BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences mathématiques
Vol. XXVII, No. 1, 1979

GENERAL TOPOLOGY

Extenders from βX—X to βX

by

Jan van MILL*)

Presented by C. RYLL-NARDZEWSKI on February 15, 1978

Summary. For any space X let $\tau(X)$ denote the topology of X. If $X \subset Y$ then a function $\kappa: \tau(X) \to \tau(Y)$ is called an extender provided that $\kappa(U) \cap X = U$ for all $U \in \tau(X)$. We prove that for each locally compact nonpseudocompact space X and for each extender $\kappa: \tau(\beta X - X) \to \tau(\beta X)$ there is a collection $\mathscr{B}$ of 2^{ω} pairwise disjoint nonvoid open subsets of $\beta X - X$ such that $\bigcap_{B \in \mathscr{B}} \kappa(B) \neq \emptyset$. In particular, there is no retraction from βX onto $\beta X - X$.

0. Introduction. All topological spaces under discussion are assumed to be Tychonoff.

For any topological space X let $\tau(X)$ denote the topology of X. If $X \subset Y$ then a function $\kappa: \tau(X) \to \tau(Y)$ is called an *extender* provided that $\kappa(U) \cap X = U$ for all $U \in \tau(X)$. In addition, X is said to be K_n-embedded in Y (cf. van Douwen [2]) provided there is an extender $\kappa: \tau(X) \to \tau(Y)$ such that

(*) if $n=0$ then $\kappa(\emptyset) = \emptyset$ and $\kappa(V) \cap \kappa(W) = \kappa(V \cap W)$ for all $V, W \in \tau(X)$; if $n > 0$ then $\kappa(V_0) \cap \ldots \cap \kappa(V_n) = \emptyset$ whenever $V_i \cap V_j = \emptyset$ for $0 \leqslant i < j \leqslant n$ and $V_0, \ldots, V_n \in \tau(X)$.

The extender κ is called a *K_n-function* (cf. van Douwen [2]). We call an extender $\kappa: \tau(X) \to \tau(Y)$ a *K_0^*-function* if it is a K_0-function and has the additional property that $\kappa(V) \cup \kappa(W) = Y$ whenever $V \cup W = X$ $(V, W \in \tau(X))$. If $r: Y \to X$ is a retraction then the function $\kappa: \tau(X) \to \tau(Y)$ defined by $\kappa(U) = r^{-1}(U)$ is easily seen to be a K_0^*-function.

Our interest in K_n-functions was motivated by the recent result of van Douwen [3] that *if there is a retraction from βX onto $\beta X - X$, then X is locally compact and pseudocompact* (under *CH* this was proved earlier by Comfort [1]). This theorem suggests the question whether for locally compact nonpseudocompact X there can be a K_n-function from $\tau(\beta X - X)$ to $\tau(\beta X)$ for some $n \geqslant 0$. The answer to this question is in the negative.

0.1. THEOREM. *Let X be a locally compact and nonpseudocompact space and let $\kappa: \tau(\beta X - X) \to \tau(\beta X)$ be an extender. Then there is a collection $\mathscr{B}$ of 2^{ω} pairwise disjoint nonvoid open subsets of $\beta X - X$ such that $\bigcap_{B \in \mathscr{B}} \kappa(B) \neq \emptyset$.*

*) Research supported by the Netherlands Organization for the Advancement of Pure Research (Z. W. O.); Juliana van Stolberglaan 148, 's-Gravenhage, The Netherlands.

This theorem shows that for locally compact nonpseudocompact spaces X the open sets of $\beta X - X$ cannot be extended to open sets of βX in a decent way; in particular it follows that there is no retraction from βX onto $\beta X - X$. The local compactness of X is the above result is essential since there is a noncompact Lindelöf space Z such that there is a K_0^*-function $\kappa: \tau(\beta Z - Z) \to \tau(\beta Z)$.

In section **1** of this note we prove some theorems about extenders which have interest in their own rights. Section **2** contains the proof of theorem 0.1.

1. Extenders. For any topological space X let $CZ(X)$ denote the collection of cozero sets of X. Recall that $CZ(X)$ is a base for X and that it is closed under countable unions (cf. Gillman and Jerison [4]).

