DENDRONS

by

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1. INTRODUCTION

Let X be a compact connected Hausdorff space. We say that X is a *dendron* provided that for every two distinct points $x,y \in X$ there exists a point $z \in X$ which separates x from y, i.e. $X\backslash\{z\} = U \cup V$ where U and V are disjoint open subsets of X such that $x \in U$ and $y \in V$. Dendrons are natural generalizations of linearly orderable continua. In the last decade several results concerning dendrons have been proved and the aim of this paper is to collect some of these results and to present them in such a way that the underlying ideas which led to these results will be recognized.

2. CONNECTIVITY PROPERTIES

In this section we collect some basic facts which will be important throughout the remaining part of this paper. *The letter* D *will always denote a given dendron.*

LEMMA 2.1. *Take* $x \in D$. *If* C *is a component of* $D\backslash\{x\}$, *then* C *is open.*

PROOF. Assume that A and B are disjoint open sets of D and that $A \cup B = D\backslash\{x\}$. We claim that $A \cup \{x\}$ is connected. Suppose not, then there exists a pair of clopen subsets U and V in $A \cup \{x\}$ such that $U \cap V = \emptyset$ and $U \cup V = A \cup \{x\}$. If $x \notin U$, then U is an open subset of the open set A and hence open in D. U is closed in set $A \cup \{x\}$ and hence closed in D. If $x \notin V$ the same arguments hold. This contradicts the connectivity of D and we conclude that $A \cup \{x\}$ is connected.

Next we assume that some quasi-component Q (i.e. the intersection of a maximal collection of clopen subsets) of $D\backslash\{x\}$ is not open. Then Q contains a point q which is in the closure of $D\backslash(Q \cup \{x\})$. Assume that z separates q and x. If $z \notin Q$ then there is a pair of disjoint open subsets A and B such that $z \in A$ and $q \in B$ and $A \cup B = D\setminus\{x\}$. However, we have seen that $B \cup \{x\}$ is connected and so we conclude that $z \in Q$. From the same argument we find that $C \cup \{x\}$ is connected for every clopen subset $C \subset D\setminus\{x\}$ which misses Q. Therefore

$$\bigcup\{C \cup \{x\} \mid C \text{ clopen in } D\setminus\{x\} \text{ and } C \cap Q = \emptyset\} = D\setminus Q$$

is connected. However, q is a member of the closure of $D\setminus Q$ and hence $\{q\} \cup D\setminus Q$ is connected and contains both q and x. Therefore z does not separate q and x. This contradiction shows that Q is open.

Finally, Q is connected, since if Q_1 and Q_2 would be a partition of Q into two clopen parts, then each of those members would be clopen in $D\setminus\{x\}$ and Q would not be a quasi-component. So the collection of quasi-components coincides with the collection of components and the components of $D\setminus\{x\}$ are open. □

COROLLARY 2.2. *The collection*

$$\mathcal{U}(D) = \{U \subset D \mid \exists x \in D \text{ such that } U \text{ is a component of } D\setminus\{x\}\}$$

is an open subbase for the topology of D.

PROOF. If $x,y \in D$ are distinct, then, since D is a dendron there are disjoint $U,V \in \mathcal{U}(D)$ with $x \in U$ and $y \in V$. By compactness this easily implies that $\mathcal{U}(D)$ is an open subbase. □

Elements of $\mathcal{U}(D)$ rae called *cutpoint components*. Define

$$\mathcal{J}(D) = \{D\setminus U \mid U \in \mathcal{U}(D)\}.$$

Observe that $\mathcal{J}(D)$ is a subbase for the closed subsets of D.

LEMMA 2.3. $\mathcal{J}(D)$ *consists of connected sets.*

PROOF. Follows directly from the proof of Lemma 2.1. □

A collection L of subsets of a set X is called *cross-free* provided that for all $L_0,L_1 \in L$ it is true that $L_0 \subset L_1$ or $L_1 \subset L_0$ or $L_0 \cap L_1 = \emptyset$ or $L_0 \cup L_1 = X$.

LEMMA 2.4. $\mathcal{U}(D)$ *is cross-free.*

PROOF. Assume that U_1 and U_2 are cutpoint components of $D\setminus\{x_1\}$ (resp. $D\setminus\{x_2\}$). If $x_1 = x_2$ then U_1 and U_2 are clearly either disjoint or equal, and both those possibilities are permitted by the definition of cross-free collections. If $x_1 \neq x_2$ then we distinguish three subcases:

(a) $x_1 \in U_2$ and $x_2 \in U_1$. Now each cutpoint component C of $D\setminus\{x_1\}$ which does not contain x_2 is a connected subset of D and hence, by connectivity (Lemma 2.1), is contained in U_2. So $U_2 \cup U_1 = D$.

(b) $x_1 \notin U_2$. This means that U_2 is a connected subset of $D\setminus\{x_1\}$ and hence either is contained in or disjoint from the cutpoint component U_1 of $D\setminus\{x_1\}$.

(c) $x_2 \notin U_1$. This case is similar to the previous one. □

COROLLARY 2.5. *$\mathcal{J}(D)$ is cross-free.* □

A collection of subsets $\mathcal{L}$ of a set X is called *normal* provided that for all disjoint $L_0, L_1 \in \mathcal{L}$ there are $S_0, S_1 \in \mathcal{L}$ with

$$L_0 \cap S_1 = \emptyset = S_0 \cap L_1 \quad \text{and} \quad S_0 \cup S_1 = X.$$

The sets S_0 and S_1 are called a *screening* of L_0 and L_1. A collection of subsets $\mathcal{L}$ of a set X is called *connected* if there is no partition of X by two non-empty members of $\mathcal{L}$.

LEMMA 2.6. *Every cross-free closed subbase $\mathcal{J}$ for a connected Hausdorff space X is normal and hence $\mathcal{J}(D)$ is normal.*

PROOF. Take two disjoint non-empty members T_0 and T_1 from $\mathcal{J}$. Since T_0 is closed and X is connected there exists a point $t_0 \in T_0 \cap (X\setminus T_0)^-$ and similarly we find a point $t_1 \in T_1 \cap (X\setminus T_1)^-$. Since X is Hausdorff we can find two basic closed sets B_0 and B_1 such that $B_0 \cup B_1 = X$, $t_0 \notin B_1$ and $t_1 \notin B_0$. Moreover,

$$B_0 = F_0 \cup F_1 \cup \ldots \cup F_m \quad \text{and} \quad B_1 = F_{m+1} \cup F_{m+2} \cup \ldots \cup F_n,$$

for a suitably chosen finite subcollection $F_0,\ldots,F_n$ of $\mathcal{J}$. Without loss of generality we may assume that no F_i is contained in some F_j. Assume that $t_0 \in F_i \cap F_j$. Then $t_1 \notin F_i \cup F_j$ and since $\mathcal{J}$ is cross-free we conclude that either $F_i \subset F_j$ or $F_j \subset F_i$. This means that we can have at most one F, say F_0, which contains t_0 and one F, say F_n, which contains t_1.

