The superextension of the closed unit interval is homeomorphic to the Hilbert cube

by

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Abstract. Let $X$ be a compact metric space and let $\mathcal{I} X$ be the superextension of $X$. For the closed unit interval $I$ we show that $\mathcal{I} I$ is homeomorphic to the Hilbert cube, thus answering a question of J. de Groot.

1. Introduction. One of the unsolved problems in the theory of superextensions is to determine the superextension of the closed unit interval $I$. De Groot [13], conjectured that $\mathcal{I} I$ is homeomorphic to the Hilbert cube. This paper contains a proof of this conjecture. Infinite dimensional techniques are very important in this work. We will represent $I$ as an inverse limit of a sequence of Hilbert cubes, such that the bonding maps are near-homeomorphisms. An approximation theorem for inverse limits of Brown ([16]) then is applicable, which gives us the desired result. The class of Hilbert cube factors, a subclass of the compact metric absolute retracts, has been investigated by several authors during the last years ([11], [23], [24], [25], [26]). Several of the common types of absolute retracts, have been shown to be Hilbert cube factors, e.g., contractible polyhedra [23], dendra [23], contractible cell complexes [24], and hyperspaces ([11], [19], [25]). These results and Chapman's results concerning $Q$-manifolds ([7], [8], [9]) will be of great importance for us. This paper is organized as follows: the second section recalls the definitions of supercompactness and superextensions and contains some theorems which have interest in their own rights and which will be, in the fourth section, the tools in proving our main result. The third section contains a proof that the Hilbert cube is a superextension of $I$, relative a specially chosen nice subbase. This result we need as a first step in our converse limit construction.

The author wishes to thank P. C. Baayen and A. Verbeek for their encouragement and helpful discussions.

2. Superextensions. In [13], De Groot defined a space $X$ to be supercompact provided that it possesses an open subbase $\mathcal{U}$ such that each covering of $X$ by elements of $\mathcal{U}$ contains a subcover of two elements of $\mathcal{U}$. Such a subbase is called binary. Clearly, according to the lemma of Alexander, every supercompact space is

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compact. The class of supercompact spaces contains the compact metric spaces (Strok and Szymański [20]), compact orderable spaces and compact tree-like spaces (Brouwer and Schrijver [5] or Van Mill [16]). De Groot conjectured that every compact Hausdorff space is supercompact. This was answered in the negative by Bell [4], who showed that if $X$ is not pseudocompact then $\beta X$ is not supercompact. Moreover there exists a compact separable first countable Hausdorff space which is not supercompact (van Douwen and van Mill [12]).

Let $X$ be a topological space and let $\mathcal{S}$ be a subbase for the closed subsets of $X$. $\mathcal{S}$ is defined to be:

(i) a $T_1$-subbase if for each $x_0 \in X$ and $S \in \mathcal{S}$ with $x_0 \notin S$ there exists a $T \in \mathcal{S}$ with $x_0 \in T$ and $T \cap S = \emptyset$.

(ii) a normal subbase if for each $S_0, T_0 \in \mathcal{S}$ with $S_0 \cap T_0 = \emptyset$ there exist $S_1, T_1 \in \mathcal{S}$ such that $S_0 \subseteq S_1$ and $S_1 \cap T_1 = \emptyset$.

(iii) a supernormal subbase if $\mathcal{S}$ is normal while moreover for all $S \in \mathcal{S}$ and $G \subseteq X$ with $S \cap G = \emptyset$ there exists an $S_0 \in \mathcal{S}$ such that $G \subseteq S_0$ and $S \cap S_0 = \emptyset$.

$\mathcal{S}$ is called binary if the corresponding open subbase $\mathcal{U} = \{X \setminus S | S \in \mathcal{S}\}$ is binary. A subset $A \subseteq X$ is called a linked system (ls), if every two of its members meet. A linked system $\mathcal{M} \subseteq \mathcal{S}$ is called fixed if $\bigcap \mathcal{M} \neq \emptyset$ and is called free if $\bigcap \mathcal{M} = \emptyset$. If $\mathcal{S}$ is binary then any linked system $\mathcal{M} \subseteq \mathcal{S}$ is fixed (and conversely).

A maximal linked system or mls (in $\mathcal{S}$) is a linked system not properly contained in any other linked system. By Zorn’s lemma every linked system is contained in at least one maximal linked system. The proofs of the following propositions and the proof of Theorem 1 can be found in [21].

**Proposition 1.** Let $\mathcal{M}_0, \mathcal{M}_1$ be mls’s in $\mathcal{S}$. Then:

(a) $\emptyset \notin \mathcal{M}_0$.

(b) If $S \in \mathcal{M}_0$, $T \in \mathcal{S}$ and $S \subseteq T$, then $T \in \mathcal{M}_0$.

(c) If $S \in \mathcal{S}$ then $\exists T \in \mathcal{M}_0$: $S \subseteq T \in \mathcal{S}$.

(d) $\mathcal{M}_0 \neq \mathcal{M}_1$ if $\exists S \in \mathcal{M}_0, \exists T \in \mathcal{S}$: $S \subseteq T \in \mathcal{S}$.

(e) If $S, T \in \mathcal{S}$ and $S \subseteq T$ then $S \subseteq \mathcal{M}_0$ or $T \in \mathcal{M}_0$.

Notation. $\lambda_\mathcal{S}(X) = \{S \in \mathcal{S} | \text{S is a mls in } \mathcal{S}\}$.

If $\mathcal{S}$ is a $T_1$-subbase then for each $x \in X$ the linked system $\mathcal{M}_x = \{S \in \mathcal{S} | x \in S\}$ is also maximal linked; the map

$$i: X \to \lambda_\mathcal{S}(X)$$

defined by $i(x) = \mathcal{M}_x$ is $1$-$1$. If $A$ is a subset of $X$ then we define

$$A^+ = \{\mathcal{M} \in \lambda_\mathcal{S}(X) \mid \exists S \in A: S \subseteq A\}.$$  

**Proposition 2.** (i) If $A \subseteq B \subseteq X$ then $A^+ \subseteq B^+$.

(ii) If $A, B \subseteq X$ and $A \cap B = \emptyset$ then $A^+ \cap B^+ = \emptyset$.

(iii) If $S, T \in \mathcal{S}$ then $S \cap T = \emptyset$ if $S^+ \cap T^+ = \emptyset$.

(iv) If $S, T \in \mathcal{S}$ then $S \cup T = X$ if $S^+ \cup T^+ = \lambda_\mathcal{S}(X)$.

(v) If $S \in \mathcal{S}$ then $S^+ \cup (X \setminus S)^+ = \lambda_\mathcal{S}(X)$.

As a closed subbase for a topology on $\lambda_\mathcal{S}(X)$ we take

$$\mathcal{S}^+ = \{S^+ | S \in \mathcal{S}\}.$$  

With this topology $\lambda_\mathcal{S}(X)$ is called the superextension of $X$ relative the subbase $\mathcal{S}$.

In case $\mathcal{S}$ consists of all the closed subsets of $X$, $\lambda_\mathcal{S}(X)$ is denoted by $\lambda X$ and is called the superextension of $X$.

**Theorem 1.** (i) $X$ is embeddable in $\lambda_\mathcal{S}(X)$ if $\mathcal{S}$ is a $T_1$-subbase.

(ii) $\lambda_\mathcal{S}(X)$ is $T_1$.

