq z \leq x \vee y\}$ defines an interval structure on X (see Section 3).

Theorem 2.1. *Let X be a topological space. Then: X is supercompact if and only if X is compact and possesses a (closed) subbase $\mathcal{S}$ and an interval structure I such that each $S \in \mathcal{S}$ is I-convex.*

Proof. Let X be a supercompact space and let $\mathcal{S}$ be a binary subbase for X. Define $I : X \times X \to \mathcal{P}(X)$ by

$$I(x, y) = \bigcap \{S \in \mathcal{S} \mid x, y \in S\}, \quad (x, y \in X).$$

Then it is easy to show that I is an interval structure on X and that each $S \in \mathcal{S}$ is I-convex.

Conversely, let X be a compact space with a closed subbase $\mathcal{S}$ consisting of I-convex sets, where I is an interval structure on X. We will show that $\mathcal{S}$ is binary.

Let $\mathcal{S}' \subset \mathcal{S}$ such that $\bigcap \mathcal{S}' = \emptyset$. Then, since X is compact, there exists a finite subset $\mathcal{S}'_0 \subset \mathcal{S}'$ such that $\bigcap \mathcal{S}'_0 = \emptyset$. Hence it is enough to prove the following: if $S_1, S_2, \ldots, S_k \in \mathcal{S}$ and $S_1 \cap \cdots \cap S_k = \emptyset$ then there exist i, j $(1 \leq i, j \leq k)$ such that $S_i \cap S_j = \emptyset$.

We proceed by induction with respect to k. If $k = 1$ or 2 it is trivial. Suppose that $k \geq 3$ and that for each $k' < k$ the statement is true. Define:

$$\begin{aligned} T_1 &= \phantom{S_1 \cap{}} S_2 \cap S_2 \cap S_4 \cap \cdots \cap S_k, \\ T_2 &= S_1 \cap \phantom{S_2 \cap{}} S_3 \cap S_4 \cap \cdots \cap S_k, \\ T_3 &= S_1 \cap S_2 \cap \phantom{S_3 \cap{}} S_4 \cap \cdots \cap S_k. \end{aligned}$$

If one of the T_i's is empty, then the induction hypothesis applies. Suppose therefore $T_i \neq \emptyset$ $(i = 1, 2, 3)$, and take $x \in T_1$, $y \in T_2$ and $z \in T_3$. Then

$$\begin{aligned} x, y &\in S_3 \cap S_4 \cap \cdots \cap S_k, \\ x, z &\in S_2 \cap S_4 \cap \cdots \cap S_k, \\ y, z &\in S_1 \cap S_4 \cap \cdots \cap S_k. \end{aligned}$$

and thus

$$\begin{aligned} I(x, y) &\subset S_3 \cap S_4 \cap \cdots \cap S_k, \\ I(x, z) &\subset S_2 \cap S_4 \cap \cdots \cap S_k, \\ I(y, z) &\subset S_1 \cap S_4 \cap \cdots \cap S_k. \end{aligned}$$

But

$$\begin{aligned}\emptyset \neq I(x, y) \cap I(x, z) \cap I(y, z) \\ \subset (S_3 \cap S_4 \cap \cdots \cap S_k) \cap (S_2 \cap S_4 \cap \cdots \cap S_k) \cap (S_1 \cap S_4 \cap \cdots \cap S_k) \\ = S_1 \cap S_2 \cap \cdots \cap S_k.\end{aligned}$$

This contradicts our hypothesis. □

For some related ideas see Gilmore [9].

Now we turn our attention to the announced correspondence between graphs and supercompact spaces.

A *graph* G is a pair (V, E), in which V is a set, called the set of vertices, and E is a collection of unordered pairs of elements of V, that is $E \subset \{\{v, w\} \mid v, w \in V, v \neq w\}$. Pairs in E are called *edges*. Usually a graph is represented by a set of points in a space with lines between two points if these two points form an edge. A subset V' of V is called *independent* if for all $v, w \in V'$ we have $\{v, w\} \notin E$. A *maximal independent* subset of V is an independent subset not contained in any other independent subset. Zorn's lemma tells us that every independent subset of V is contained in some maximal independent subset. We write

$$\mathscr{I}(G) := \{V' \subset V \mid V' \text{ is maximal independent}\};$$

and for each $v \in V$:

$$B_v := \{V' \in \mathscr{I}(G) \mid v \in V'\}$$

and

$$\mathscr{B}(G) := \{B_v \mid v \in V\}.$$

The *graph space* $T(G)$ of G is the topological space with $\mathscr{I}(G)$ as underlying point set and with $\mathscr{B}(G)$ as a (closed) subbase.

If $\mathscr{S}$ is a collection of sets then the non-intersection graph $G(\mathscr{S})$ of $\mathscr{S}$ is the graph with vertex-set $\mathscr{S}$ and with edges the collection of all pairs $\{S_1, S_2\}$ such that $S_1 \cap S_2 = \emptyset$. The following observation was made by De Groot [12]:

Theorem 2.2. *A space X is supercompact iff X is the graph space of a graph, in particular:*

(i) *if X has a binary subbase $\mathscr{S}$ then X is homeomorphic to the graph space of $G(\mathscr{S})$;*

(ii) *for a graph G, the graph space $T(G)$ is supercompact, with $\mathscr{B}(G)$ as a binary subbase.*

Let G_j be a graph $(j \in J)$; the *sum* $\sum_{j\in J} G_j$ of these graphs is the graph with vertex set a disjoint unoin of the vertex sets of the G_j $(j \in J)$ and edge set the corresponding union of the edge sets. These sums of graphs and products of topological spaces are related in the following theorem.

Theorem 2.3. *Let J be a set and for each $j \in J$ let G_j be a graph. Then $T(\sum_{\in J} G_j)$ is homeomorphic to $\prod_{j\in J} T(G_j)$.*

Proof. Straightforward. □

We shall now give subbase characterizations of some obvious classes of topological spaces; in Sections 3, 4 and 5 subbase characterizations of special classes of spaces are given. With each subbase characterization we also give a characterization in terms of graphs.

Proposition 2.4. *The following assertions are equivalent:*

(i) *X is a second countable supercompact space;*

(ii) *X possesses a countable binary subbase;*

(iii) *X is homeomorphic to the graph space of a countable graph.*

(A graph is called *countable* if its vertex set is countable.)