If some F contains neither t_0 nor t_1 but has an intersection with F_0 then we can choose $t_2 \in F \cap F_0$ and the same argument shows then that $F \subset F_0$ and hence F is superfluous. So we have F_0, F_n, and a collection of F's disjoint from F_0 and F_n. If there is a point $t_3 \in F$ which is not contained in $F_0 \cup F_n$ then a similar argument shows that $F_0 \cap F_n$ is empty and we have a partition of the space in three disjoint closed subsets, namely F_0, F_n and $\cup\{F_i \mid 0 < i < n\}$. This is a contradiction and we obtain that $F_0 \cup F_n = X$.

Finally we show that $F_n \cap T_0 = \emptyset$. Since $t_0 \in T_0 \setminus F_n$ amd $t_1 \in F_n \setminus T_0$, and since t_0 is neither in the interior of T_0 nor in the closure of F_n we obtain that $T_0 \cup F_n \neq X$. We conclude that $T_0 \cap F_n = \emptyset$ and similarly that $T_1 \cap F_0 = \emptyset$ which means that $\mathcal{J}$ is normal. □

A collection of subsets $\mathcal{L}$ of a set X is called *binary* provided that for all $\mathcal{M} \subset \mathcal{L}$ with $\cap\mathcal{M} = \emptyset$ there are $M,N \in \mathcal{M}$ with $M \cap N = \emptyset$.

<u>LEMMA 2.7</u>. *If* X *is a compact connected Hausdorff space and its closed subbase* $\mathcal{J}$ *is cross-free then* $\mathcal{J}$ *is binary. Consequently,* $\mathcal{J}(D)$ *is binary.*

<u>PROOF</u>. Suppose not. Assume that $\mathcal{M}$ is a subfamily of $\mathcal{J}$ in which every two members have a non-empty intersection. We have that X is compact and so $\cap\mathcal{M} = \emptyset$ implies that there is a finite subcollection of $\mathcal{M}$ containing a minimal number of sets $M_1,\ldots,M_n$ which has an empty intersection. Now if $i \neq j$ then $M_i \cap M_j \neq \emptyset$ and M_i is not contained in M_j. So $M_i \cup M_j = X$. In particular, $M_i \cup M_n$ = for X for $0 < i < n$ and hence $M_n \cup [\cap_{0<i<n} M_i] = X$. Moreover, $M_n \cap [\cap_{0<i<n} M_i] = \emptyset$ which implies that M_n is clopen, contradicting that X is connected. □

If $x,y \in X$ and if $\mathcal{J}$ is a subbase for X then put

$$I_{\mathcal{J}}(x,y) = \cap\{T \in \mathcal{J} \mid x,y \in T\}.$$

For notational simplicity, $I_{\mathcal{J}(D)}(x,y)$ will be denoted by $I(x,y)$.

<u>LEMMA 2.8</u>. *If* $C \subset D$ *is an intersection of elements of* $\mathcal{J}(D)$, *then the function* $r_C \colon D \to C$ *defined by*

$$\{r_C(x)\} = \bigcap_{c \in C} I(x,c) \cap C$$

is a retraction.

PROOF. From the binarity of $J(D)$, Lemma 2.7, it follows that

$$E = \bigcap_{c\in C} I(x,c) \cap C \neq \emptyset.$$

Suppose that there are two distinct points $e_0, e_1 \in E$. Find $T_0, T_1 \in J(D)$ with $e_0 \in T_0\setminus T_1$, $e_1 \in T_1\setminus T_0$ and $T_0 \cup T_1 = D$. If $x \in T_0$ then

$$E = \bigcap_{c\in C} I(x,c) \cap C \subset I(x,e_0) \subset T_0,$$

which is impossible since $e_1 \notin T_0$. Similarly we find that $x \notin T_1$. This contradiction shows that r_C is well-defined. Obviously, $r_C(x) = x$ for all $x \in C$.

The only remaining part is to show that r_C is continuous. Let $x \in D$ and suppose that $r_C(x) \notin A \cap C$, for some A in $J(D)$ which intersects C. Since $J(D)$ is binary there is a $c \in C$ such that $I(x,c) \cap A = \emptyset$, and we can find a $B \supset I(x,c)$ such that $B \in J(D)$ and $B \cap A = \emptyset$. Now we can find two sets S_1 and S_2 in $J(D)$ such that $S_1 \cup S_2 = D$, $S_1 \cap A = \emptyset$ and $S_2 \cap D = \emptyset$ (Lemma 2.6). For every point p of the open set $D\setminus S_2$ we obtain that $r_C(p) \notin A$ because $I(p,c) \subset S_1$ which misses A. This proves continuity. □

The retraction of Lemma 2.8 is called the *canonical retraction* of D onto C.

COROLLARY 2.9. *If* $C \subset D$ *is an intersection of elements of* $J(D)$, *then* C *is connected.* □

COROLLARY 2.10. D *is locally connected.*

PROOF. Take $x \in D$ and let U be an open neighbourhood of x. Since, by Corollary 2.2, $J(D)$ is a closed subbase for D, we can find finitely many $T_1, T_2, \ldots$ $\ldots, T_n \in J(D)$ with $x \notin \bigcup_{1\le i\le n} T_i \supset D\setminus U$. Since $J(D)$ is binary (Lemma 2.7) for each $i \le n$ we can find $T_i' \in J(D)$ with $x \in T_i'$ and $T_i' \cap T_i = \emptyset$ (observe that $\{x\} = \bigcap\{T \in J(D) \mid x \in T\}$). By the normality of $J(D)$, (Lemma 2.6) we can find for each $i \le n$ an element $T_i'' \in J(D)$ with $T_i' \subset T_i''$, $x \in \mathrm{int}(T_i'')$ and $T_i'' \cap T_i = \emptyset$. Put

$$T = \bigcap_{1\le i\le n} T_i''.$$

Then T is a neighbourhood of x which is contained in U and which, by Corollary 2.9, is connected. □

For all $x,y \in D$ define

$$S(x,y) = \{p \in D \mid p \text{ separates } x \text{ from } y\} \cup \{x,y\}.$$

We claim that $S(x,y) = I(x,y)$, where $I(x,y)$ is defined as above. We establish that claim in our next two lemmas.