(iii) $\lambda_\mathcal{S}(X)$ is $T_2$ if $\mathcal{S}$ is a normal subbase.

(iv) $\lambda_\mathcal{S}(X)$ is supercompact. A binary subbase is $\{(X \setminus S)^+ | S \in \mathcal{S}\}$.

(v) $\forall S \in \mathcal{S}: \operatorname{Int}(\operatorname{Int}(S^+)) = S$.

In case $\mathcal{S}$ is a topological embedding, we often identify $X$ and $\{X\}$. An interval structure $\mathcal{I}(X)$ on a topological space $X$ is a function $I: X \times X \to \lambda_\mathcal{S}(X)$ such that:

(i) $I(x,y) \in I(x,y)$, $(x,y) \in X$,

(ii) $I(x,y) = I(y,x)$, $(x,y) \in X$,

(iii) if $u, v \in I(x,y)$ then $I(u,v) \subseteq I(x,y)$, $(u,v,xy) \in X$,

(iv) $I(x,y) \cap I(z,x) \cap I(y,z) = \emptyset$, $(x,y,z) \in X$.

A subset $A \subseteq X$ is called $I$-closed if for all $x, y \in A$ it follows that $I(x,y) \subseteq A$.

If $X$ is a supercompact space with binary closed subbase $\mathcal{S}$, then $I_{\mathcal{S}}: X \times X \to \lambda_\mathcal{S}(X)$ defined by

$$I_{\mathcal{S}}(x,y) = \{S \in \mathcal{S} | x, y \in S\}$$

defines an interval structure on $X$. Furthermore it is clear that each $S \in \mathcal{S}$ is $I_{\mathcal{S}}$-closed.

The converse is also true. If a compact space $X$ possesses an interval structure $I$ and a closed subbase $\mathcal{S}$ consisting of $I$-closed sets, then $X$ is supercompact ([15], Theorem 1.1). In particular, $\mathcal{S}$ is a binary closed subbase for $X$.

**Theorem 2.** If $\mathcal{S}$ is a binary normal closed subbase for $X$, then $I_{\mathcal{S}}(x,y) \cap I_{\mathcal{S}}(x,z) \cap I_{\mathcal{S}}(y,z)$ is a singleton for each $x, y, z \in X$.

**Proof.** Choose $x, y, z \in X$ and let $p, q \in I_{\mathcal{S}}(x,y) \cap I_{\mathcal{S}}(x,z) \cap I_{\mathcal{S}}(y,z)$, with $p \neq q$. As $\mathcal{S}$ is a binary normal closed subbase, it is a normal $T_1$-closed subbase ([16], Lemma 1) and therefore there exist $S_0, S_1 \in \mathcal{S}$ such that $p \in S_0 \setminus S_1$ and $q \in S_1 \setminus S_0$ and $S_0 \cup S_1 = X$. We have to consider two cases:

(i) Suppose first that $x \in S_0$. We again distinguish two subcases:

(a) $y \in S_0$. Then $I_{\mathcal{S}}(x,y) \subseteq S_0$ and consequently $q \in S_0$, which is a contradiction.

(b) $y \in S_1$. If $z \in S_0$, then we can derive the same contradiction as in (a). If $x \in S_1$, then $I_{\mathcal{S}}(y,z) \subseteq S_1$ and consequently $p \in S_1$, which is a contradiction.

(ii) Suppose that $x \in S_1$. This can be treated in the same way as case (i).
Lemma 1. If \( S \) is a binary normal closed subbase for \( X \), then the map \( f: X \times X \times X \to X \) defined by
\[
\{ (x, y, z) \} = I_{P}(x, y) \cap I_{P}(x, z) \cap I_{P}(y, z)
\]
is a continuous surjection.

Proof. As a first step we will prove that \((x, y, z) \notin f^{-1}[S] \iff I_{P}(x, y) \cap S = \emptyset \) or \( I_{P}(x, z) \cap S = \emptyset \) or \( I_{P}(y, z) \cap S = \emptyset \) (\( S \in S' \)).

If \( S \neq \emptyset \) and \( I_{P}(x, y) \cap S \neq \emptyset \) and \( I_{P}(x, z) \cap S \neq \emptyset \) and \( I_{P}(y, z) \cap S \neq \emptyset \).

Then \( \nabla = \{ (x, y) \in S' \mid x \neq y \} \) and \( (x, y) \in S \) is open and \( S \cap S \) is a linked system and consequently, since \( S' \) is binary, \( \cap \nabla \neq \emptyset \). As \( \cap \nabla = I_{P}(x, y) \cap I_{P}(x, z) \cap I_{P}(y, z) \cap S = \emptyset \) it would follow that \( f(x, y, z) \in S \), which is a contradiction

If \( I_{P}(x, y) \cap S = \emptyset \), then \( I_{P}(x, y) \cap I_{P}(x, z) \cap I_{P}(y, z) \cap S = \emptyset \) and so \( f(x, y, z) \notin S \).

From Theorem 1 it follows that \( f \) is well-defined. To prove that \( f \) is continuous, choose \( S \in S' \) and let \((x, y, z) \notin f^{-1}[S] \). Without loss of generality we may assume that \( I_{P}(x, y) \cap S = \emptyset \). Using the fact that \( S' \) is binary and that \( I_{P}(x, y) \) is an intersection of subbase elements it follows that there exists an \( S_{0} \in S' \) such that \( I_{P}(x, y) \subseteq S_{0} \) and \( S_{0} \cap S = \emptyset \). The normality of \( S' \) implies the existence of \( S_{1}, S_{2} \subseteq S' \) such that \( S_{0} \subseteq S_{1} \subseteq S_{2} \subseteq S \) and \( S_{0} \cap S_{2} = \emptyset \) and \( S_{2} \subseteq X \). Then \( x, y \in S_{2} \subseteq X \). Define \( U = X \setminus S_{1} \). Then \( \Pi_{1}(x, y) \subseteq \Pi_{1}(U) \cap \Pi_{1}(U)^{-1}[U] \). Furthermore \( \Pi_{1}(U) \cap \Pi_{1}(U)^{-1}[U] \cap f^{-1}[S] = \emptyset \), for suppose to the contrary that there exists a point \( (x_{0}, y_{0}, z_{0}) \in \Pi_{1}(U) \cap \Pi_{1}(U)^{-1}[U] \cap f^{-1}[S] \). Then \( x_{0} \in U \) and \( y_{0} \in U \) and consequently \( I_{P}(x_{0}, y_{0}) \subseteq S \). Hence it follows that \( I_{P}(x_{0}, y_{0}) \cap S = \emptyset \) and consequently \( (x_{0}, y_{0}, z_{0}) \notin f^{-1}[S] \), which is a contradiction. Therefore \( f \) is continuous. To prove that \( f \) is onto, choose \( x \in X \). Then
\[
\{ (x, x, x) \} = I_{P}(x, y) \cap I_{P}(x, z) \cap I_{P}(y, z) = \{ x \}.
\]

Lemma 2. \( g = f \mid X \times \{ y \} \times X \) is a retraction of \( X \) onto \( I_{P}(x, y) \).