Proof. Note that each subbase of a second countable space contains a countable subcollection which also is a subbase. □

A subbase $\mathcal{S}$ for X is called *weakly normal* if for each $S_0, S_1 \in \mathcal{S}$ with $S_0 \cap S_1 = \emptyset$ there exists a finite covering $\mathcal{M}$ of X by elements of $\mathcal{S}$ such that each element of $\mathcal{M}$ meets at most one of S_0 and S_1. A graph (V, E) is called *weakly normal* if for each $\{v, w\} \in E$ there are $v_1, \ldots, v_k, w_1, \ldots, w_l \in V$ $(k, l \geq 0)$ such that:

$$\{v, v_1\}, \ldots, \{v, v_k\}, \{w, w_1\}, \ldots, \{w, w_l\} \in E$$

and if

$$v'_1, \ldots, v'_k, w'_1, \ldots, w'_l \in V$$

with

$$\{v_1, v'_1\}, \ldots, \{v_k, v'_k\}, \{w_1, w'_1\}, \ldots, \{w_l, w'_l\} \in E,$$

then

$$\{v'_1, \ldots, v'_k, w'_1, \ldots, w'_l\}$$

is not independent.

Theorem 2.5. *Let X be a supercompact space with binary subbase $\mathcal{S}$ and let X be the graph space of the graph G. The following assertions are equivalent:*

(i) *X is a Hausdorff space;*

(ii) *$\mathcal{S}$ is a weakly normal subbase;*

(iii) *G is a weakly normal graph.*

Proof. (i)$\Rightarrow$(ii). Take $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 = \emptyset$. As X is normal (compact Hausdorff) there exist closed sets C and D with

$$C \cap S_1 = \emptyset = S_2 \cap D \quad \text{and} \quad C \cup D = X.$$

Since X is compact and C and D are intersections of finite unions of sets in $\mathcal{S}$, we can take C and D to be finite intersections of finite unions of sets in $\mathcal{S}$, or, what is the same, finite unions of finite intersections of sets in $\mathcal{S}$.

Since $C \cap S_1 = \emptyset$, each of the finite intersections composing C has an empty intersection with S_1. Now $\mathcal{S}$ is binary and therefore we can replace these finite intersections by single sets of $\mathcal{S}$. Hence we may suppose that C is a finite union of elements of $\mathcal{S}$. Similarly we can take D as a finite union of elements of $\mathcal{S}$.

(ii)$\Rightarrow$(i). This is a consequence of a theorem of De Groot and Aarts [13].

(i)$\Leftrightarrow$(iii). The simple proof is left to the reader. $\square$

This theorem now implies the following remarkable fact, which was first observed by De Groot [12].

Theorem 2.6. *The following assertions are equivalent*:

(i) *X is compact metric*;

(ii) *X has a countable weakly normal binary subbase*;

(iii) *X is homeomorphic to the graph space of a countable weakly normal graph.*

Proof. This is a consequence of the deep result of Strok and Szymański [16] that every compact metric space is supercompact. $\square$

Using this theorem we can derive a rather remarkable characterization of the Cantor discontinuum C. We call a graph (V, E) *locally finite* if for all $v \in V$ the set $\{w \in V \mid \{v, w\} \in E\}$ is finite.

Theorem 2.7. *The following assertions are equivalent*:

(i) *X is homeomorphic to the Cantor discontinuum*;

(ii) *X is homeomorphic to the graph space of a countable locally finite graph with infinitely many edges.*

Proof. (i)$\Rightarrow$(ii). By Theorem 2.3 X is homeomorphic to the graph space of the following graph (cf. De Groot [12]):

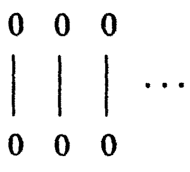

Fig. 1.

(ii)$\Rightarrow$(i). We are going to show that X is a compact metric totally disconnected space without isolated points, whence it will follow that X is homeomorphic to the

Cantor discontinuum. Let G be a countable locally finite graph with infinitely many edges. We will first show that the closed subbase $\mathscr{B}(G)$ of $T(G)$ consists of clopen sets.

Take $v \in V$. Since G is locally finite, there are $w_1, w_2, \ldots, w_n \in V$ such that

$$\{w_1, \ldots, w_n\} = \{w \in V \mid \{v, w\} \in E\}.$$

Now for all $i = 1, 2, \ldots, n$ the set B_{w_i} is closed, hence $\bigcup_{i=1}^{n} B_{w_i}$ is closed too. it is obvious that

$$X \setminus \bigcup_{i=1}^{n} B_{w_i} = B_v,$$

and hence B_v is open.

Since it now follows that $T(G)$ is Hausdorff ($T(G)$ being T_1 and totally disconnected), compact and second countable, $T(G)$ is compact metric.

Finally we show that $T(G)$ has no isolated points. For suppose there is a $V' \in \mathscr{I}(G)$ such that $\{V'\} = \bigcap_{i=1}^{m} B_{v_i}$. That is, if $V'' \in \mathscr{I}(G)$ and $\{v_1, v_2, \ldots, v_m\} \subset V''$ then $V' = V''$. Let W be the set

$$\{w \in V \mid \{v_i, w\} \in E \quad \text{for some } i \in \{1, 2, \ldots, m\}\}.$$

Since G is locally finite, W is finite. Now the set

$$E' = \{\{v, w\} \in E \mid w \in W, v \in V\}$$

also is finite. Since E is infinite there is an edge $\{a, b\} \in E \setminus E'$. It is easy to see that $a \notin W$ and $b \notin W$, hence $\{v_1, \ldots, v_m, a\}$ and $\{v_1, \ldots, v_m, b\}$ both are independent sets of vertices, and hence both are contained in a maximal independent set, say in V''_a and V''_b respectively. As $\{v_1, \ldots, v_m\} \subset V''_a$ and $\{v_1, \ldots, v_m\} \subset V''_b$ it follows that $V''_a = V''_b = V'$; hence $a, b \in V'$. But $\{a, b\} \in E$, hence V' is not independent which is a contradiction. □

The following corollary was suggested to us by the referee.

Corollary 2.8. *X is homeomorphic to 2^κ for some infinite $\kappa \Leftrightarrow X$ is homeomorphic to the graph space of a locally finite graph with infinitely many edges.*

Proof. To show $\Leftarrow$, note that the graph breaks up into the sum (in the sense of Theorem 2.2) of graphs G_α each with countably many edges. If G_α has infinitely many edges, its graph space is homeomorphic to the Cantor set (Theorem 2.7); if finitely many, its graph space is a finite discrete space. By the axiom of choice we can lump these graphs together so that each one of the resulting graphs has $\aleph_0$ edges, hence the graph space is homeomorphic to a product of Cantor sets. □

Finally we call attention to the fact that there is a natural relation between superextensions and graphs (cf. De Groot [12]).