LEMMA 2.11.

$$S(x,y) \subset I(x,y).$$

PROOF. Take $p \in S(x,y)\setminus\{x,y\}$. Then $D\setminus\{p\} = U \cup V$, where U and V are disjoint open subsets of D with $x \in U$ and $y \in V$. Since $I(x,y)$ is connected (Corollary 2.9) and since $x \in I(x,y) \cap U$, $y \in I(x,y) \cap V$, this implies that $p \in I(x,y)$.□

LEMMA 2.12.

$$I(x,y) \subset S(x,y).$$

PROOF. Let $p \in I(x,y)\setminus S(x,y)$. Suppose that $q \in S(x,y)$ and that $U_x(q)$ (resp. $U_y(q)$) are the cutpoint components of x (resp. y) in $D\setminus\{q\}$. If $p \notin U_x(q) \cup U_y(q)$ then there is a cutpoint component $U_p(q)$ and x and y are both in $D\setminus U_p(q)$, which means that $p \notin I(x,y)$. Therefore every $q \in S(x,y)$ either separates x and p or y and p and $S(x,y) \subset S(x,p) \cup S(y,p)$.

Conversely, if $q \in S(x,p)$ then no cutpoint component of $D\setminus\{q\}$ contains both x and y, since in that case $D\setminus U_p(q)$ contains both x and y in contradiction with $p \in I(x,y)$. So $q \in S(x,y)$ and $S(x,p) \subset S(x,y)$. Similarly $S(p,y) \subset S(x,y)$. Therefore

$$S(x,y) = S(x,p) \cup S(p,y).$$

Define

$$A_x = \bigcup_{q \in S(x,p)} U_x(q) \quad\text{and}\quad A_y = \bigcup_{q \in S(y,p)} U_y(q)$$

Then A_x and A_y are both open. Define

$$A_p = D\setminus(A_x \cup A_y \cup \{p\}).$$

We claim that A_p is open. Let $a \in A_p$ and separate a and p with a point s. Then $s \notin (S(x,p) \cup S(y,p))$. If $U_a(s) \cap A_x \neq \emptyset$ then $\exists r \in S(x,p)$ such that:

$$U_a(s) \cap U_x(r) \neq \emptyset, \quad p \notin U_a(s) \cup U_x(r),$$

$$a \in U_a(s)\backslash U_x(r), \quad x \in U_x(r)\backslash U_a(s),$$

which contradicts Lemma 2.4. Therefore $U_a(s) \cap A_x = \emptyset$, and $U_a(s) \cap A_y = \emptyset$. $\bigcup_{a\in A_p} U_a(s) = A_p$ so we obtain that A_x, A_y and A_p are a partition of $D\backslash\{p\}$ into open parts, i.e. p is a cutpoint which separates x and y. This contradicts the assumption that $p \notin S(x,y)$ which proves the lemma. □

COROLLARY 2.13. *If* $x,y \in D$, *then* $I(x,y) = S(x,y)$. □

COROLLARY 2.14. *If* $C \subset D$ *is a subcontinuum, then* $C = \bigcap\{T \in \mathcal{J}(D) \mid C \subset T\}$.

PROOF. Take $x \notin C$ and $c \in C$ arbitrarily. Since $I(x,c)$ is connected and $x \notin C$ there has to be a point $y \in I(x,c)\backslash C$ different from x. By Corollary 2.13, y separates c from x. Let U be the component of $D\backslash\{y\}$ containing x. Since C is connected and U is open, $D\backslash(U \cup \{y\})$ is open. Since $y \notin C$ we may conclude that $C \cap U = \emptyset$. Consequently, $T = D\backslash U \in \mathcal{J}(D)$ contains C but misses x. □

COROLLARY 2.15.

(1) $S(x,y) = \bigcap\{C \subset D \mid x,y \in C \text{ and } C \text{ is a continuum}\}$.

(2) *Each subcontinuum* $C \subset D$ *is a retract of* D *under the retraction* $r_C\colon D \to C$ *defined by*

$$\{r_C(x)\} = \bigcap_{c\in C} S(x,c) \cap C.$$

(3) *The intersection of an arbitrary family of subcontinua of* D *is either empty or is a continuum.*

PROOF. Combine Corollary 2.14 and, respectively, Corollary 2.13 and Lemma 2.8. □

The retraction r_C is called the *canonical retraction of* D *onto* C.

LEMMA 2.16. *If* $a,b,c \in D$ *then* $S(a,b) \cap S(a,c) \cap S(b,c)$ *is a singleton.*

PROOF. By Corollary 2.13 and the binarity of $\mathcal{J}(D)$ (Lemma 2.7), we have

$$E = S(a,b) \cap S(b,c) \cap S(a,c) \neq \emptyset.$$

Assume that there are distinct $x,y \in E$. Find $S,T \in J(D)$ with $x \in S\setminus T$, $y \in T\setminus S$ and $T \cup S = D$. At least two points of $\{a,b,c\}$ must be contained in S or T. So, without loss of generality, $a,b \in S$. Then

$$E \subset S(a,b) = I(a,b) \subset S,$$

which is a contradiction since $y \in E\setminus S$. □

<u>LEMMA 2.17</u>. *If* $x,y \in D$ *are distinct,* $p \in I(x,y)$ *and* $q \in I(x,y)\setminus I(x,p)$, *then* $q \in I(p,y)$.

<u>PROOF</u>. Clearly $q \neq x$ and if $q = y$ then there is nothing to prove. So assume that $q \neq y$. Write $D\setminus\{q\} = U \cup V$ where U and V are disjoint and open, $x \in U$ and $y \in V$. Since $q \notin I(x,p)$ and since $I(x,p)$ is connected (Lemma 2.8) we conclude that $I(x,p) \subset U$. Therefore, by the connectivity of $I(p,y)$ this implies that $q \in I(p,y)$. □

<u>COROLLARY 2.18</u>. *If* $x,y \in D$ *are distinct, then* $S(x,y)$ *is a linearly ordered continuum with order defined by* $p \leq q$ *iff* p *separates* x *from* q.