Proof. \( g \) is continuous, and furthermore it is clear that
\[
g(\{ x \} \times \{ y \} \times X) = I_{P}(x, y).
\]
Choose \( z \in I_{P}(x, y) \). Then
\[
\{ z \} = \{ f(x, y, z) \} = I_{P}(x, y) \cap I_{P}(x, z) \cap I_{P}(y, z) = \{ x \},
\]
since \( z \in I_{P}(x, y) \) (Theorem 2). This proves that \( g \) is a retraction.

Corollary 1. If \( S' \) is a binary normal closed subbase for the topological space \( X \), then the following properties are equivalent:
(i) \( X \) is connected.
(ii) \( X \times Y \times X \) is connected.
(iii) Each intersection of elements of \( S' \) either is void or is connected.

Proof. (i) => (ii) This is a consequence of Lemma 2.
(ii) => (i) Suppose that \( X \) is not connected. Then there exist open non-empty \( U \) and \( V \) in \( X \) such that \( U \cap V = X \) and \( U \cap V = \emptyset \). Choose \( x \in U \) and \( y \in V \). Then \( I_{P}(x, y) \) is not connected, which is a contradiction.

(iii) => (i) Obvious.

(iii) => (ii) Let \( A \) be a subsystem of \( S' \) such that \( \bigcup A \neq \emptyset \) and \( \bigcap A \neq \emptyset \). Choose \( x, y \in A \) and open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( \bigcup A \subseteq U \cap V \) and \( U \cap V \subseteq \bigcap A \). Then for each \( A \in A \) the interval \( I_{P}(x, y) \) is contained in \( A \) and consequently \( I_{P}(x, y) = \bigcap A \). This is a contradiction.

A mean \( m \) is a continuous map \( m: X \times X \to X \) such that
(i) \( m(x, x) = x \) for all \( x \in X \).
(ii) \( m(x, y) = m(y, x) \), for all \( x, y \in X \).

Theorem 3. Any topological space which possesses a binary normal closed subbase, also has a mean.

Proof. Let \( S' \) be a binary normal closed subbase for the topological space \( X \). Let \( f \) be defined as in Lemma 1. Choose \( p \in X \) and define \( m: X \times X \to X \) by \( m = f \mid \{ p \} \times X \). Then \( m \) is a continuous map of \( X \times X \) onto \( X \). Furthermore
\[
m(x, y) = I_{P}(x, y) \cap I_{P}(x, p) \cap I_{P}(y, x) = \{ x \}
\]
and
\[
m(x, y) = I_{P}(x, y) \cap I_{P}(x, p) \cap I_{P}(y, p) = I_{P}(x, y) \cap I_{P}(y, p) \cap I_{P}(x, p) = \{ m(x, y) \}.
\]

Of course there are many spaces which possess a binary normal closed subbase. Examples are products of compact orderable spaces, products of compact tree-like spaces ([16]) and superextensions of normal spaces. Theorem 3 gives us many easy examples of spaces which are supercompact, but which do not possess a binary normal closed subbase. For example the supercompact space
\[
Y = \{ (0, y) \mid -1 \leq y \leq 1 \} \cup \{ (x, \sin(1/x)) \mid 0 < x \leq 1 \}
\]
possesses no binary normal closed subbase, since this space has no mean ([33]). That \( Y \) is supercompact is not trivial. To prove this, define for each \( n \in \{ 0, 1, 2, \ldots \} \)
\[
x_{n} = \frac{2}{(2n+1)\pi}.
\]
Notice that \( \sin(1/x_{n}) = 1 \) if \( n \) is even and that \( \sin(1/x_{n}) = -1 \) if \( n \) is odd. Let \( r \) be a retraction of \( Y \) onto \( \{ 0 \} \times [-1, 1] \) defined by
\[
r(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in \{ 0 \} \times [-1, 1], \\ (0, y) & \text{if } (x, y) \notin \{ 0 \} \times [-1, 1].
\end{cases}
\]
It can be shown that

\[ \{ r^{-1}([0, x, 1]) \cup C \mid -1 \leq \varepsilon \leq 1 \text{ and } C \text{ is a component of } r^{-1}([0, x, 1]) \cup r^{-1}([0, -1, x]) \cup C \text{ is a component of } \{ r^{-1}([0, -1, x]) \cup \{ (x, \sin(1/3))/x \in \{ x, p \}, \text{ where } x \leq p \leq x_{n-1} ; n \in [0, 1, 2] \} \} \]

is a binary closed subbase for \( Y \). Moreover it is obvious that this subbase is not normal. That \( Y \) possesses no binary normal closed subbase can also be derived from a rather deep theorem of Verbeek [21]: if a connected space possesses a binary normal closed subbase then this space must be locally connected, as shown in Corollary 211, III. 4.1 Corollary]. Clearly \( Y \) is not locally connected and therefore the result also follows. However, this argument cannot be used in the class of connected and locally connected spaces. Then our theorem applies. It is well-known for example that the \( n \)-spheres \( S^n \) are supercompact, but do not have a mean ([22]) and consequently they cannot possess a binary normal closed subbase. It is unknown whether there exists a contractible locally connected example. We will prove that in the class of metric spaces each continuum which possesses a binary normal closed subbase must be an AR (absolute retract), a theorem which has wide applications.

For a compact metric space \( X \), let \( 2^X \) be the space of all nonempty closed subsets of \( X \) with the Vietoris topology, i.e., the topology induced by the Hausdorff metric. This space is called the hyperspace of \( X \). A basis for the open sets consists of all sets

\[ \langle O_1, O_2, \ldots, O_n \rangle = \{ G \in 2^X \mid G \subseteq \bigcup_{i=1}^n O_i \text{ and } G \cap O_i \neq \emptyset \text{ for } i = 1, 2, \ldots, n \} \]

where \( O_1, O_2, \ldots, O_n \) denotes an arbitrary finite sequence of open subsets of \( X \). For most strong results concerning hyperspaces see [11], [19] and [25].

**Lemma 3.** Let \( p \in X \) and let \( G \) be a closed subset of \( X \). Then \( h(G) = \{ A \subseteq X \mid A = G \cap A \text{ if } A \in G \text{ or } A \in A \text{ and } G \cap A \neq \emptyset \} \) is an mls.

**Proof.** Verbeek [21], I. 1.3(c). \( \square \)

**Theorem 4.** Let \( X \) be a connected metric space, which possesses a binary normal closed subbase. Then \( X \) is an AR.

**Proof.** Fix a point \( p \in X \) and define a map \( h: 2^X \to X \) by \( h(G) = h(G) \) where \( h(G) \) is defined in Lemma 3. We will prove that \( h \) is continuous. Let \( O \) be an open set in \( X \) and let \( G \subseteq h^{-1}(O) \).

Case 1. \( p \in O \). Define \( U = \langle O, X \rangle = \{ H \in 2^X \mid H \cap O \neq \emptyset \} \). As \( G \subseteq h^{-1}(O^c) \), it follows that \( h(G) \in O^c \) and consequently \( G \cap O \neq \emptyset \), since \( G \subseteq h(G) \). Therefore \( G \subseteq U \).

Choose \( H \subseteq U \). Then \( H \cap O \neq \emptyset \) and so there exists a \( q \in H \cap O \). Since \( p \in O \), it follows that \( \{ p, q \} \subseteq O \). However it is clear that \( \{ p, q \} \subseteq h(H) \), and consequently \( H \subseteq h^{-1}(O^*) \). Therefore \( U \) is an open neighborhood of \( G \) which is contained in \( h^{-1}(O^*) \).