3. Lattices and bipartite graphs

In this section we give a correspondence between spaces induced by a lattice and graph spaces obtained from bipartite graphs. Let $(X, \leq)$ be a lattice with universal bounds 0 and 1. If a and b are elements of X then $[a, b]$ will denote the set

$$[a, b] = \{x \in X \mid a \leq x \leq b\}.$$

The *interval space* of X is the topological space X the topology of which is generated by the subbase

$$\mathcal{S} = \{[0, x] \mid x \in X\} \cup \{[x, 1] \mid x \in X\}.$$

Spaces obtained in this way are called *lattice spaces.* According to a theorem of Frink (cf. Birkhoff [3]) *the interval space of a lattice $(X, \leq)$ is compact iff $(X, \leq)$ is complete.*

Theorem 3.1. *Every compact lattice is supercompact.*

Proof. Let $(X, \leq)$ be a complete lattice and define an interval structure (cf. Section 2) I on X by

$$I(x, y) := [x \wedge y, x \vee y].$$

This is easily seen to be an interval structure while moreover the subbase $\mathcal{S}$ for X defined above consists of I-convex sets; consequently X is super compact by Theorem 2.1. □

A graph (V, E) is called *bipartite* if V can be partitioned in two sets V_0 and V_1 such that each edge consists of an element of V_0 and an element of V_1. A well-known and easily proved theorem in graph theory, see e.g. Wilson [19]; tells us that *a graph (V, E) is bipartite if and only if each circuit is even,* that is, whenever

$$\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$$

are edges in E, then k is even (this characterization uses a weak form of the axiom of choice).

We call a collection $\mathcal{S}$ of subsets of a set X bipartite if the non-intersection graph $G(\mathcal{S})$ is bipartite.

Theorem 3.2. *The following assertions are equivalent:*

(i) *X is homeomorphic to a compact lattice space;*

(ii) *X possesses a binary bipartite subbase;*

(iii) *X is homeomorphic to the graph space of a bipartite graph.*

Proof. (i)⇒(ii). Let $(X, \leq)$ be a complete lattice; the subbase

$$\mathcal{S} = \{[0, x] \mid x \in X\} \cup \{[x, 1] \mid x \in X\}$$

is bipartite and binary.

(ii)$\Rightarrow$(i). Let X be a topological space with a binary bipartite subbase $\mathcal{S}$; let $\mathcal{S}=\mathcal{S}_0\cup\mathcal{S}_1$, such that $\mathcal{S}_0\cap\mathcal{S}_1=\emptyset$ and $\bigcap\mathcal{S}_0\neq\emptyset\neq\bigcap\mathcal{S}_1$ (this is possible since $\mathcal{S}$ is binary and bipartite). Define an order "$\leqslant$" on X by

$$x\leqslant y \quad \text{iff} \quad y\in S \quad \text{whenever } x\in S\in\mathcal{S}_1.$$

The relation "$\leqslant$" is reflexive and transitive; "$\leqslant$" is anti-symmetric too. For suppose that $x\neq y$ and $x\leqslant y\leqslant x$. Since X is T_1, there exists an $S\in\mathcal{S}$ such that $x\in S$ and $y\notin S$. However, this implies that there also exists a $T\in\mathcal{S}$ such that $y\in T$ and $T\cap S=\emptyset$, since $\mathcal{S}$ is binary. From this it follows that either $S\in\mathcal{S}_1$ or $T\in\mathcal{S}_1$. If $S\in\mathcal{S}_1$ then $y\in S$, since $x\leqslant y$, which is a contradiction. If $T\in\mathcal{S}_1$, then $x\in T$, since $y\leqslant x$, which also is a contradiction.

We will show that "$\leqslant$" defines a complete lattice by proving that for each $X'\subset X$ there is a $z\in X$ such that $z=\sup X'$.

Let $X'\subset X$. Define

$$\mathcal{S}_0'=\{S\in\mathcal{S}_0\,|\,X'\subset S\}$$

and

$$\mathcal{S}_1'=\{T\in\mathcal{S}_1\,|\,S\cap T\neq\emptyset \quad \text{for all } S\in\mathcal{S}_0'\}.$$

Now $\bigcap\mathcal{S}_0'\cap\bigcap\mathcal{S}_1'\neq\emptyset$, since $\bigcap\mathcal{S}_0'\neq\emptyset\neq\bigcap\mathcal{S}_1'$ and also $S\cap T\neq\emptyset$ for all $S\in\mathcal{S}_0'$ and $T\in\mathcal{S}_1'$ (notice that $\mathcal{S}$ is binary!). Choose $z\in\bigcap\mathcal{S}_0'\cap\bigcap\mathcal{S}_1'$. This point z is an upper bound for X', for let $x\in X'$ and let $x\in T\in\mathcal{S}_1$; then $T\in\mathcal{S}_1'$ and hence $z\in T$. Therefore $x\leqslant z$ for all $x\in X'$.

Suppose now that $x\leqslant z'$ for all $x\in X'$ and that $z\not\leqslant z'$. Then there exists a $T\in\mathcal{S}_1$ with the properties $z\in T$ and $z'\notin T$. As $\mathcal{S}$ is binary and bipartite, there is an $S\in\mathcal{S}_0$ such that $S\cap T=\emptyset$ and $z'\in S$. Now, $X'\subset S$, since otherwise there must be an $x_0\in X'$ and a $T'\in\mathcal{S}_1$ with the properties $x_0\in T'$ and $T'\cap S=\emptyset$. Then, since $x_0\leqslant z'$ we have that $z'\in T'$, which contradicts the fact that $S\cap T'=\emptyset$. Therefore $X'\subset S$, which implies that $S\in\mathcal{S}_0'$. But $z\notin S$, which cannot be the case since $z\in\bigcap\mathcal{S}_0'\cap\bigcap\mathcal{S}_1'$.

Finally the topology induced by the lattice-ordering $\leqslant$ coincides with the original topology of the space X. Indeed, for $x\in X$ we have that

$$[x,1]=\bigcap\{S\in\mathcal{S}_1\,|\,x\in S\},$$

as can easily be seen.

Furthermore

$$[0,x]=\bigcap\{S\in\mathcal{S}_0\,|\,x\in S\},$$

for suppose that $y\leqslant x$ and that $y\notin S$ for some $S\in\mathcal{S}_0$ with $x\in S$. Then there exists a $T\in\mathcal{S}_1$ such that $S\cap T=\emptyset$ and $y\in T$. Hence $x\in T$, contradicting the fact that $S\cap T=\emptyset$.