<u>PROOF</u>. From Corollary 2.13 the relation $\leq$ can also be defined by $p \leq q$ iff $p \in I(x,q)$. If $p \leq q$ and $q \leq p$ then $p \in I(x,q)$, consequently

$$p \in I(x,p) \cap I(p,q) \cap I(x,q).$$

Similarly

$$q \in I(x,p) \cap I(p,q) \cap I(x,q).$$

This implies that $p = q$ (Lemma 2.16). Now we show that $\leq$ is a partial order. If $p \leq q$ and $q \leq r$ then $p \in I(x,q)$ and $q \in I(x,r)$. Therefore $p \in I(x,q) \subset I(x,r)$ or equivalently, $p \leq r$. Let us now show that $\leq$ is linear. Take $p,q \in I(x,y)$ such that $p \nleq q$ and $q \nleq p$. Then $p \notin I(x,q)$, hence $p \in I(q,y)$ (Lemma 2.17). Similarly, $q \in I(p,y)$. Therefore

$$p \in I(p,q) \cap I(p,y) \cap I(q,y)$$

and

$$q \in I(p,q) \cap I(p,y) \cap I(q,y),$$

consequently by Lemma 2.16, $p = q$ which is a contradiction.

Let us now show that $\le$ generates the topology of $I(x,y)$. Clearly

$$\{q \in I(x,y) \mid q \le p\} = I(x,p)$$

and by Lemma 2.17,

$$\{q \in I(x,y) \mid p \le q\} = I(p,y).$$

Therefore the initial segments are closed in $I(x,y)$. By the compactness of $I(x,y)$ this implies that $\le$ generates the topology of $I(x,y)$. □

NOTES. (for Section 2). Lemma 2.1 (that cutpoint components are open) is due to KOK [9]; see also WARD [23].

The fact that the intersection of an arbitrary family of subcontinua of D is a subcontinuum and that each set of the form $S(x,y)$ is orderable by the order of 2.18 is well-known. See HOCKING & YOUNG [8], MOORE [16], and WHYBURN [27]. The approach developed in this section is implicit in VAN MILL & SCHRIJVER [11], VAN MILL & VAN DE VEL [12] and VAN MILL [10]. The Corollaries 2.10 and 2.14 and some other results are related to the results of GURIN [7], PROIZVOLOV [18], and WARD [23].

3. THE THEOREM OF CORNETTE AND BROUWER

In this section we will show that each dendron is a continuous image of an ordered continuum. We will assume that the reader is familiar with the theory of inverse systems and inverse limits.

Let L and M be ordered continua. A continuous *surjection* $f\colon L \to M$ is called *order preserving* if $f(x) \le f(y)$ for all $x,y \in L$ with $x \le y$.

LEMMA 3.1. *Let* $(L_\alpha, f_{\alpha\beta}, \alpha \in A)$ *be an inverse system of ordered continua such that each* $f_{\alpha\beta}$ *is order preserving. Then* $\varprojlim(L_\alpha, f_{\alpha\beta}, \alpha \in A)$ *is an ordered continuum.*

PROOF. For each $\alpha \in A$ let $\pi_\alpha\colon L \to L_\alpha$ be the projection. Define an order $\le$ on L by putting

$$x \le y \text{ iff } \forall\alpha \in A: \pi_\alpha(x) \le \pi_\alpha(y).$$

It is clear that ≤ is a linear order on L which generates the topology of L. It is well-known that the inverse limit of an inverse system consisting of continua is a continuum. Hence L is an ordered continuum. □

LEMMA 3.2. *Let* D *be a dendron and let* κ *be an ordinal. For each* $\alpha < \kappa$ *let* $D_\alpha \subset D$ *be a subcontinuum such that* $\beta < \alpha$ *implies that* $D_\beta \subset D_\alpha$. *If* $r_{\alpha\beta}: D_\alpha \to D_\beta$ *denotes the canonical retraction, then*

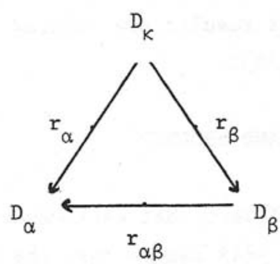
$$\varprojlim(D_\alpha, r_{\alpha\beta}, \alpha < \kappa)$$

is homeomorphic to the closure of $\bigcup_{\alpha<\kappa} D_\alpha$.

PROOF. Let D_κ denote the closure of $\bigcup_{\alpha<\kappa} D_\alpha$ and for each $\alpha < \kappa$ let $r_\alpha: D_\kappa \to D_\alpha$ be the canonical retraction. It is easy to see that for each $\alpha < \beta < \kappa$ the diagram below commutes, which implies, by compactness, that the function

$$\psi: D_\kappa \to \varprojlim(D_\alpha, r_{\alpha\beta}, \alpha < \kappa)$$

defined by $\psi(x)_\alpha = r_\alpha(x)$ is a continuous surjection. It therefore suffices



to show that ψ is one to one. To this end, take distinct $x, y \in D_\kappa$. Let V and W be disjoint and connected neighbourhoods of, respectively, x and y (Corollary 2.10). It is clear that for some $\alpha < \kappa$ we have that $V \cap D_\alpha \neq \emptyset \neq D_\alpha \cap W$. Take a point $s \in V \cap D_\alpha$ and a point $t \in W \cap D_\alpha$. Since V is a continuum,

$$I(x,s) \subset V$$

which implies that

$$\{r(x)\} = \bigcap_{d \in D_\alpha} I(x,d) \cap D_\alpha \subset I(x,s) \subset V$$

(Corollary 2.15). We conclude that $r_\alpha(x) \in V$ and, similarly, $r_\alpha(y) \in W$. Consequently, $r_\alpha(x) \neq r_\alpha(y)$. Therefore $\psi(x) \neq \psi(y)$ and ψ is one-to-one. □

We now come to the main result of this section.

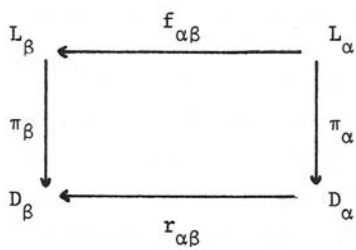
THEOREM 3.3. *Let* D *be a dendron. Then* D *is a continuous image of an ordered continuum.*

PROOF. Let $\kappa = |D|$ and let

$$\{d_\alpha \mid \alpha < \kappa \text{ and } \alpha \text{ is a successor}\},$$

enumerate D.

By transfinite induction, for every $\alpha < \kappa$ we will construct a subcontinuum $D_\alpha \subset D$ and an ordered continuum L_α and for each $\beta < \alpha$ an order preserving map $f_{\alpha\beta}: L_\alpha \to L_\beta$ and a continuous surjection $\pi_\alpha: L_\alpha \to D_\alpha$ such that for each $\beta < \alpha$ the diagram below commutes. Here $r_{\alpha\beta}$ denotes the canonical retraction.