Suppose that $X \subset Y$. Let $\mathscr{B} \subset \tau(X)$ be such that it contains $\emptyset$ and X and in addition is closed under finite intersections and finite unions. A function $\kappa: \mathscr{B} \to \tau(Y)$ is called

an *extender* provided that $\kappa(B) \cap X = B$ for all $B \in \mathscr{B}$;

a K_0*-function* if it is an extender and moreover satisfies the following conditions:
(i) $\kappa(\emptyset) = \emptyset$,
(ii) $\kappa(B_0 \cap B_1) = \kappa(B_0) \cap \kappa(B_1)$ for all $B_0, B_1 \in \mathscr{B}$.

a K_0^**-function* if it is a K_0-function which satisfies:

$$\kappa(B_0) \cup \kappa(B_1) = Y \text{ if } B_0 \cup B_1 = X (B_0, B_1 \in \mathscr{B})$$

a *Boolean extender* if it is a K_0^*-function such that

$$\kappa(B_0) \cup \kappa(B_1) = \kappa(B_0 \cup B_1) \text{ for all } B_0, B_1 \in \mathscr{B}.$$

1.1. THEOREM. *Let X be a subspace of Y. The following statements are equivalent:*
(i) *there is a K_0^*-function $\kappa: CZ(X) \to \tau(Y)$;*
(ii) *there is a Boolean extender $\kappa: CZ(X) \to \tau(Y)$;*
(iii) *for every compact space Z and for every continuous mapping $f: X \to Z$ there is a continuous mapping $\bar{f}: Y \to Z$ such that $\bar{f} \restriction X = f$.*

Proof. (ii)⇒(i) is obvious; hence it suffices to establish (i)⇒(iii) and (iii)⇒(ii).

Indeed, assume that there is a K_0^*-function $\kappa: CZ(X) \to \tau(Y)$ and that $f: X \to Z$ is continuous. Fix $y \in Y$ and define

$$\mathscr{U}_y = \{T \in CZ(Z) | y \in \kappa f^{-1}(T)\}.$$

Then $\mathscr{U}_y \neq \emptyset$ since $Z \in \mathscr{U}_y$. Also, notice that $f^{-1}(T) \in CZ(X)$ for all $T \in CZ(Z)$. Now the fact that κ is a K_0-function implies that $\mathscr{U}_y$ is closed under finite intersections. Also $\emptyset \notin \mathscr{U}_y$. Consequently, by the fact that Z is compact, we have that

$$\bigcap_{T \in \mathscr{U}_y} \mathrm{cl}_Z(T) \neq \emptyset.$$

We claim that this intersection contains precisely one point. Indeed, to the contrary, assume that it contains two distinct points, say p and q. Take $T_0, T_1 \in CZ(Z)$ such that $p \in T_0 - \mathrm{cl}_Z(T_1)$, $q \in T_1 - \mathrm{cl}_Z(T_0)$ and $T_0 \cup T_1 = Z$. Then $\kappa f^{-1}(T_0) \cup \kappa f^{-1}(T_1) = Y$ and consequently, without loss of generality, we may assume that $y \in \kappa f^{-1}(T_0)$. Then $T_0 \in \mathscr{U}_y$ and hence $q \in \mathrm{cl}_Z(T_0)$, which is a contradiction.

Now define $\bar{f}\colon Y \to Z$ by

$$\{\bar{f}(y)\} = \bigcap_{T \in \mathscr{U}_y} \mathrm{cl}_Z(T).$$

It is clear that $\bar{f} \restriction X = f$ and hence we need only check the continuity of $\bar{f}$. Indeed, take $T \in CZ(Z)$ and let $y \in \bar{f}^{-1}(T)$. Then since $\mathscr{U}_y$ is closed under finite intersections and since Z is compact there is a $T_0 \in \mathscr{U}_y$ such that

$$\mathrm{cl}_Z(T_0) \subset T.$$

Then $\kappa f^{-1}(T_0)$ is a neighborhood of y which is entirely contained in $\bar{f}^{-1}(T)$.

To prove that (iii) implies (ii) assume that $f\colon Y \to \beta X$ is such that f restricted to X is the identity. Then the function $\kappa\colon CZ(X) \to \tau(Y)$ defined by

$$\kappa(B) = f^{-1}\big(\beta X - \mathrm{cl}_{\beta X}(X - B)\big)$$

is easily seen to be a Boolean extender.