Also if $T\in\mathcal{S}_1$, let

$$\mathcal{S}_0'=\{S\in\mathcal{S}_0\,|\,S\cap T\neq\emptyset\}.$$

Then $T \cap \bigcap \mathcal{S}'_0 \neq \emptyset$, since $\mathcal{S}$ is binary. Choose $z \in T \cap \bigcap \mathcal{S}'_0$. We will show that

$$[z, 1] = T.$$

If $z \leq y$, then $y \in T$ since $z \in T$. If $y \in T$ and $z \not\leq y$, then there exists an $S \in \mathcal{S}_0$ such that $y \in S$ and $z \notin S$. However, $S \cap T \neq \emptyset$ and consequently $S \in \mathcal{S}'_0$ and $z \in S$, which is a contradiction.

Conversely, if $S \in \mathcal{S}_0$ let

$$\mathcal{S}'_1 = \{T \in \mathcal{S}_1 \mid S \cap T \neq \emptyset\}.$$

Then $S \cap \bigcap \mathcal{S}'_1 \neq \emptyset$, since $\mathcal{S}$ is binary. Choose $z \in S \cap \bigcap \mathcal{S}'_1$. We will show that

$$[0, z] = S.$$

If $y \leq z$ and $y \notin S$ then $y \in T$ for some $T \in \mathcal{S}$, with $S \cap T = \emptyset$. Hence $z \notin T$, which contradicts the fact that $y \leq z$. If $y \in S$ and $y \not\leq z$ then there is some $T \in \mathcal{S}_1$ such that $y \in T$ and $z \notin T$. Then $S \cap T \neq \emptyset$ and $T \in \mathcal{S}'_1$. Hence $z \in T$, contradicting the fact that $z \notin T$.

(ii)$\Rightarrow$(iii). Let X be a space with a binary bipartite subbase $\mathcal{S}$. By definition $G(\mathcal{S})$ is bipartite and, by theorem 2.2, X is homeomorphic to the graph space of $G(\mathcal{S})$.

(iii)$\Rightarrow$(ii). Let G be a bipartite graph. It is easy to see that the binary subbase $\mathcal{B}(G)$ for the graph space of G is bipartite. □

4. Tree-like spaces and weakly comparable graphs

We now turn our attention to compact tree-like spaces, which are characterized with the help of weakly comparable subbases and graphs.

A *tree-like space* is a connected space in which every two distinct points x and y can be seperated by a third point z, i.e. x and y lie in different components of $X \setminus \{z\}$. Obviously every connected orderable space is tree-like; however, the class of tree-like space is much bigger, see e.g. Kok [15].

A collection $\mathcal{S}$ of subsets of a set X is called *normal* if for every $S_0, S_1 \in \mathcal{S}$ with $S_0 \cap S_1 = \emptyset$ there exist $T_0, T_1 \in \mathcal{S}$ with $S_0 \cap T_1 = \emptyset = T_0 \cap S_1$ and $T_0 \cup T_1 = X$. Clearly a normal collection is weakly normal, cf. Section 1. In addition $\mathcal{S}$ is called *weakly comparable* if for all $S_0, S_1, S_2 \in \mathcal{S}$ satisfying $S_0 \cap S_1 = \emptyset = S_0 \cap S_2$ it follows that $S_1 \subset S_2$ or $S_2 \subset S_1$ or $S_1 \cap S_2 = \emptyset$ (the notion comparable will be defined in Section 5).

A collection $\mathcal{S}$ of subsets of a set X is called *connected* (*strongly connected*) if there is no partition of X in two (finitely many) elements of $\mathcal{S}$.

Lemma 4.1. *Let $\mathcal{S}$ be a weakly comparable collection of subsets of the set X. Then the following properties are equivalent*:

(i) *$\mathcal{S}$ is normal and connected*;

(ii) *$\mathcal{S}$ is weakly normal and strongly connected.*

Proof. (i)$\Rightarrow$(ii). Let $\mathcal{S}$ be weakly comparable, normal and connected. Clearly $\mathcal{S}$ is weakly normal. Suppose $\mathcal{S}$ is not strongly connected and let k be the minimal number such that there are pairwise disjoint sets $S_1, \ldots, S_k$ in $\mathcal{S}$ with union X. Since $\mathcal{S}$ is connected, $k \geq 3$. As $S_1 \cap S_2 = \emptyset$ there exist, by the normality of $\mathcal{S}$, T_1 and T_2 in $\mathcal{S}$ such that $S_1 \cap T_2 = \emptyset = T_1 \cap S_2$ and $T_1 \cup T_2 = X$. Now S_3 intersects either T_1 or T_2. We may suppose $S_3 \cap T_1 \neq \emptyset$. Hence since $S_2 \cap T_1 = \emptyset = S_2 \cap S_3$, by the weak comparability of $\mathcal{S}$, $S_3 \cap T_1 = \emptyset$ or $T_1 \subset S_3$ or $S_3 \subset T_1$. Since the first two cases cannot occur, it follows that $S_3 \subset T_1$. In the same way one proves that for each $j = 4, \ldots, k$ either $S_j \subset T_1$ or $S_j \cap T_1 = \emptyset$. Hence there exists a smaller number of pairwise disjoint sets in $\mathcal{S}$ covering X.

(ii)$\Rightarrow$(i). Let $\mathcal{S}$ be a weakly normal, strongly connected, weakly comparable collection of subsets of X. We need only show that $\mathcal{S}$ is normal. To prove this let $T_0, T_1 \in \mathcal{S}$ such that $T_0 \cap T_1 = \emptyset$. Let k be the minimal number such that there are $S_1, \ldots, S_k$ in $\mathcal{S}$ covering X and such that each S_i meets at most one of T_0 and T_1. By the minimality of k we may suppose that no two of these subsets $S_1, \ldots, S_k$ are contained in each other. If $k = 2$ we are ready.

Suppose therefore $k \geq 3$. We prove that the sets $S_1, \ldots, S_k$ are pairwise disjoint. Without loss of generality we prove only that $S_1 \cap S_2 = \emptyset$. Suppose that $S_1 \cap S_2 \neq \emptyset$. By the weak comparability they are neither both disjoint from T_0 nor are they both disjoint from T_1. We may suppose therefore $S_1 \cap T_0 \neq \emptyset \neq S_2 \cap T_1$. Since now $S_1 \cap T_1 = \emptyset = T_1 \cap T_0$ it follows that either $S_1 \subset T_0$ or $T_0 \subset S_1$. If $S_1 \subset T_0$ then $T_0 \cap S_2 \supset S_1 \cap S_2 \neq \emptyset$, which cannot be the case since $T_0 \cap S_2 = \emptyset$. It follows that $T_0 \subset S_1$ and similarly $T_1 \subset S_2$. We may suppose that $S_3 \cap T_0 = \emptyset$. Since also $S_2 \cap T_0 = \emptyset$ we have $S_3 \cap S_2 = \emptyset$. From this it follows that $S_3 \cap T_1 = \emptyset$ and since also $S_1 \cap T_1 = \emptyset$, we have $S_3 \cap S_1 = \emptyset$. Now from the weak comparability it follows from $S_3 \cap S_2 = \emptyset = S_3 \cap S_1$ that $S_2 \cap S_1 = \emptyset$, which is a contradiction.