Case 2. \( p \notin O \). Define \( U = \langle O \rangle = \{ H \in 2^X \mid H \cap O \neq \emptyset \} \). Choose \( H \subseteq U \). Then \( H \subseteq O \) and consequently \( G \subseteq h^{-1}(O^*) \). Conversely, if \( G \subseteq h^{-1}(O^*) \), then \( H \subseteq O \), since \( p \notin O \). The combination of these two results yields \( h^{-1}(O^*) \subseteq U \), and therefore \( h^{-1}(O^*) \) is an open neighborhood of \( G \).

Now, since \( X \) possesses a binary normal closed subbase \( \mathcal{F} \) there is a retraction \( r: \lambda \to X \) defined by \( \{ r(\mathcal{F}) \} = \{ S \subseteq \mathcal{F} \mid S \in \mathcal{F} \} \) (Verbeek [21], II. 4.5) and consequently the map \( \xi: 2^X \to X \) defined by \( \xi = r \circ h \) is a continuous surjection. We will show that \( \xi \) also is a retraction. Choose \( x \in X \). Then

\[ \xi(x) = r(h(x)) = r(\{ A \subseteq X \mid A \text{ is closed and } x \in A \text{ or } p \in A \text{ and } x \in A \} = r(\{ A \subseteq X \mid A \text{ is closed and } x \in A \} = r(x) = x, \]

since \( r \) is a retraction. Now the connectedness of \( X \) implies that \( X \) is a Peano continuum (Verbeek [21], III. 4.1 Corollary) and consequently \( 2^X \) is homeomorphic to the Hilbert cube \( 2^X \) (Curtis and Schori [11]). It now follows that \( X \) is an AR. \( \square \)

The above theorem may seem to be a deep theorem, since we use the Curtis and Schori result: \( 2^X \neq X \) is a Peano continuum. However we only need that \( 2^X \) is an AR iff \( X \) is a Peano continuum, since a retract of an AR is again an AR, and this was proved by Wojdyslawski [27] in 1939. The superextension \( \lambda \) of any normal space \( X \) possesses a binary normal closed subbase and therefore we immediately obtain the following corollary:

**Corollary 2.** The superextension \( \lambda M \) of a strongly infinite dimensional AR is a non degenerate metrizable continuum.

**Proof.** This immediately follows from Verbeek's theorem [21], IV. 2.6: \( \lambda M \) is a strongly infinite dimensional Peano continuum iff \( M \) is a non degenerate metrizable continuum. \( \square \)

This answers a question of Verbeek raised in [21].

**Question 1.** Is the converse of Theorem 4 also true?)

A counter example to this question cannot be obtained within the class of one dimensional AR's, since this class consists of dendrites, which possess binary normal subbases ([18]).

A surprising consequence of Theorem 4, for which no direct proof is known, is that the superextension of any metrizable continuum is contractible.

We will now derive some results concerning supernormal subbases.

**Lemma 4.** Let \( \mathcal{F} \) be a closed supernormal \( T_1 \)-subbase for \( X \) and let \( \mathcal{U} \) be a closed \( T_1 \)-subbase such that \( \mathcal{F} \subseteq \mathcal{U} \). Then \( \forall \mathcal{A} \in \lambda \mathcal{U}(X) = \{ S \subseteq \mathcal{F} \} \subseteq \mathcal{F} \) is an mls in \( \mathcal{F} \).

(0) This was answered in the negative recently by A. Szymański.
Proof. Let \( \mathcal{M} \in \lambda_{\mathcal{P}}(X) \) and define \( P_{\mathcal{M}} = \{ \mathcal{S} \in \mathcal{P} | \mathcal{S} \subseteq \mathcal{M} \} \). From the normality of \( \mathcal{P} \) it follows that \( P_{\mathcal{M}} \neq \emptyset \), and therefore \( P_{\mathcal{M}} \) is a linked system. Suppose that \( P_{\mathcal{M}} \) is not maximal linked. Then there exists an \( \mathcal{S}_0 \in \mathcal{P} \) such that \( P_{\mathcal{M}} \cup \{ \mathcal{S}_0 \} \) is linked and \( \mathcal{S}_0 \notin P_{\mathcal{M}} \). Then \( \mathcal{S}_0 \notin \mathcal{P} \) and there exists an \( \mathcal{M} \in \mathcal{P} \) with \( \mathcal{M} \cap \mathcal{S}_0 = \emptyset \). Since \( \mathcal{P} \) is supernormal there is an \( \mathcal{S}^* \in \mathcal{P} \) with \( \mathcal{M} \subseteq \mathcal{S}^* \) and \( \mathcal{S}^* \cap \mathcal{S}_0 = \emptyset \). This is a contradiction, since \( \mathcal{M} \in \mathcal{P} \) implies \( \mathcal{S}^* \in \mathcal{P} \) and therefore \( \mathcal{S}^* \in P_{\mathcal{M}} \). \( \blacksquare \)

THEOREM 5 (G. A. Jensen; cf. [14]). Let \( \mathcal{G} \) be a \( T_1 \)-subbase for \( X \), let \( \mathcal{P} \) be a normal \( T_1 \)-subbase for \( Y \) and let \( f \) be a continuous map \( f: X \rightarrow Y \) such that \( \forall \mathcal{T} \in \mathcal{P} : f^{-1}(\mathcal{T}) \in \mathcal{G} \). Then \( f \) can be extended to a continuous map \( \overline{f}: \lambda_{\mathcal{P}}(X) \rightarrow \lambda_{\mathcal{G}}(Y) \). Moreover, if \( f \) is onto then \( \overline{f} \) is onto. If \( f \) is 1-1 and \( \forall \mathcal{S} \in \mathcal{P} : f(\mathcal{S}) \in \mathcal{G} \) then \( \overline{f} \) is an embedding.

The construction of the map \( \overline{f} \) is very simple. If \( \mathcal{M} \in \lambda_{\mathcal{P}}(X) \), then

\[ P_{\mathcal{M}} = \{ \mathcal{T} \in \mathcal{P} | f^{-1}(\mathcal{T}) \in \mathcal{M} \} \]

is contained in precisely one \( \mathcal{S} \) in \( \mathcal{G} \). This \( \mathcal{S} \) is denoted by \( P_{\mathcal{M}} \) and the map \( \overline{f} \) is defined by \( \overline{f}(\mathcal{M}) = P_{\mathcal{M}} \). These mappings will be called Jensen mappings.

For another solution of the extension problem see [17].

COROLLARY 3. Let \( \mathcal{G} \) be a closed supernormal \( T_1 \)-subbase for \( X \) and let \( \mathcal{P} \) be a closed \( T_1 \)-subbase such that \( \mathcal{P} \subseteq \mathcal{G} \). Then \( \lambda_{\mathcal{P}}(X) \) is a Hausdorff quotient of \( \lambda_{\mathcal{G}}(X) \) under the map \( f \) defined by \( f(\mathcal{M}) = (\{ \mathcal{S} \in \mathcal{P} | \mathcal{S} \subseteq \mathcal{M} \}) \). Moreover, \( f \) is the identity on \( X \).

The definition of subbases which are supernormal seems to be pathological, since in compactification theory a closed subbase almost always fails to have this property. In our construction for \( \lambda \) however, subbases which are supernormal appear in a natural way. Therefore it is worth the trouble to study some elementary properties of these subbases first.