1.2. Corollary. *Let X be a compact subspace of Y. Then the following statements are equivalent*:

(i) *X is a retract of Y;*

(ii) *there is a K_0^*-function $\kappa\colon CZ(X) \to \tau(Y)$.*

1.3. Corollary. *Up to equivalence βX is the unique compactification of X for which there is a K_0^*-function from the cozero sets of X to the topology of βX.*

The following theorem gives another surprising property of K_0^*-functions.

1.4. Theorem. *Let X be such that there is a K_0^*-function $\kappa\colon CZ(\beta X - X) \to \tau(\beta X)$. Then for every compactification γX of X there is a K_0^*-function $\mu\colon CZ(\gamma X - X) \to \tau(\gamma X)$*

Proof. Let $f\colon \beta X \to \gamma X$ be the unique continuous mapping which extends the identity on X. Now define $\mu\colon CZ(\gamma X - X) \to \tau(\gamma X)$ by

$$\mu(T) = T \cup (\kappa f^{-1}(T) \cap X).$$

First observe that $\mu(T)$ is open since

$$f^{-1}(\mu(T)) = f^{-1}(T) \cup (\kappa(f^{-1}(T)) \cap X) = \kappa(f^{-1}(T)).$$

Hence μ is well-defined. The easy check that μ indeed is a K_0^*-function is left to the reader.

1.5. Corollary. *Let X be such that there is a retraction from βX onto $\beta X - X$. Then for every compactification γX of X there is a retraction from γX onto $\gamma X - X$.*

Proof. Apply theorem 1.1 and Theorem 1.4.

1.6. Remark. If there is a retraction $r\colon \beta X \to \beta X - X$ then the retraction $s\colon \gamma X \to \gamma X - X$ obtained from Theorem 1.1 and Theorem 1.4 can in fact be described as follows

$$\begin{cases} s(x) = x \text{ if } x \in \gamma X - X \\ s(x = fr(x) \text{ if } x \in X \end{cases}$$

(here $f\colon \beta X \to \gamma X$ and extends id_X). Of course it can also be checked directly that s defined in this way is continuous.

2.

2.1. Proof of Theorem 0.1. Since x is not pseudocompact, by a theorem in Gillman and Jerison [4] we may assume that the set of natural numbers $N \subset X$ and N is C-embedded in X. For each $n \in N$ let C_n be a cozero set of X such that

(i) $n \in C_n$ and $\mathrm{cl}_X(C_n)$ is compact;

(ii) $n < m$ implies that $\mathrm{cl}_X(C_n) \cap \mathrm{cl}_X(C_m) = \emptyset$;

(iii) $\bigcup_{n \in N} \mathrm{cl}_X(C_n)$ is closed in X.

Let $\{A_\alpha | \alpha < 2^\omega\}$ be a collection of infinite subsets of N such that $\alpha < \gamma$ implies that $A_\alpha \cap A_\gamma$ is finite (there is such a collection, see Gillman and Jerison [4]). For each $\alpha < 2^\omega$ define $B_\alpha \subset \beta X - X$ by

$$B_\alpha = \big(\beta X - \mathrm{cl}_{\beta X}(X - \bigcup_{n \in A_\alpha} C_n)\big) \cap (\beta X - X).$$

Then B_α is a nonvoid open subset of $\beta X - X$.

CLAIM. *$\bigcup_{n \in A_\alpha} C_n - \kappa(B_\alpha)$ has compact closure in X.*

Assume, to the contrary, that $\bigcup_{n \in A_\alpha} C_n - \kappa(B_\alpha)$ has no compact closure in X. Then there is an infinite $E \subset A_\alpha$ and for each $n \in E$ a point $x_n \in C_n - \kappa(B_\alpha)$. For each $n \in E$ choose a compact zero set Z_n such that $x_n \in Z_n \subset C_n$ and an Urysohn mapping $f_n\colon X \to I$ such that $f_n^{-1}(1) = Z_n$ and $f_n(X - C_n) = 0$. Then the mapping $f\colon X \to I$ defined by $f = \sum_{n \in E} f_n$ is easily seen to be continuous. In addition $f^{-1}(1) = \bigcup_{n \in E} Z_n$ which implies thet $\bigcup_{n \in E} Z_n$ is a zero set of X. This shown that the sets $\bigcap_{n \in A_\alpha} (X - C_n)$ and $\bigcup_{n \in E} Z_n$ have disjoint closures in βX. We conclude that all cluster points of $\{x_n | n \in E\}$ in βX are elements of B_α. Since $\{x_n | n \in E\}$ is not compact there is such a point, say p. Then $\kappa(B_\alpha)$ is a neighborhood of p which misses $\{x_n | n \in E\}$. This is a contradiction.