Since there are no pairwise disjoint sets $S_1, \ldots, S_k$ in $\mathcal{S}$ with union X, it cannot be the case that $k \geq 3$. Hence $\mathcal{S}$ is normal. $\square$

A graph (V, E) is called *normal* if for each edge $\{v, w\} \in E$ there are edges $\{v, v'\}$ and $\{w, w'\}$ in E such that whenever $\{v', v''\}$ and $\{w', w''\}$ are edges then also $\{v'', w''\}$ is an edge (see Figure 2).

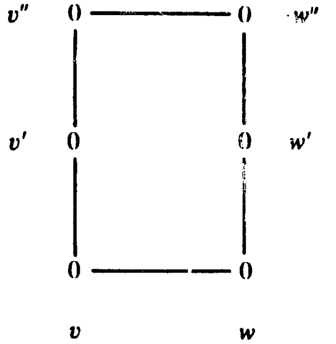

Fig. 2.

Clearly each normal graph is a weakly normal graph (see Section 1).

A graph (V, E) is called *weakly comparable* if for each "path" $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}$ of edges either $\{v_1, v_3\} \in E$ or $\{v_0, v_3\} \in E$ or $\{v_1, v_4\} \in E$ (see Fig. 3).

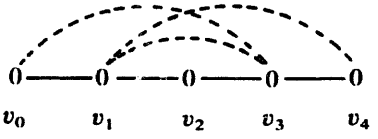

Fig. 3.

A graph (V, E) is called *contiguous* (Bruijning [4]) if for each edge $\{v, w\} \in E$ there exist edges $\{v, v'\}$ and $\{w, w'\}$ such that $\{v', w'\} \notin E$.

A graph (V, E) is *connected* if for each two vertices $v, w \in V$ there is a path of edges $\{v, v_1\}, \{v_1, v_2\}, \ldots, \{v_k, w\}$.

Finally, we call a collection $\mathcal{S}$ of subsets of a set X *graph-connected* if the corresponding non-intersection graph $G(\mathcal{S})$ is connected.

Lemma 4.2. *Let $\mathcal{S}$ be a binary collection of subsets of the set X with non-intersection graph $G(\mathcal{S})$. Then*

(i) *$\mathcal{S}$ is normal iff $G(\mathcal{S})$ is normal*;

(ii) *$\mathcal{S}$ is weakly comparable iff $G(\mathcal{S})$ is weakly comparable*;

(iii) *$\mathcal{S}$ is connected iff $G(\mathcal{S})$ is contiguous.*

Proof. Note that $S_1 \cup \cdots \cup S_k = X$ ($S_i \in \mathcal{S}$, $i \in \{1, 2, \ldots, k\}$) if and only if in $G(\mathcal{S})$ for each $S_1', \ldots, S_k'$ with $\{S_i, S_i'\}$ is an edge of $G(\mathcal{S})$ it follows that $\{S_1', S_2', \ldots, S_k'\}$ is not independent. $\square$

If X is a tree-like space then a subset A of X is called a *segment* if A is a component of $X \setminus \{x_0\}$ for certain $x_0 \in X$. Kok [15] has shown that *every segment in a tree-like space is open.* In particular every tree-like space is Hausdorff.

Theorem 4.3. *Let X be a topological space. Then the following properties are equivalent*:

(i) *X is compact tree-like.*

(ii) *X possesses a binary normal connected (closed) subbase $\mathcal{T}$ such that for all $T_0, T_1 \in \mathcal{T}$ we have that $T_0 \subset T_1$ or $T_1 \subset T_0$ or $T_0 \cap T_1 = \emptyset$ or $T_0 \cup T_1 = X$.*

(iii) *X is homeomorphic to the graph space of a connected normal contiguous weakly comparable graph.*

Proof. (i)$\Rightarrow$(ii). Let X be compact tree-like and let $\mathcal{U}$ denote the collection of segments of X. Since every two distinct points of X are contained in disjoint segments, the compactness of X implies that $\mathcal{U}$ is an open subbase for the topology of

X. We will show that for all $U_0, U_1 \in \mathcal{U}$ either $U_0 \cup U_1 = X$ or $U_0 \cap U_1 = \emptyset$ or $U_0 \subset U_1$ or $U_1 \subset U_0$. To prove this, take $U_0, U_1 \in \mathcal{U}$ and suppose that U_i is a component of $X \setminus \{x_i\}$ ($i \in \{0, 1\}$). Without loss of generality we may assume that $x_0 \neq x_1$. Suppose that $X \setminus \{x_i\} = U_i + U_i^*$ ($i \in \{0, 1\}$) (this means $U_i \cap U_i^* = \emptyset$ and $X \setminus \{x_i\} = U_i \cup U_i^*$). We have to consider two cases:

(a) suppose first that $x_1 \in U_0$. We again distinguish two subcases:

(a$^{(i)}$) $x_0 \in U_1$. It then follows that $\mathrm{cl}_X(U_0^*) = U_0^* \cup \{x_0\} \subset U_1$, since $\mathrm{cl}_X(U_0^*)$ is connected. This implies $U_0 \cup U_1 = X$.

(a$^{(ii)}$) $x_0 \in U_1^*$. The $\mathrm{cl}_X(U_1) \subset U_0$, since $\mathrm{cl}_X(U_1)$ is connected. Therefore $U_1 \subset U_0$.

(b) suppose that $x_1 \in U_0^*$. We distinguish two subcases:

(b$^{(i)}$) $x_0 \in U_1$. This implies that $\mathrm{cl}_X(U_0) \subset U_1$, since $\mathrm{cl}_X(U_0)$ is connected. Hence $U_0 \subset U_1$.

(b$^{(ii)}$) $x_0 \in U_1^*$. Now we have $\mathrm{cl}_X(U_0) \subset U_1^*$, since $\mathrm{cl}_X(U_0)$ is connected.

Therefore $U_0 \subset U_1^*$ and consequently $U_0 \cap U_1 = \emptyset$.