PROPOSITION 3. Let \( \{ \mathcal{G}_i \}_{i \in I} \) be a collection of closed \( T_1 \)-subbases for the topological space \( X \), which all are supernormal. Then \( \bigcup_{i \in I} \mathcal{G}_i \) is a closed \( T_1 \)-subbase which is supernormal. Moreover \( \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \) can be embedded in \( \prod_{i \in I} \lambda_{\mathcal{G}_i}(X) \).

Proof. That \( \bigcup_{i \in I} \mathcal{G}_i \) is a closed \( T_1 \)-subbase which is supernormal is obvious. To prove the embedding property, for all \( x \in X \) let

\[ f_x: \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \rightarrow \lambda_{\mathcal{G}_i}(X) \]

be the Jensen mapping. Note that these mappings exist. Let

\[ e: \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \rightarrow \prod_{i \in I} \lambda_{\mathcal{G}_i}(X) \]

be the evaluation map defined by \( (e(x))_i = f_x(x) \). We will show that \( e \) is an embedding and for this it suffices to show that \( e \) is one to one. Choose \( \mathcal{M}_0, \mathcal{M}_1 \in \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \) such that \( \mathcal{M}_0 \neq \mathcal{M}_1 \). Then there exist \( M_i \in \mathcal{G}_i \) \( (i = 1, 0) \) such that \( M_i \in \mathcal{M}_i \) \( (i = 0, 1) \) and \( M_0 \cap M_1 = \emptyset \). Choose \( x_0 \in I \) such that \( M_0 \in \mathcal{G}_{x_0} \). Then, since \( \mathcal{G}_{x_0} \) is supernormal, we may assume without loss of generality that \( M_1 \in \mathcal{G}_{x_0} \). However, Corollary 3 shows that \( M_1 \in f_{x_0}(\mathcal{M}_{x_0}) \) \( (i = 0, 1) \) and consequently \( f_{x_0}(\mathcal{M}_{x_0}) \neq f_{x_0}(\mathcal{M}_{x_0}) \). \( \blacksquare \)

If the conditions of Proposition 3 are satisfied, then we will always identify \( \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \) and \( e(\lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X)) \). It then is useful to characterize those points of \( \prod_{i \in I} \lambda_{\mathcal{G}_i}(X) \) which belong to \( \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \). Note that a point \( x = (x_i)_{i \in I} \) of \( \prod_{i \in I} \lambda_{\mathcal{G}_i}(X) \) is a point of which the coordinates are maximal linked systems so that we can speak of \( \bigcup_{i \in I} x_i \).

LEMMA 5. Let \( \{ \mathcal{G}_i : i \in I \} \) be a collection of closed \( T_1 \)-subbases for the topological space \( X \), which all are supernormal. Then \( x = \prod_{i \in I} \lambda_{\mathcal{G}_i}(X) \) is an element of \( \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \) if and only if \( \bigcup_{i \in I} x_i \) is linked.

Proof. If \( x = \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \), then \( x = \bigcup_{i \in I} x_i \), so that \( \bigcup_{i \in I} x_i \) is linked. If \( \bigcup_{i \in I} x_i \) is linked, then it also is maximal linked (in \( \mathcal{G}_i \)) for suppose to the contrary that there exists an \( \mathcal{S} \in \mathcal{G}_i \) such that \( \bigcup_{i \in I} x_i \cap \{ \mathcal{S} \} \) is linked, but \( \mathcal{S} \notin \bigcup_{i \in I} x_i \). Choose \( x_0 \in I \) such that \( \mathcal{S} \in \mathcal{G}_{x_0} \). It then follows that \( x_{x_0} \cap \{ \mathcal{S} \} \) is linked and consequently, since \( x_{x_0} \) is a maximal linked system, \( \mathcal{S} \in x_{x_0} \cap \bigcup_{i \in I} x_i \). This is a contradiction. Hence \( \bigcup_{i \in I} x_i \subseteq \lambda_{\bigcup_{i \in I} \mathcal{G}_i}(X) \) and now it is not hard to see that \( e(\bigcup_{i \in I} x_i) = x \). \( \blacksquare \)

The importance of Proposition 3 and Lemma 5 is that one can study the behaviour of a superextension relative the union of certain subbases, in a product of superextensions. We will demonstrate this by an example. Let \( I \) denote the real number interval \([0, 1]\) and let \( I \) be embedded in \( \mathbb{R}^2 \) as indicated in Figure 1. Define

\[ \mathcal{G} = \{ \mathcal{A} \subseteq \mathbb{R}^2 : \mathcal{A} = \Pi_{i=1}^3 [0, x] \} \cup \{ \mathcal{A} = \Pi_{i=1}^3 [x, 1] : x \in I \} \]
Then $\mathcal{F}$ is a binary normal closed subbase for $I^2$. We are interested in $\lambda_{\mathcal{F}}(I)$ where $\mathcal{F}$ is defined by

$$\mathcal{F} = \{ T \cap I \mid T \in \mathcal{F} \}.$$  

(Here $I$ denotes the embedded copy of $I$ in $I^2$.)

It is easy to see that $\mathcal{F}$ is a subbase which is supernormal. We assert that $\lambda_{\mathcal{F}}(I)$ is homeomorphic to the space $X$ indicated in Figure 2. To prove this define an interval structure $I_x$ on $X$ by

$$I_x(x, y) = (\bigcap \{ T \in \mathcal{F} \mid x, y \in T \}) \cap X.$$  

The verification that $I_x$ indeed is an interval structure is routine and follows immediately from Figure 2, since for all $x, y, z \in X$ we have

$$I_x(x, y) \cap I_x(y, z) \subseteq I_x(x, z).$$  

Consequently, each element of $\mathcal{F} \cap X = \{ T \cap X \mid T \in \mathcal{F} \}$ is $I_x$-closed and therefore $\mathcal{F} \cap X$ is a binary closed subbase for $X$. We now take recourse to the following theorem.

THEOREM 6 ([17]). Let $X$ be a subspace of the topological space $Y$. Then $Y$ is homeomorphic to a superextension of $X$ if $Y$ possesses a binary closed subbase $\mathcal{F}$ such that for all $T_0, T_1 \in \mathcal{F}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$.

In particular, under these conditions $Y \cong \lambda_{\mathcal{F}}(X)$. As an application of this theorem it follows that in the example under discussion $X \cong \lambda_{\mathcal{F}}(I)$, since for all $T_0, T_1 \in \mathcal{F}$ with $T_0 \cap T_1 \cap X \neq \emptyset$ we have that $T_0 \cap T_1 \cap I \neq \emptyset$, as can easily be seen. The homeomorphism between $X$ and $\lambda_{\mathcal{F}}(I) = \lambda_{\mathcal{F} \cap X}(I)$ is very "direct". For instance the point $p$ in Figure 3 represents the $\mathcal{F} \cap X$ mls $\mathcal{M}$ for which

$$[(0, e), [e, 1], [a, b] \cup [c, d], [0, a] \cup [b, c] \cup [d, 1]]$$

is a pre-mls ($\mathcal{M}$ is a linked system which is contained in precisely one mls).

Now, if one takes two different embeddings of $I$ in $I^2$ of the above type, then there arise two different superextensions $\lambda_{\mathcal{F}}(I)$ and $\lambda_{\mathcal{F}'}(I)$. What about $\lambda_{\mathcal{F} \cup \mathcal{F}'}(I)$?