In addition we will construct the D_α's in such a way that $d_\alpha \in D_\alpha$ for each successor $\alpha < \kappa$. The construction is a triviality.

Let $D_0 = L_0 = \{d_0\}$ and let π_0 be the identity. Suppose that we have constructed everything for all $\beta < \alpha$. If α is a limit put

$$D_\alpha = (\bigcup_{\beta<\alpha} D_\beta)^- \quad \text{and} \quad L_\alpha = \varprojlim(L_\beta, f_{\beta\eta}, \beta < \alpha)$$

and define all maps in the obvious way (applying the Lemmas 3.1 and 3.2). If α is a successor and if $d_\alpha \in D_{\alpha-1}$ then we don't do anything, i.e. put $D_\alpha = D_{\alpha-1}$, etc. So suppose that $d_\alpha \notin D_{\alpha-1}$. Let $r: D_\alpha \to D_{\alpha-1}$ be the canonical retraction and put

$$D_\alpha = D_{\alpha-1} \cup I(d_\alpha, r(d_\alpha)).$$

Observe that $D_{\alpha-1} \cap I(d_\alpha, r(d_\alpha)) = \{r(d_\alpha)\}$. Take a point $y \in L_{\alpha-1}$ with $\pi_{\alpha-1}(y) = r(d_\alpha)$. In $L_{\alpha-1}$ replace $\{y\}$ be an "interval" which maps onto $I(d_\alpha, r(d_\alpha))$ in such a way that the endpoints of this interval are mapped onto $r(d_\alpha)$ (one can take for example two copies of $I(d_\alpha, r(d_\alpha))$ with the points corresponding to d_α identified).

Let L_α be the resulting space and let $\pi\colon L_\alpha \to D_\alpha$ be a map with the property that

$$\pi_\alpha(x) = \pi_{\alpha-1}(x) \text{ if } x \in L_{\alpha-1}\backslash\{\text{the endpoints of the added interval}\}.$$

In addition, let $f_{\alpha,\alpha-1}\colon L_\alpha \to L_{\alpha-1}$ be the map which collapses the added interval to the point y. It is clear that everything defined in this way is as required. Now put

$$L = \varprojlim(L_\alpha, f_{\alpha\beta}, \alpha < \kappa).$$

By Lemma 3.1, L is an ordered continuum which, by the diagram, maps onto D.□

COROLLARY 3.4. *Every dendron is hereditarily normal.*

NOTES. (for Section 3). Theorem 3.3 was first shown by CORNETTE [3] and independently, but later, by A.E. BROUWER [1]. Our proof is a simplification of their ideas; see also PEARSON [17] and WARD [26].

A Souslin dendron is a dendron D which satisfies the countable chain condition, is not separable, and which moreover has the property that each countable subset is contained in a metrizable subcontinuum of D. If the above program is carried out with some extra care, it can be shown that each Souslin dendron is a continuuous image of a Souslin continuum. In addition, each Souslin continuum can be mapped onto a Souslin dendron. Notice that a Souslin continuum (= a linearly orderable CCC non-separable continuum) is not a Souslin dendron. For details see VAN MILL & WATTEL [13].

Lemma 3.1 is due to CAPEL [2], and Corollary 3.4 is due to GURIN [7], see also PROIZVOLOV [19].

4. THE FIXED POINT PROPERTY

In this section we show that every dendron has the fixed point property.

LEMMA 4.1. *Let* L *be an ordered continuum. Then* L *has the fixed point property.*

PROOF. Let $f: L \to L$ be any self map and put

$$U = \{x \in L \mid x < f(x)\}, \quad \text{and} \quad V = \{x \in L \mid f(x) < x\}$$

respectively. Then U and V are clearly open. Suppose that f has no fixed point. Then $U \cup V = L$ and hence, since $U \cap V = \emptyset$, by connectivity, either $U = \emptyset$ or $V = \emptyset$. If $U = \emptyset$, then $f(\min(L)) < \min(L)$, and if $V = \emptyset$ then $\max(L) < f(\max(L))$, which is impossible. □

Let D be a dendron. A point $x \in D$ is called an *endpoint* if $D\setminus\{x\}$ is connected. A *finite dendron* is a dendron with only a finite number of endpoints. Note that a finite dendron is nothing but a finite connected acyclic graph.

LEMMA 4.2. *Let* D *be a finite dendron. Then* D *has the fixed point property.*

PROOF. Let E denote the set of endpoints of D. We induct on $|E|$. If $|E| \leq 2$ then use Lemma 4.1. So assume that the lemma is true for n and assume that $|E| = n+1$; list E as $\{e_1,\dots,e_{n+1}\}$. Put

$$D' = \bigcup\{I(e_i,e_j) \mid i,j \in \{1,2,\dots,n\}\}.$$

Then D' is a subcontinuum of D and hence D' is a dendron (Corollary 2.15(1)). Also D' has precisely n endpoints. Let $r_{D'}: D \to D'$ be the canonical retraction (Corollary 2.15(2)) and put $x = r_{D'}(e_{n+1})$. Observe that

$$I(e_{n+1},x) \cap D' = \{x\} \quad \text{and that} \quad I(e_{n+1},x) \cup D' = D.$$

By Corollary 2.18, $I(e_{n+1},x)$ is an ordered continuum. Let $f: D \to D$ be any self-map. Assume that f has no fixed points. If $f(x) \in D'$ then define $g: D' \to D'$ by

$$g(t) = f(t) \quad \text{if } f(t) \in D'$$

$$g(t) = x \quad \text{if } f(t) \notin D'$$

(we just collapse the interval $I(e_{n+1},x)$ to the point x). By induction hypothesis, g has a fixed point. This point cannot be x and hence must be a fixed point of f. If $f(x) \in I(e_{n+1},x)$ then we collapse D' to the point x and proceed in the same way. This gives us the required contradiction. □

We now come to the main result of this section.