Now define $\mathcal{T} = \{X \setminus U \mid U \in \mathcal{U}\}$. Then $\mathcal{T}$ is a closed subbase such that for all $T_0, T_1 \in \mathcal{T}$ either $T_0 \cup T_1 = X$ or $T_0 \cap T_1 = \emptyset$ or $T_0 \subset T_1$ or $T_1 \subset T_0$. In particular $\mathcal{T}$ is weakly comparable. To show that $\mathcal{T}$ is binary it suffices to show that each covering of X by elements of $\mathcal{U}$ contains a subcover of two elements of $\mathcal{U}$. Indeed, let $\mathcal{A}$ be an open cover of X by elements of $\mathcal{U}$. By the compactness of X there already are finitely many elements of $\mathcal{A}$ covering X, say

$$U_1 \cup U_2 \cup \cdots \cup U_n = X \quad (U_i \in \mathcal{A}, i \in \{1, 2, \ldots, n\}).$$

In addition, we may assume that $\emptyset \neq U_i \not\subset U_j$ for $i \neq j$. We claim that for each $U_i \in \{U_1, U_2, \ldots, U_n\}$ there exists a $U_j \in \{U_1, U_2, \ldots, U_n\}$ such that $U_i \cap U_j \neq \emptyset$, for assume to the contrary for some fixed i it were true that $U_i \cap U_j = \emptyset$ for all $j \neq i$. As $\{U_1, U_2, \ldots, U_n\}$ is a covering of X it would follow that X is not connected, which is a contradiction. Therefore $U_i \cup U_j = X$. Consequently $\mathcal{T}$ is a binary subbase.

As X is Hausdorff, by Theorem 2.5, $\mathcal{T}$ is weakly normal, which implies that $\mathcal{T}$ is normal by Lemma 4.1, since trivially $\mathcal{T}$ is strongly connected (notice that $\mathcal{T}$ consists of closed sets).

(ii)$\Rightarrow$(i). Since $\mathcal{T}$ is a binary subbase we have that X is compact. Therefore we need only prove that X is tree-like. First we will show that X is connected. Suppose that X is not connected. Then there are closed disjoint sets G and H such that $G \cup H = X$ and $G \neq \emptyset \neq H$. G and H are intersections of finite unions of subbase elements. Since G and H are closed, G and H are even finite intersections of finite unoins of subbase elements, or, what is the same, finite unions of intersections. Let m be the minimal number such that there are $G_1, \ldots, G_m$ such that

(α) $G_1, \ldots, G_m$ are non-void and intersections of subbase elements;

(β) $G_1 \cup \cdots \cup G_m = X$;

(γ) there is an $I \subset \{1, 2, \ldots, m\}$ such that

$$\bigcup_{i \in I} G_i \neq \emptyset \neq \bigcup_{j \notin I} G_j \quad \text{and} \quad \bigcup_{i \in I} G_i \cap \bigcup_{j \notin I} G_j = \emptyset.$$

We first prove that $G_i \cap G_j = \emptyset$ if $i \neq j$. Suppose that $G_i \cap G_j \neq \emptyset$ for $i \neq j$. We claim that $G_i \cup G_j = \bigcap\{T \in \mathcal{T} \mid G_i \cup G_j \subset T\}$. For take $x \notin G_i \cup G_j$. Then, since G_i and G_j are intersections of subbase elements there are T_0 and T_1 in $\mathcal{T}$ such that $G_i \subset T_0$, $x \notin T_0$, $G_j \subset T_1$ and $x \notin T_1$. Now since $T_0 \cap T_1 \supset G_i \cap G_j \neq \emptyset$ and $T_0 \cup T_1 \neq X$ ($x \notin T_0 \cup T_1$!) it follows that either $T_0 \subset T_1$ or $T_1 \subset T_0$. Therefore $x \notin T$ for some $T \in \mathcal{T}$ with $G_i \cup G_j \subset T$. It now follows that m is not the minimal number of sets with the above property, which is a contradiction.

Second, we prove that each G_i is an element of $\mathcal{T}$. Suppose that some $G_i \notin \mathcal{T}$. Let $j \neq i$. Then since G_i is an intersection of subbase elements and $\mathcal{T}$ is binary, there is a $T \in \mathcal{T}$ such that $G_i \subset T$ and $T \cap G_j = \emptyset$. The sequence $G_1, \ldots, G_{i-1}, T, G_{i+1}, \ldots, G_m$ is also a sequence with the above properties (α), (β) and (γ). So again $T \cap G_k = \emptyset$ if $k \neq i$, hence $G_i \subset T \subset X \setminus \bigcup_{k \neq i} G_k$, which implies that $G_i = T$ and therefore $G_i \in \mathcal{T}$. Hence there is a collection $G_1, \ldots, G_m$ of pairwise disjoint subbase elements covering X and as $\mathcal{T}$ is weakly comparable, and hence by Lemma 4.1 is strongly connected, this is a contradiction. This proves that X is connected.

We will now show that every two distinct points can be separated by a third point. Let $x, y \in X$ such that $x \neq y$. As X is a T_1-space we have that $\{z\} = \bigcap\{T \in \mathcal{T} \mid z \in T\}$ for all $z \in X$ and consequently, since $\mathcal{T}$ is binary, there exist $T_0, T_1 \in \mathcal{T}$ such that $x \in T_0$, $y \in T_1$ and $T_0 \cap T_1 = \emptyset$. The normality of $\mathcal{T}$ implies the existence of $T'_0, T'_1 \in \mathcal{T}$ such that $T'_0 \cup T'_1 = X$ and $T_0 \cap T'_1 = \emptyset = T'_0 \cap T_1$. Define $\mathcal{A} = \{T \in \mathcal{T} \mid T \cup T'_0 = X\}$. Since X is connected we have that $\mathcal{A} \cup \{T'_0\}$ is a linked system and consequently $T'_0 \cap \bigcap \mathcal{A} \neq \emptyset$. We claim that this intersection consists of one point. Assume to the contrary that $z_0, z_1 \in T'_0 \cap \bigcap \mathcal{A}$ with $z_0 \neq z_1$. In the same way as above there exist $S_0, S_1 \in \mathcal{T}$ such that $z_0 \in S_0 \setminus S_1$ and $z_1 \in S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. Since $z_0 \notin S_1$ we have that $S_1 \notin \mathcal{A}$ and consequently $T'_0 \cup S_1 \neq X$. Hence $T'_0 \subset S_1$ or $S_1 \subset T$ for $S_1 \cap T'_0 = \emptyset$ is impossible since $z_1 \in S_1 \cap T'_0$. However, this implies that $S_1 \subset T'_0$, since $z_0 \notin S_1$. With the same technique one proves that $S_0 \subset T'_0$; but this is a contradiction since $T'_0 \neq X$. Let $\{z_0\} = T'_0 \cap \bigcap \mathcal{A}$. Then z_0 is a separation point of x and y, since T'_0 and $\bigcap \mathcal{A}$ are closed subsets of X such that $T'_0 \cup (\bigcap \mathcal{A}) = X$ and $x \in T'_0$ and $y \in \bigcap \mathcal{A}$. This proves that X is compact tree-like.