Proposition 3 shows that $\lambda_{\mathcal{F} \cup \mathcal{F}'}(I)$ can be embedded in $\lambda_{\mathcal{F}}(I) \times \lambda_{\mathcal{F}'}(I)$, which is contained in $I^2 \times I^2$, so that in any case $\lambda_{\mathcal{F} \cup \mathcal{F}'}(I)$ is finite dimensional. The normality of $\mathcal{F} \cap \mathcal{F}'$ implies that $\lambda_{\mathcal{F} \cup \mathcal{F}'}(I)$ is connected (Verbeek [21], III, 4.1 Corollary) and consequently $\lambda_{\mathcal{F} \cup \mathcal{F}'}(I)$ is an AR (Theorem 4). Moreover Lemma 5 shows that the points of $I^2 \times I^2$ which belong to $\lambda_{\mathcal{F} \cup \mathcal{F}'}(I)$ are completely characterized in a simple way. We will see that there is much more to say about $\lambda_{\mathcal{F} \cup \mathcal{F}'}(I)$.

3. The Hilbert cube as a superextension of $I$. The Hilbert cube $Q$ is the topological product of infinitely many copies of $I$. A Hilbert cube is a topological space which is homeomorphic to $Q$. In [17] it was shown that $Q$ belongs to the class of superextensions of $I$, however this was not a satisfying result because we could not describe the defining subbase well. We will present another subbase $\mathcal{F}$ for $I$ such that $\lambda_{\mathcal{F}}(I) \cong Q$.

As in Section 2, let $\mathcal{F}$ be the canonical binary subbase for $I^2$,

$$\mathcal{F} = \{ A \subseteq I^2 \mid A = \bigcap_{i \in I} [0, x] \cup A = \bigcap_{i \in I} [x, 1] \cup \{ 0 \} \cup \{ 1 \}; x \in I \}.$$  

Define

$$E = \{-2^k \mid k = 0, 1, 2, \ldots\}$$

and for each $n \in E$ let $I$ be embedded in $I^2$, preserving arc-length, as indicated in Figure 4.

All angles are $\frac{1}{2} \pi$ except the one at $(0, 0)$ which is $\frac{1}{4} \pi$. Define $\mathcal{M}_n$ by

$$\mathcal{M}_n = \{ T \cap I \mid T \in \mathcal{F} \}.$$  

Then $\lambda_{\mathcal{M}_n}(I)$ is the convex-hull of the embedded copy of $I$ in $I^2$. We will show that $\lambda_{\mathcal{M}_n}(I)$ is homeomorphic to $Q$.

LEMMA 6. $\lambda_{\mathcal{M}_n}(I)$ is a convex subspace of $\lambda_{\mathcal{M}_n}(I)$.

Proof. Suppose that $\lambda_{\mathcal{M}_n}(I)$ is not a convex subspace of $\lambda_{\mathcal{M}_n}(I)$. Then there exist $x, y \in \lambda_{\mathcal{M}_n}(I)$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1, \alpha > 0, \beta > 0$ such that $\alpha x + \beta y \notin \lambda_{\mathcal{M}_n}(I)$. But then...
Since for all \( i \in E \) the point \( ax_i + \beta y_i \in \lambda_{\mathcal{E}}(I) \) it follows that \( \cup_{i \in E} (ax_i + \beta y_i) \) is not linked (Lemma 5). Note that we identify \( ax_i + \beta y_i \) and its mirror which is represented by \( ax_i + \beta y_i \). Now, there exist indices \( i_0 \) and \( i_1 \) such that \( (ax_{i_0} + \beta y_{i_0}) \cup (ax_{i_1} + \beta y_{i_1}) \) is not linked. Hence there exist an \( M \in ax_{i_0} + \beta y_{i_0} \) and an \( N \in ax_{i_1} + \beta y_{i_1} \) such that \( M \cap N = \emptyset \). If in the copy of \( I^2 \) corresponding to \( i_0 \) we draw a horizontal

line through \( x_{i_0} \) and determine its intersection \( p_0 \) with the embedded copy of \( I \), and we do the same in the copy of \( I^2 \) corresponding to \( i_1 \), then \( p_0 \) and \( p_1 \) are derived from the same point of \( I \); for it not, then it is easy to see that \( x_{i_0} \cup x_{i_1} \) is not linked. In the same way, straight horizontal lines through \( y_{i_0} \) and \( y_{i_1} \) must determine the same point on the embedded copies of \( I \) and consequently the same is true for horizontal lines through \( ax_{i_0} + \beta y_{i_0} \) and \( ax_{i_1} + \beta y_{i_1} \) because of the specially chosen embeddings of \( I \). Hence it follows that the situation drawn in Figure 5 is the only possibility (except for interchanging \( i_0 \) and \( i_1 \)).

![Figure 5](image)

---

**Remarks.** (i) \( M \) meets any set of the form \( \Pi_0^{-1}[\{x\}, I] \) with \( x \in \Pi_0(ax_{i_0} + \beta y_{i_0}) \) in point \( O \) of the embedded copy of \( I \).

(ii) \( N \) meets any set of the form \( \Pi_0^{-1}[\{x\}, I] \) with \( x \in \Pi_0(ax_{i_0} + \beta y_{i_0}) \) in the point \( \frac{1}{2} \) of the embedded copy of \( I \).

(iii) It is possible that an element of \( ax_{i_0} + \beta y_{i_0} \), containing \( M \), and an element of \( ax_{i_1} + \beta y_{i_1} \), containing \( N \), have a void intersection. In that case of course the sets \( M \) and \( N \) also have a void intersection.

(iv) In Figure 5 we have drawn the points \( x_{i_0}, y_{i_0}, x_{i_1}, y_{i_1} \) in such a way that \( \Pi_0 x_{i_0} < \Pi_0 y_{i_0} \) and \( \Pi_0 x_{i_1} < \Pi_0 y_{i_1} \). This is done because in the cases \( \Pi_0 x_{i_0} = \Pi_0 y_{i_0} \) or \( \Pi_0 x_{i_1} = \Pi_0 y_{i_1} \) or \( \Pi_0 x_{i_0} > \Pi_0 y_{i_0} \) or \( \Pi_0 x_{i_1} > \Pi_0 y_{i_1} \) it is easy to see that \( (ax_{i_0} + \beta y_{i_0}) \cup (ax_{i_1} + \beta y_{i_1}) \) is linked, as the reader can easily verify.

Without loss of generality we may assume that \( \Pi_0 y_{i_0} - \Pi_0 x_{i_1} \leq \Pi_0 y_{i_0} - \Pi_0 x_{i_0} \). It then follows that

\[
\Pi_0^{-1}[\{x\}, I] \cap I \subseteq \Pi_0^{-1}[\{y\}, I] \cap I
\]

since \( N \subseteq \Pi \mathcal{E} \) and since

\[
\Pi_0(ax_{i_0} + \beta y_{i_0}) - \Pi_0 x_{i_1} \leq \Pi_0(ax_{i_0} + \beta y_{i_0}) - \Pi_0 x_{i_0}.
\]

Moreover, this is a contradiction since \( x_{i_0} \cup x_{i_1} \) is linked.