THEOREM 4.3. *Let* D *be a dendron. Then* D *has the fixed point property.*

PROOF. Let $f\colon D \to D$ be any self-map. If f has no fixed point then, by compactness and by the local connectedness of D (Corollary 2.10), there is a finite cover $\mathcal{U}$ of D by non-empty subcontinua such that for every $U \in \mathcal{U}$ we have that

$$U \cap f(U) = \emptyset.$$

Let $F \subset X$ be finite such that for all $U \in \mathcal{U}$ both $F \cap U$ and $F \cap f(U)$ are non-empty. Define

$$D' = \bigcup\{I(x,y) \mid x,y \in F\}.$$

Observe that D' is a finite dendron. Define $g\colon D' \to D'$ by

$$g(x) = r_{D'}(f(x)),$$

where $r_{D'}\colon D \to D'$ is the canonical retraction (Corollary 2.15(2)). We claim that g has no fixed points which contradicts Lemma 4.2. Take $x \in D'$. There is a $U \in \mathcal{U}$ containing x. Then $f(x) \in f(U)$. Since $f(U)$ is a continuum that intersects D' (observe that $F \subset D'$), by Corollary 2.15(2),

$$r_{D'}(f(x)) \in f(U),$$

consequently, $g(x) \neq x$ since $U \cap f(U) = \emptyset$. □

NOTES. (for Section 4). Lemma 4.1 is well-known. Theorem 4.3 was first shown by SCHERRER [20] and generalized by WALLACE [22], see also WARD [24], [25].

5. A CHARACTERIZATION OF DENDRONS

In this section we show that a Hausdorff continuum X is a dendron if and only if X possesses a cross-free closed subbase.

LEMMA 5.1. *Let* X *be a* T_1 *space and let* $\mathcal{J}$ *be a binary closed subbase for* X. *Then for any distinct* $x,y \in X$ *there are disjoint* $T_0,T_1 \in \mathcal{J}$ *with* $x \in T_0$ *and* $y \in T_1$.

PROOF. Observe that, since X is T_1 and since $\mathcal{J}$ is a closed subbase, for every point $z \in X$ it is true that

$$\{z\} = \cap\{T \in \mathcal{J} \mid z \in T\}.$$

Consequently, the desired result follows directly from the binarity of $\mathcal{J}$. □

We now come to the main result in this section.

THEOREM 5.2. *Let X be a Hausdorff continuum. Then X is a dendron iff X possesses a cross-free closed subbase.*

PROOF. For the implication "dendron ⇒ ∃ cross-free closed subbase" see Section 2. So let X be a Hausdorff continuum and let $\mathcal{J}$ be a cross-free closed subbase for X. Let $x,y \in X$ such that $x \neq y$. Let $x \in T_0$ and $y \in T_1$ such that $T_0,T_1 \in \mathcal{J}$ and $T_0 \cap T_1 = \emptyset$, (cf. 5.1). According to Lemma 2.6 we can find $S_0,S_1 \in \mathcal{J}$ such that $S_0 \cup S_1 = X$, and $S_0 \cap T_1 = \emptyset = S_1 \cap T_0$.

Define

$$\mathcal{A} = \{T \in \mathcal{J} \mid T \cup S_0 = X\}.$$

Since X is connected we have that $\mathcal{A} \cup \{S_0\}$ has the property that every two of its elements meet and consequently, by binarity of $\mathcal{J}$ (Lemma 2.7), $(\cap\mathcal{A}) \cap S_0 \neq \emptyset$. We claim that this intersection consists of one point.

Assume to the contrary that $z_0,z_1 \in (\cap\mathcal{A}) \cap S_0$ such that $z_0 \neq z_1$. In the same way as above there are $R_0,R_1 \in \mathcal{J}$ such that $z_0 \in R_0\setminus R_1$ and $z_1 \in R_1\setminus R_0$ and $R_0 \cup R_1 = X$. Since $z_0 \notin R_1$ and $z_0 \in \cap\mathcal{A}$ we have that $R_1 \notin \mathcal{A}$ and consequently $R_1 \cup S_0 \neq X$. Hence $S_0 \subset R_1$ or $R_1 \subset S_0$ because $R_1 \cap S_0 = \emptyset$ is impossible since $z_1 \in R_1 \cap S_0$. However, this implies that $R_1 \subset S_0$ since $z_0 \notin R_1$. With the same technique one shows that $R_0 \subset S_0$; but this is a contradiction because $S_0 \neq X$. Let $z_0 = (\cap\mathcal{A}) \cap S_0$, then z_0 is a separation point of x and y, since S_0 and $\cap\mathcal{A}$ are closed subsets of X such that $(\cap\mathcal{A}) \cup S_0 = X$ and $x \in S_0$ and $y \in \cap\mathcal{A}$. This proves that X is a dendron. □

NOTES. (for Section 5). Theorem 5.3 is due to VAN MILL & SCHRIJVER [11] and is related to a characterization of ordered spaces in VAN DALEN & WATTEL [4].

6. A CHARACTERIZATION OF SUBSPACES OF DENDRONS

In this section we will use the results of the previous sections to show that a Hausdorff space X can be embedded in a dendron iff X has a cross-free closed subbase. We first show how to modify a given cross-free closed subbase to one with certain additional pleasant properties. Then we use this modified subbase to obtain embeddings into dendrons.

A closed subbase S for a space X is called a T_1-*subbase* provided that for all $x \in X$ and $S \in S$ not containing x there exists an element $T \in S$ with $x \in T$ and $T \cap S = \emptyset$.

LEMMA 6.1. *Let* X *be a Hausdorff space with a cross-free closed subbase S. Then there is a cross-free closed subbase for* X *which in addition is normal and* T_1.

PROOF. First of all we extend S to a larger subbase S^t by taking:

$$S^t = S \cup \{\{p\} \mid p \in X\}$$

(i.e. we add all singletons to the subbase). In this case S^t is still cross-free because $\{p\} \cap \{q\} = \emptyset$ for all $p \neq q$ and either $\{p\} \cap S = \emptyset$ or $\{p\} \subset S$ for each $S \in S$. Clearly the subbase S^t is a T_1 collection.

Next we add for each clopen $S \in S^t$ also its complement and obtain

$$S^n = S^t \cup \{X \setminus S \mid S \in S^t \text{ and S is clopen}\}.$$

Also S^n is a T_1 collection which is cross-free since if $S, R \in S^t$ then

$$\begin{aligned} &S \subset R \text{ implies } X\setminus S \supset X\setminus R \text{ and } (X\setminus S) \cup R = X,\\ &R \subset S \text{ implies } X\setminus S \subset X\setminus R \text{ and } (X\setminus S) \cap R = \emptyset,\\ &R \cap S = \emptyset \text{ implies } (X\setminus S) \cup (X\setminus R) = X \text{ and } R \subset X\setminus S,\\ &R \cup S = X \text{ implies } (X\setminus S) \cap (X\setminus R) = \emptyset \text{ and } X\setminus S \subset R. \end{aligned}$$

We now show that S^n is not only cross-free but is in addition normal.