(ii)$\Rightarrow$(iii). Let X be a space possessing a binary normal connected subbase $\mathcal{T}$ such that for all $T_0, T_1 \in \mathcal{T}$ we have that either $T_0 \subset T_1$ or $T_1 \subset T_0$ or $T_0 \cap T_1 = \emptyset$ or $T_0 \cup T_1 = X$. We may suppose that $\emptyset \notin \mathcal{T}$ and $X \notin \mathcal{T}$. Then the non-intersection graph $G(\mathcal{T})$ is normal. $G(\mathcal{T})$ is weakly comparable since $\mathcal{T}$ is weakly comparable, as is easy to show. $G(\mathcal{T})$ is contiguous since $\mathcal{T}$ is connected. So we need only to prove that $G(\mathcal{T})$ is connected. Let $T_0, T_1 \in \mathcal{T}$, then either

(a) $T_0 \cap T_1 = \emptyset$; hence there is an edge in $G(\mathcal{T})$ between T_0 and T_1; or

(b) $T_0 \cup T_1 = X$: hence there are T'_0 and T'_1 in $\mathcal{T}$ such that $T_0 \cap T'_0 = T'_0 \cap T'_1 = T'_1 \cap T_1 = \emptyset$, forming a path in $G(\mathcal{T})$ connecting T_0 and T_1; or

(c) $T_0 \subset T_1$; hence there is a $T_2 \in \mathcal{T}$ such that $T_0 \cap T_2 = \emptyset = T_2 \cap T_1$, giving again a path connecting T_0 and T_1; or

(d) $T_1 \subset T_0$; this case is similar to case (c).

(iii)$\Rightarrow$(ii). Let X be the graph space of a connected normal contiguous weakly connected graph $G=(V,E)$. We will prove that the subbase $\mathcal{B}(G)$ for the graph space satisfies the conditions of (ii). $\mathcal{B}(G)$ clearly is binary, normal and connected. Suppose that $v, w\in G$; we must show that either $B_v\subset B_w$, $B_w\subset B_v$, $B_v\cap B_w=\emptyset$ or $B_v\cup B_w=X$. Pick a path of minimal number k of edges from v to w. By connectedness and weak comparability we have that $k=1$, 2 or 3.

Case 1. $k=1$, i.e. $\{v,w\}\in E$ so that $B_v\cap B_w=\emptyset$;

Case 2. $k=2$, say $\{v,v'\}\in E$ and $\{v',w\}\in E$. It now follows that $\{v,w\}\notin E$ (otherwise $k=1$) and therefore $B_v\subset B_w$ or $B_w\subset B_v$, for if not, there would be edges $\{v,v''\}$, $\{w,w''\}\in E$ such that $\{v,w'\}\notin E$ and $\{w,v'\}\notin E$, contradicting the weak comparability of G;

Case 3. $k=3$, say $\{v,v_1\},\{v_1,v_2\},\{v_2,w\}\in E$. By Case 2 we have $B_v\subset B_{v_2}$ or $B_{v_2}\subset B_v$. In the former case $B_v\cap B_w=\emptyset$ (but then $k=1$), so we have $B_{v_2}\subset B_v$ and similarly $B_{v_1}\subset B_w$. Now suppose that $B_v\cup B_w\neq X$; then we conclude that $B_v\cup B_{v_1}\cup B_{v_2}\cup B_w\neq X$ and consequently we may pick a maximal independent set M such that $v, v_1, v_2, w\notin M$. By maximality there is a $t_1\in M$ with $\{t_1,v\}\in E$. Since $\{v,v_2\}\notin E$ (otherwise $k=2$) and $\{v,w\}\notin E$ (otherwsie $k=1$), we have, by weak comparability, that $\{t_1,v_2\}\in E$. But then, by Case 2, $B_w\subset B_{t_1}$ (then $B_v\cap B_w=\emptyset$) or $B_{t_1}\subset B_w$. But the latter case contradicts $M\in B_{t_1}\backslash B_w$. □

Corollary 4.4. *Each compact tree-like space is supercompact.* □

Corollary 4.5. *Let X be a topological space. Then the following properties are equivalent*:

(i) *X is a product of compact tree-like space.*

(ii) *X possesses a binary normal connected weakly comparable closed subbase.*

(iii) *X is homeomorphic to the graph space of a normal contiguous weakly comparable graph.*

Proof. Notice that each graph is the sum of its components. Then apply Theorem 2.3 and Theorem 4.3. □

An interesting application of this corollary is the following. In [11], De Groot proved a topological characterization of the n-cell I^n, and of the Hilbert cube I^∞ by means of a binary subbase of a special kind (cf. Theorem 5.5). Anderson [1] has proved that the *product of a countably infinite number of dendra is homeomorphic to the Hilbert cube*, where a *dendron* is defined to be a nondegenerate, uniquely arcwise connected Peano continuum. It is well known, however, that a dedron is simply a compact metric tree-like space (cf. Whyburn [18]). Since the dimension of a dendron is 1, using our characterization of products of compact tree-like spaces, we are able to give a new characterization of the Hilbert cube, thus generalizing the result of De Groot, mentioned above, for the case of the Hilbert cube.

Theorem 4.6. *A topological space X is homeomorphic to the Hilbert cube I^∞ if and only if X has the following properties*:

(i) *X is infinite dimensional*;

(ii) *X possesses a countable binary, connected normal weakly comparable subbase.*

Proof. The necessity follows from Corollary 4.5, since the Hilbert cube is a product of compact tree-like spaces. The sufficiency follows from the fact that by Corollary 4.5 X is homeomorphic to a countable product of dendra. As X is infinite dimensional this must be a countable infinite product. Hence X is homeomorphic to the Hilbert cube. □

5. Ordered spaces and comparable subbases

Finally we treat the relations between ordered spaces and comparable subbases and graphs. Note that an ordered space is the interval space of a totally ordered set (cf. Section 3). Hence clearly every ordered space is a lattice space while moreover a connected ordered space is tree-like.