Now for every $\alpha < 2^\omega$ we can choose an infinite $D_\alpha \subset A_\alpha$ such that

$$\bigcup_{n \in D_\alpha} C_n \subset \kappa(B_\alpha).$$

For each $n \in N$ let $L_n = \{\alpha < 2^\omega | n \in D_\alpha\}$. Since $|N| = \omega$ and $\mathrm{cf}(2^\omega) \geqslant \omega_1$ (this is well known, see for instance Juhász [6]) we conclude that one of the L_n's must have cardinality 2^ω. This completes the proof, since the B_α's are clearly pairwise disjoint.

2.2. Example. A noncompact Lindelöf space Z such that there is a K_0^*-function $\kappa\colon \tau(\beta Z - Z) \to \tau(\beta Z)$.

Let X be the absolute of the closed unit segment $I = [0, 1]$ (in the sense of Iliadis [5]). Let $\pi\colon X \to I$ be the natural projection of X onto I (this mapping sends each open ultrafilter onto its limit). Let

$$Z = \{x \in X | \pi(x) \text{ is rational}\},$$

and

$$T=\{x \in X \mid \pi(x) \text{ is irrational}\}.$$

Since π is irreducible (cf. Iliadis [5]) both Z and T are dense in X. In addition, it is clear that both Z and T are Lindelöf and hence normal. Also $\beta Z = X = \beta T$ since both Z and T are dense in the extremally disconnected compact space X. Hence $\beta Z - Z = T$ and since T is normal and since $\beta T = X$ we see that the function $\kappa: \tau(T) \to \tau(X)$ defined by

$$\kappa(U) = X - \mathrm{cl}_X(T - U)$$

is a K_0^*-function.

It must be noticed that theorem 1.1 implies that there is no K_0^*-function $\kappa: \tau \times (\beta Q - Q) \to \tau(\beta Q)$, where Q denotes the space of the rationals. The space Z of example 2.2 can be mapped by an irreducible perfect mapping onto Q. This observation leads to the following corollary.

2.3. COROLLARY. *For a space X the property of having a K_0^*-function $\kappa: \tau(\beta X - X) \to \tau(\beta X)$ is not preserved under perfect irreducible surjections.*

I am indebted to Evert Wattel for some helpful comments during the preparation of this note.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706 (USA)

REFERENCES

[1] W. W. Comfort, *Retractions and other continuous maps from βX onto $\beta X - X$*, Trans Amer. Math. Soc., **114** (1965), 1–9.

[2] E. K. van Douwen, *Simultaneous linear extension of continuous functions*, Gen. Top. Appl., **5** (1975), 297–319.

[3] —, *Retractions from βX onto $\beta X - X$* [to appear].

[4] L. Gillman, M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

[5] S. Iliadis, *Absolutes of Hausdorff spaces*, Soviet Math. Dokl., **4** (1963), 295–298.

[6] I. Juhász, *Cardinal functions in topology*, Mathematical Centre Tract 34, Amsterdam, 1971.

Ян Фан Милл, **Продолжители из βX-X в βX**

Содержание. Пусть $\tau(X)$ обозначает топологию пространства X. Для $X \subset Y$ функция $\kappa: \tau(X) \to \tau(Y)$ называется продолжителем, если $\kappa(U) \cap X = U$ для всех $U \in \tau(X)$. Доказывается, что для каждого локально бикомпактного не псевдокомпактного пространства X и для каждого продолжителя $\kappa: \tau(\beta X - X) \to \tau(\beta X)$ существует система $\mathscr{B}$, состоящая из 2^{ω} непересекающихся непустых открытых множеств в $\beta X - X$ таких, что $\bigcap_{B \in \mathscr{B}} \kappa(B) \neq \emptyset$. В частности, не существует ретракции из βX на $\beta X - X$.