**Lemma 7.** \( \lambda_{\mathcal{E}}(I) \) is infinite dimensional.

**Proof.** We will show that \( \lambda_{\mathcal{E}}(I) \) contains a copy of the Hilbert cube. For each \( n \in E \), let \( I_n \) be defined by

\[
I_n = \left( \frac{1}{2^n + \frac{1}{2}}, \frac{1}{2^n + \frac{1}{2}} \right).
\]

Define a map \( \psi: \prod_{n \in E} I_n \to I^2 \) by

\[
(\psi(x))_n = \left( x_n, \frac{1}{2^n} \right).
\]

Note that for each \( i \in E \), \( (\psi(x))_n \) is an element of \( \lambda_{\mathcal{E}}(I) \) for all \( x \in \prod_{n \in E} I_n \). Furthermore it is obvious that \( \psi \) is an embedding. It suffices to show that \( \lambda_{\mathcal{E}}(I) \) is contained in \( \lambda_{\mathcal{E}}(I) \) and for this it suffices to show that \( \cup_{n \in E} (\psi(x))_n \) is linked for all \( x \in \prod_{n \in E} I_n \) (Lemma 5). Assume to the contrary that for some \( x \in \prod_{n \in E} I_n \), \( \cup_{n \in E} (\psi(x))_n \) were not linked. Then there exist indices \( n_0, n_1, n_2 \) such that \( (\psi(x))_{n_0} \cup (\psi(x))_{n_1} \) is not linked and therefore there exists an \( M \in (\psi(x))_{n_0} \) and an \( N \in (\psi(x))_{n_1} \) such that \( M \cap N = \emptyset \). Then there are two possibilities drawn in Figure 6 and Figure 7.

Without loss of generality we may assume that \( n_1 < n_0 \). This shows that

\[
\Pi_0^{-1}[\{y\}, I] \cap I \subseteq \Pi_0^{-1}[\{y\}, I] \cap I.
\]

Since \( n_1 < n_0 \) it follows that

\[
\sqrt{2} \Pi_0(\psi(x))_{n_0} < \frac{1}{n_1} \leq \frac{1}{n_0} \leq \sqrt{2} \Pi_0(\psi(x))_{n_0} - \frac{1}{n_1}.
\]
and therefore
\[ \sqrt{2} |\Pi_0(\varphi(x))_n - 1| < \sqrt{2} |\Pi_0(\varphi(x))_n - 1| \]
which shows that the component containing 0 of \( \Pi_0^{-1}[\frac{1}{2}, \Pi_0(\varphi(x))_n] \cap I \) cannot be contained in the component containing 0 of \( \Pi_0^{-1}[\frac{1}{2}, \Pi_0(\varphi(x))_n] \cap I \), contradiction. \( \square \)

The superextension of the closed unit interval is homeomorphic to the Hilbert cube. \( \square \)

Proof. According to a theorem of Keller [15] each infinite dimensional compact convex subspace of the Hilbert space is homeomorphic to the Hilbert cube. \( \square \)

As noted in the introduction, we need \( \lambda_{\mathcal{U}}(I) \) as first step in an inverse limit representation of \( M \). This will be demonstrated in the next section.

4. \( \mathcal{U} \) is a Hilbert cube.

Definition. Let \( \mathcal{F} \) and \( \mathcal{S} \) be two families of closed sets in \( X \). Then \( \mathcal{F} \) separates \( \mathcal{S} \) if for any \( T_0, T_1 \in \mathcal{F} \) with \( T_0 \cap T_1 = \emptyset \), there exist \( S_0, S_1 \in \mathcal{S} \) such that \( T_0 \subseteq S_0 (i = 0, 1) \) and \( S_0 \cap S_1 = \emptyset \).

Notation. \( \mathcal{F} \subseteq \mathcal{S} \).

For the closed unit interval \( I \), define \( \mathcal{F} = \{G \in I | G \text{ is the union of a finite number of closed intervals with rational endpoints}\} \).

It is clear that \( \mathcal{F} \) separates the collection of closed subsets of \( I \) so that each mls \( \mathcal{A} \in \mathcal{U} \) is completely determined by its trace on \( \mathcal{F} \). In fact it can be proved that \( \mathcal{U} \) and \( \lambda_{\mathcal{U}}(I) \) are equivalent ([17]), which means, homeomorphic under a homeomorphism which on \( I \) is the identity. Furthermore it should be noticed that \( \mathcal{F} \) is a countable subbase. Define

\[ \mathcal{F} = \{ (S_0, S_1) | S_i \in \mathcal{F} (i = 0, 1) \text{ and } S_0 \cap S_1 = \emptyset \} \]

Then \( \mathcal{F} \) again is countable; we enumerate \( \mathcal{F} \), using a bijection of \( N \setminus \{1\} \) onto \( \mathcal{F} \). If \( (S_0, S_1) \in \mathcal{F} \), then \( \varepsilon = d(S_0, S_1) > 0 \) and also \( \delta = \frac{\varepsilon}{\sqrt{2}} > 0 \). Consider the following embedding, depending on \( (S_0, S_1) \) of \( I \) in \( I^2 \) (see Fig. 8). All angles are \( \frac{\pi}{4} \) except the one at \((\frac{1}{2}, 0)\) which is \( \frac{\pi}{2} \). Furthermore \( b - \alpha = \delta \) and \( S_0 = \Pi_0^{-1}[0, a] \cap I \) and \( S_1 = \Pi_0^{-1}[b, 1] \cap I \). Since \( S_0 \) and \( S_1 \) are finite unions of intervals, such an embedding always is possible. In the embedding of \( I \) in \( I^2 \) we will not use more angles than necessary. As in Section 2 define

\[ \mathcal{F} = \{ A \in I^2 | A = \Pi_0^{-1}[0, x] \cap I \} (i \in \{0, 1\}, x \in I) \]
This time, $\lambda_{\mathcal{F},c}(I)$ is not the convex hull of the embedded copy of $I$ in $I^*$, but it is the space designed in Figure 9.

If $(S_n, S_1)$ is the $n$th element of $\mathcal{F}$, let $\lambda_{\mathcal{F},c}(I)$ be the superextension of $I$, as indicated in Figure 9. In addition, put $\mathcal{F}_1 = \bigcup_{i \in E} \mathcal{F}_1$, where the $\mathcal{F}_i$'s are defined as in Section 3.

A $Q$-factor is a space the product of which with the Hilbert cube is a Hilbert cube. A $Q$-manifold is paracompact Hausdorff space modelled on $Q$, i.e., a space which admits an open cover by sets homeomorphic to open subsets of $Q$. $Q$-manifolds are locally compact and metrizable. The hardest part of our program is to show that for each $n \in N$ the superextension $\lambda_{\mathcal{F},c}(I)$ is a $Q$-manifold, the proof of which will be postponed till Section 5.

**Lemma 8.** For each $n \in N$, $\lambda_{\mathcal{F},c}(I)$ is a $Q$-manifold.

Now, an interesting theorem of Chapman is applicable to show that $\lambda_{\mathcal{F},c}(I)$ is even a Hilbert cube.