Let X be a set and let $\mathcal{S}$ be a collection of subsets of X. The collection $\mathcal{S}$ is called *comparable* (De Groot [11]) if for all $S_0, S_1, S_2 \in \mathcal{S}$ with $S_0 \cap S_1 = \emptyset = S_2 \cap S_0$ it follows that either $S_1 \subset S_2$ or $S_2 \subset S_1$. A graph (V, E) is called *comparable* if for each path $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}$ of edges it follows that either $\{v_0, v_3\} \in E$ or $\{v_1, v_4\} \in E$ (cf. Fig. 4).

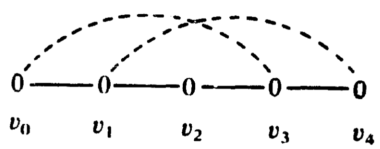

Fig. 4.

Lemma 5.1. (i) *A graph G is comparable iff G is weakly comparable and bipartite.*

(ii) *Each comparable graph is normal.*

(iii) *A collection $\mathcal{S}$ of subsets of a set X is comparable iff it is weakly comparble and bipartite.*

(iv) *A comparable collection $\mathcal{S}$ of subsets of a set X is normal if it satisfies the following condition: for each $x \in X$ and each $S \in \mathcal{S}$ with $x \notin S$ there exists an $S_0 \in \mathcal{S}$ with $x \in S_0$ and $S_0 \cap S = \emptyset$.*

Proof. The simple proof is left to the reader. □

Theorem 5.2. *Let X be a topological space. The following assertions are equivalent*:

(i) *X is compact orderable*;

(ii) *X possesses a binary graph-connected comparable subbase*;
(iii) *X is homeomorphic to the graph space of a connected comparable graph.*

Proof. (i)$\Rightarrow$(ii). Let $(X, \leq)$ be a complete totally-ordered set, with universal bounds 0 and 1. Clearly the subbase

$$\mathcal{S}=\{[0, 1]\mid x\in X, 0\leq x<1\}\cup\{[x, 1]\mid x\in X, 0<x\leq 1\}$$

is binary, graph-connected and comparable.

(ii)$\Rightarrow$(i). Let X be a space with a binary graph-connected comparable subbase $\mathcal{S}$. Since X is bipartite (Lemma 5.1), $\mathcal{S}$ induces a lattice ordering $\leq$ on X, such as in the proof of Theorem 3.2 (ii)$\Rightarrow$(i). We only have to prove that this order is a total order. Suppose that $\leq$ is not total, that is for some $x, y\in X$ we have $x\not\leq y$ and $y\not\leq x$. Consequently there are $S, T\in\mathcal{S}_1$ such that:

$$x\in S,\ y\notin S,\ y\in T \quad\text{and}\quad x\notin T.$$

Since $\mathcal{S}$ is graph-connected and bipartite there are $S_1,\ldots,S_k$ such that

$$S\cap S_1=S_1\cap S_2=\cdots=S_{k-1}\cap S_k=S_k\cap T=\emptyset$$

with k odd. Suppose that k is the smallest number for which such a path in $G(\mathcal{S})$ exists. If $k\geq 3$ then $S_1\cap S_2=\emptyset=S_2\cap S_3$ and hence $S_1\subset S_3$ or $S_3\subset S_1$. If $S_1\subset S_3$ then

$$S\cap S_1=S_1\cap S_4=S_4\cap S_5=\cdots=S_k\cap T=\emptyset,$$

which gives a shorter path from S to T.

The case $S_3\subset S_1$ can be treated similarly.

Hence $k=1$ and consequently $S\cap S_1=\emptyset=S_1\cap T$. Since $\mathcal{S}$ is comparable, $S\subset T$ or $T\subset S$. This means that either $x\in T$ or $y\in S$, which both are contradictions.

(ii)$\Rightarrow$(iii). Let X be a space with a binary graph-connected comparable subbase $\mathcal{S}$. Then X is homeomorphic to the graph space of the graph $G(\mathcal{S})$, while moreover it is easy to see that $G(\mathcal{S})$ is connected and comparable.

(iii)$\Rightarrow$(ii). Let X be the graph space of a connected comparable graph $G=(V, E)$. $\mathcal{B}(G)$ is graph-connected since G is connected. $\mathcal{B}(G)$ is comparable, for suppose that $B_{v_1}, B_{v_2}, B_{v_3}\in\mathcal{I}(G)$ and

$$B_{v_1}\cap B_{v_2}=\emptyset=B_{v_2}\cap B_{v_3}$$

and $B_{v_1}\not\subset B_{v_3}$ and $B_{v_3}\not\subset B_{v_1}$.

Hence $\{v_1, v_2\}\in E$ and $\{v_2, v_3\}\in E$; and there are V' and $V''\in\mathcal{I}(G)$ such that $V'\in B_{v_1}\backslash B_{v_3}$ and $V''\in B_{v_3}\backslash B_{v_1}$.

As $v_3\notin V'$ there is a $v_4\in V'$ such that $\{v_3, v_4\}\in E$. As $v_1\notin V''$ there is a $v_0\in V''$ such that $\{v_0, v_1\}\in E$. Now

$$\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\in E$$

and also $\{v_0, v_3\}\notin E$ (for $v_0, v_3\in V''$) and $\{v_1, v_4\}\notin E$ (for $v_1, v_4\in V'$). This contradicts the comparability of the graph G.

Hence the graph space $T(G)$ of G has a binary comparable graph-connected subbase $\mathscr{B}(G)$.

This completes the proof of the theorem. □

Corollary 5.3 (De Groot & Schnare [14]). *Let X be a topological sapce. Then the following statements are equivalent:*

(a) *X is a product of compact orderable spaces;*

(ii) *X possesses a binary comparable subbase;*

(iii) *X is homeomorphic to the graph space of a comparable graph.*

Proof. Apply Theorem 5.2 and Theorem 2.3. □

Corollary 5.4. *Let X be a topological space. Then the following statements are equivalent.*

(i) *X is connected compact orderable;*

(ii) *X possesses a connected graph-connected comparable subbase;*

(iii) *X is homeomorphic to the graph space of a connected contiguous comparable graph.*

Proof. Apply Theorem 5.2 and Theorem 4.3. □

Corollary 5.5. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is a product of connected compact orderable spaces;*

(ii) *X possesses a connected comparable subbase;*

(iii) *X is homeomorphic to the graph space of a contiguous comparable graph.*

Proof. Combine Corollary 5.5 and Theorem 2.3. □

Adding countability conditions on the subbases and graphs one easily obtains characterizations of (products of) (connected) compact subsets of the real line (cf. De Groot [12], Bruijning [4]).

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Note added in proof

Recently Van Douwen and Mills independently gave elementary proofs of the supercompactness of compa[illegible]tric spaces. In addition, Mills has shown that every compact topological gro[illegible]ercompact.

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