Let R and S be two disjoint members of S^n. If S is clopen then also $X\setminus S$ is in S^n and we obtain a screening between S and R by S and $X\setminus S$, and the same holds for R. If neither S nor R is clopen then we can find a point $r \in R$ and a point $s \in S$ such that $r \in Cl_X(X\setminus R)$ and $s \in Cl_X(X\setminus S)$.

Next we will derive a screening of $\{s\}$ and $\{r\}$ by means of two subbase members. Since X is Hausdorff we can find two basic closed subsets B_s and B_r such that $B_s \cup B_r = X$, $r \notin B_s$ and $s \notin B_r$. B_r is a finite union of subbase members $F_{r1},\ldots,F_{rn}$, and B_s is a finite union of $F_{s1},\ldots,F_{sm}$.

Define $\mathcal{F} = \{F_{si}\} \cup \{F_{rj}\}$ and $\mathcal{F}_s = \{F_{sj} \mid s \in F_{sj}\}$, then for F_{si} and $F_{sj} \in \mathcal{F}_s$ we have that

$$s \in F_{si} \cap F_{sj} \quad \text{and} \quad r \notin F_{si} \cup F_{sj}$$

hence either $F_{si} \subset F_{sj}$ or $F_{sj} \subset F_{si}$ and so there exists a largest member $F_s = \cup\mathcal{F}_s \in \mathcal{F}$. In the same way there is a maximal F_r in $\mathcal{F}$ which contains r. We now have two cases. If $F_s \cup F_r = X$ then we have obtained our screening with two members of $\mathcal{S}$.

In the other case we can find a point x in $X\backslash(F_s \cup F_r)$. Let F_x be the maximal member of $\mathcal{F}$ containing x. Since

$$r \notin F_x \cup F_s; \quad s \in F_s\backslash F_x \quad \text{and} \quad x \in F_x\backslash F_s$$

we have

$$F_x \cap F_s = \emptyset \text{ and similarly } F_x \cap F_r = \emptyset \text{ and } F_s \cap F_r = \emptyset.$$

Consequently, we obtain a partition of the space into three disjoint closed parts: F_s, F_r and $\cup\{F_x \mid x \notin F_s \cup F_r\}$. (The last collection is closed since it is the union of a finite collection because $\mathcal{F}$ is finite.) This means that F_s is clopen and X F_s is in $\mathcal{S}^n$.

Anyway we obtain a screening of s and r by means of two subbase members, call them F_s' and F_r'. Now S does not contain a neighbourhood of s and F_r' is closed and does not contain s and hence $S \cup F_r' \neq X$. Moreover, $s \in S\backslash F_r'$ and $r \in F_r'\backslash S$ and therefore $F_r' \cap S = \emptyset$ and similarly $F_s' \cap R = \emptyset$. Since $F_s' \cup F_r' = X$ we have $R \subset F_r'$ and $S \subset F_s'$ and we obtained a screening of R and S. □

<u>REMARK 6.2</u>. In the previous lemma the Hausdorff property cannot be omitted since in an infinite space with the cofinite topology the collection of all singletons is a cross-free T_1 subbase, but it cannot have a T_1 normal subbase since a space with a T_1 normal subbase is completely regular (cf. [5]).

A collection $\mathcal{S}$ of subsets of a set X is called *strongly connected* provided that X cannot be partitioned into finitely many non-empty elements of $\mathcal{S}$.

LEMMA 6.3. *Let X be a set and let S be cross-free and connected. Then S is strongly connected.*

PROOF. From 6.1 it follows that S is normal and T_1. Assume that there exists a number n with the property that there is a minimal collection $S_1, S_2, \ldots, S_n$ of mutually disjoint sets such that $\bigcup_{1\le i\le n} S_i = X$, but for every number smaller than n there is no such partition of X with members of S. Since S_1 and S_n are disjoint there are two subsets T_1 and T_n in S such that $T_1 \cap S_n = \emptyset$ and $T_n \cap S_1 = \emptyset$ and $T_n \cup T_1 = X$. Let $1 < j < n$ then either $S_j \cap T_1 \neq \emptyset$ or $S_j \cap T_n \neq \emptyset$, say $S_j \cap T_1 \neq \emptyset$. Then $S_j \cup T_1 \neq X$ because S_n is disjoint from both, and therefore $S_j \subset T_1$. Let $J = \{j \mid S_j \subset T_1\}$. Then $\bigcup_{i\notin J} S_i \cup T_1 = X$, is a disjoint cover of X with less than n members. This contradiction shows our lemma. □

COROLLARY 6.4. *Let X be a compact Hausdorff space and let S be a cross-free connected subbase for X. Then X is connected (and consequently, X is a dendron).*

PROOF. Suppose that X is equal to $G \cup H$ with $G \cap H = \emptyset$ and G and H are closed. Then H is an intersection of a collection of closed base members $\{B_\alpha\}_{\alpha\in A}$ for some index set A. Since $\cap B_\alpha \cap G = \emptyset$ and since X is compact there is a finite subcollection of B_α's which misses G and therefore G and H are both finite intersections of finite unions of members of S. We could also write G and H as finite unions of finite intersections of subbasic closed sets. Let m be the minimal number such that there are $G_1, \ldots, G_m$ such that:

(a) $G_1, \ldots, G_m$ are non-void intersections of finitely many subbase members;

(b) $G_1 \cup \ldots \cup G_m = X$;

(c) There is a number $k < m$ such that

$$\bigcup_{1\le i\le k} G_i \neq \emptyset \neq \bigcup_{k<i\le m} G_i$$

and

$$\Big(\bigcup_{1\le i\le k} G_i\Big) \cap \Big(\bigcup_{k<i\le m} G_i\Big) = \emptyset.$$

We claim that $G_i \cap G_j = \emptyset$ for $i \neq j$, (w.l.o.g. $G_i, G_j \subset G$). Suppose not. Take a point $x \notin G_i \cup G_j$. Then there are subbase members S_i and S_j such that $G_i \subset S_i$ and $G_j \subset S_j$ but $x \notin S_i \cup S_j$. Now $S_i \cap S_j \neq \emptyset$ and $S_i \cup S_j \neq X$, so either $S_i \subset S_j$ or $S_j \subset S_i$ and in both cases the largest of the two contains

$G_i \cup G_j$. Therefore

$$G_i \cup G_j = \bigcap\{S \in \mathcal{S} \mid G_i \cup G_j \subset S\}.$$

But now we can decrease the number m by taking a finite intersection of this collection which misses H, instead of both G_i and G_j. Next we prove that each G_i is a member of $\mathcal{S}$. Suppose that $G_i \notin \mathcal{S}$, and let $m \neq i$. Then there is a member $T \in \mathcal{S}$ such that $T \cap G_m = \emptyset$ and $G_i \subset T$. The sequence $G_1,\ldots,G_{i-1},T,$ $G_{i+1},\ldots,G_m$ is also a sequence which satisfies (a), (b) and (c) and we conclude that $T \cap G_j = \emptyset$ whenever $1 \le j \le m$ with $j \neq i$ and $G_i \subset T$, so $G_i = T$. We found a finite collection of pairwise disjoint members of $\mathcal{S}$ which cover X. This contradicts Lemma 6.3. □

Let $\mathcal{S}$ be a subbase for a space X. The superextension $\lambda(X,\mathcal{S})$ has an underlying set, the set of all maximal linked systems in $\mathcal{S}$ with topology generated by taking the collection

$$\mathcal{S}^+ = \{S^+ \mid S \in \mathcal{S}\},$$

where

$$S^+ = \{\mathcal{M} \mid \mathcal{M} \in \lambda(X,\mathcal{S}) \text{ and } S \in \mathcal{M}\},$$

as a (closed) subbase. The following facts are well-known and easy to prove:
- $\mathcal{S}^+$ is binary (as a consequence, $\lambda(X,\mathcal{S})$ is compact);
- if $\mathcal{S}$ is normal then $\lambda(X,\mathcal{S})$ is Hausdorff;
- if $\mathcal{S}$ is a T_1 collection then the function $i\colon X \to \lambda(X,\mathcal{S})$ defined by $i(x) = \{S \in \mathcal{S} \mid x \in S\}$ is an embedding;
- $\mathcal{S}$ is connected iff $\mathcal{S}^+$ is connected.

For details, see [21]. Superextensions were introduced by DE GROOT [6].

<u>LEMMA 6.5</u>. *Let* X *be a space and let* $\mathcal{S}$ *be a closed subbase of* X *with the following properties:*

(a) $\mathcal{S}$ is a T_1 collection;

(b) $\mathcal{S}$ is normal;

(c) $\mathcal{S}$ is cross-free.

Then X *can be embedded in a dendron* T.

PROOF. If $\mathcal{S}$ is a connected subbase then $\lambda(X,\mathcal{S})$ is a compact space with a cross-free connected subbase $\mathcal{S}^+$, and now it follows from 6.4 and 5.2 that $\lambda(X,\mathcal{S})$ is a dendron which contains X.

If $\mathcal{S}$ is not connected, then we extend X to a space Y and $\mathcal{S}$ to a subbase $\mathcal{S}^\sim$ in such a way that $\mathcal{S}^\sim$ is a connected subbase for Y, and since $\lambda(Y,\mathcal{S}^\sim)$ contains X as a subspace we have that X is a subspace of a dendron.

Let $\{\langle H_\alpha,K_\alpha\rangle \mid \alpha \in A\}$ enumerate all the pairs $\langle H,K\rangle \in \mathcal{S}*\mathcal{S}$ such that $K = X\backslash H$ (in such a way that $\langle H,K\rangle$ and $\langle K,H\rangle$ do not both occur). Let $\mathcal{H} = \{H_\alpha \mid \alpha \in A\}$ and $\mathcal{K} = \{K_\alpha \mid \alpha \in A\}$. Define

$$Y = X \cup (I\times A), \text{ where I is the open unit interval } (0,1).$$

For $\alpha \in A$ we define

$$A_0(\alpha) = \{\beta \in A\backslash\{\alpha\} \mid H_\beta \subset H_\alpha \text{ or } K_\beta \subset H_\alpha\},$$

and

$$A_1(\alpha) = \{\beta \in A\backslash\{\alpha\} \mid H_\beta \supset H_\alpha \text{ or } K_\beta \supset H_\alpha\}.$$

Thus $A = A_0(\alpha) \cup A_1(\alpha) \cup \{\alpha\}$. For $\alpha \in A$ define

$$H_\alpha^\sim = H_\alpha \cup (I\times A_0(\alpha)), \quad K_\alpha^\sim = K_\alpha \cup (I\times A_1(\alpha)).$$

Then for $r \in I$ we define

$$H_{\alpha r}^\sim = H_\alpha^\sim \cup ((0,r]\times\{\alpha\}) \text{ and } K_{\alpha r}^\sim = K_\alpha^\sim \cup ([r,1)\times\{\alpha\}).$$

For each $S \in \mathcal{S}\backslash(\mathcal{H}\cup\mathcal{K})$, let

$$A(S) = \{\alpha \in A \mid H_\alpha \subset S \text{ or } K_\alpha \subset S\};$$

then let

$$S^\sim = S \cup (I\times A(S)).$$

Finally, set

It is easily verified that $S^{\sim}$ is a connected cross-free subbase satisfying (a) and (b). □

We now come to the main result of this section.

THEOREM 6.6. *A Hausdorff space* X *can be embedded in a dendron iff* X *possesses a cross-free closed subbase.*

PROOF. Corollary 2.5 states that a dendron has a cross-free closed subbase, if we restrict ourselves to a subspace X then the collection of all restrictions of subbase members is still cross-free. Conversely, if X possesses a cross-free closed subbase, then Lemma 6.1 states that X possesses a cross-free closed subbase which is both normal and T_1. From Lemma 6.5 it follows that X can be embedded in a dendron. □

NOTES. (for Section 6). Lemma 6.3 and Corollary 6.4 are due to VAN MILL & SCHRIJVER [11]. All other results in this section can be found in VAN MILL & WATTEL [14].

In [15] the authors showed that for compact X the following statements are equivalent:

(1) X is orderable;

(2) X has a weak selection;

(X has a weak selection iff there is a map $s: X^2 \to X$ such that $s(x,y) = s(y,x) \in \{x,y\}$ for all $x,y \in X$.)

This result suggests the natural question whether for dendrons there is a similar characterization, i.e. is there a natural number $n \in \mathbb{N}$ and algebraic conditions on a map $s: X^n \to X$ such that a continuum X is a dendron if and only if X has such a map? For this question Ward has given a satisfactory solution in [24], in which he states:

A compact Hausdorff space is a dendron if and only if there exists a continuous function $m: X \times X \to X$ such that

(i) m is idempotent, i.e. $m(x,x) = x$;

(ii) m is associative;

(iii) m is commutative, i.e. $m(x,y) = m(y,x)$;

(iv) m is monotone;

(v) if $m(a,x) = a$ and $m(b,x) = b$, then $m(a,b) \in \{a,b\}$.

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