**Proposition 4.** For each $n \in N$, $\lambda_{\mathcal{F},c}(I)$ is a Hilbert cube.

**Proof.** The normality of $\bigcup_{i = 1} \mathcal{F}_i$ implies that $\lambda_{\mathcal{F},c}(I)$ is connected (Verbeek [21], III. 4.1 Corollary) and consequently $\lambda_{\mathcal{F},c}(I)$ is an AR (Theorem 4). Therefore $\lambda_{\mathcal{F},c}(I)$ is a compact contractible $Q$-manifold by Lemma 8. However, a compact contractible $Q$-manifold is a Hilbert cube, by a theorem of Chapman [7].

Consider the following inverse limit system:

$$\lambda_{\mathcal{F}}(I) \leftarrow \lambda_{\mathcal{F},c}(I) \leftarrow \lambda_{\mathcal{F},c}(I) \leftarrow \lambda_{\mathcal{F},c}(I) \leftarrow \lambda_{\mathcal{F},c}(I) \leftarrow \ldots$$

where all the bonding maps are the Jensen mappings.

The superextension of the closed unit interval is homeomorphic to the Hilbert cube.

**Lemma 9.** $\lambda I$ is homeomorphic to $\bigcup_{i = 1} \mathcal{F}_i(I)$.

**Proof.** Since all subbases in the inverse limit system are supernormal and therefore are normal there exists for each $n \in N$ a Jensen mapping

$$\xi_n : \lambda I \to \lambda_{\mathcal{F},c}(I).$$

From the definition of the Jensen mappings it follows that for each $n \in N$ the diagram

$$\begin{array}{ccc}
\lambda I & \xrightarrow{\xi_n} & \lambda_{\mathcal{F},c}(I) \\
\downarrow & & \downarrow \\
\lambda I & \xrightarrow{\xi_{n+1}} & \lambda_{\mathcal{F},c}(I)
\end{array}$$

commutes, and therefore the map

$$e : \lambda I \to \bigcup_{i = 1} \mathcal{F}_i(I)$$

defined by $(e(\mathcal{F}_i))_n = \xi_n(\mathcal{F}_i)$ $(n \in N)$ is a continuous closed surjection. It remains to show that $e$ is one to one. Choose $\mathcal{F}_i, \mathcal{F}_j \in \lambda I$ such that $\mathcal{F}_i \neq \mathcal{F}_j$. Then there exist $M \in \mathcal{F}_i$ and $N \in \mathcal{F}_j$ such that $M \cap N = \emptyset$. Since $\mathcal{F}_1$ separates the closed subsets of $I$, there exist $S_0, S_1 \in \mathcal{F}$ with $M \subseteq S_0, N \subseteq S_1$, and $S_0 \cap S_1 = \emptyset$. Of course it follows that $S_0 \in \mathcal{F}_i$ and $S_1 \in \mathcal{F}_j$. Now, $(S_0, S_1) \in \mathcal{F}_1$, say the $n$th element, and therefore $S_0$ and $S_1$ are separated by elements of $\mathcal{F}_n$, and consequently $\xi_n(\mathcal{F}_i) \neq \xi_n(\mathcal{F}_j)$, since $\mathcal{F}_n \subseteq \mathcal{F}_i$. This proves that $e$ is one to one consequently $e$ is a homeomorphism.

An onto map between homeomorphic compact metric spaces is called a near-homeomorphism if it is the uniform limit of homeomorphisms. An approximation theorem for inverse limits of Brown [6], often used in infinite dimensional topology, says that if $Y = \lim X_n$, where $\{X_n\}$ denotes an inverse sequence, and the $X_n$ are all homeomorphic to a compact metric space $X$ and each bonding map is a near-homeomorphism, then $Y$ is homeomorphic to $X$.

If $X$ and $Y$ are locally compact, then a map $f : X \to Y$ is called proper if the inverses of compact subsets of $Y$ are compact in $X$. A proper map $f$ is called cell-like or cellular (CE), if it is onto and point inverses have trivial shape. Chapman announced a theorem that characterizes near-homeomorphisms between Hilbert cubes as being those continuous surjections with the property that the inverse image of each point has trivial shape. This result is a consequence of his papers [8] and [9]. This theorem makes Brown’s approximation theorem applicable in our situation.

**Lemma 10.** Let $g_{i+1} : \lambda_{\mathcal{F},c}(I) \to \lambda_{\mathcal{F},c}(I)$ be the Jensen mapping. Then $g_{i+1}$ is monotone.
Proof. We will show that point inverses of \( g_{n-1} \) are closed under the interval structure of \( \lambda_{\lambda_{i}}(I) \), which suffices to show that \( g_{n-1} \) is monotone (Corollary 1), since \( \lambda_{\lambda_{i}}(I) \) is connected. Choose \( \mathcal{N} \in \lambda_{\lambda_{i}}(I) \) and let \( \mathcal{M}_{0}, \mathcal{M}_{1} \in g_{n-1}^{-1}(\mathcal{N}) \).

Suppose there exists an \( \mathcal{M}_{2} \in \lambda_{\lambda_{i}}(\mathcal{M}_{0}, \mathcal{M}_{1}) \), then \( g_{n-1}(\mathcal{M}_{2}) \neq \mathcal{N} \) and therefore there exist \( N_{0}, N_{1} \in \bigcup_{i=1}^{n-1} \mathcal{S}_{i} \) such that \( N_{0} \in g_{n-1}(\mathcal{M}_{2}) \) and \( N_{1} \in \mathcal{N} \) and \( N_{0} \cap N_{1} = \emptyset \). However \( \bigcup_{i=1}^{n-1} \mathcal{S}_{i} \) is supernormal and, therefore \( \mathcal{N} \in \mathcal{M}_{i}(i \in \{0,1\}) \) and \( g_{n-1}(\mathcal{M}_{2}) \subseteq \mathcal{M}_{i} \) (Corollary 3). This proves that \( N_{0} \in \mathcal{M}_{0} \) and \( N_{1} \in \mathcal{M}_{1} \) and therefore

\[
\bigcup_{i=1}^{n-1} \lambda_{\lambda_{i}}(\mathcal{M}_{0}, \mathcal{M}_{1}) \subseteq N_{i},
\]

which is a contradiction since \( \mathcal{M}_{2} \in \bigcup_{i=1}^{n-1} \lambda_{\lambda_{i}}(\mathcal{M}_{0}, \mathcal{M}_{1}) \). ■

Lemma 11. Let \( g_{n-1}: \lambda_{\lambda_{i}}(I) \rightarrow \lambda_{\lambda_{i}}^{-1}(I) \) be the Jensen mapping. Then each point inverse either is a point or is homeomorphic to an interval. In particular \( g_{n-1} \) is cellular.

Proof. Let \( f_{c} \) be the Jensen mapping of \( \lambda_{\lambda_{i}}^{-1}(I) \) onto \( \lambda_{\lambda_{i}}(I) \). Let \( \mathcal{N} \in \lambda_{\lambda_{i}}^{-1}(I) \).

Choose \( \mathcal{M}_{0}, \mathcal{M}_{1} \in g_{n-1}^{-1}(\mathcal{N}) \), such that \( \mathcal{M}_{0} \neq \mathcal{M}_{1} \). Then there exists an \( M_{0} \in \mathcal{M}_{0} \) and an \( M_{1} \in \mathcal{M}_{1} \), such that \( M_{0} \cap M_{1} = \emptyset \). Since \( \bigcup_{i=1}^{n-1} \mathcal{S}_{i} \) is supernormal, it follows that \( M_{0} \) and \( M_{1} \) are both elements of \( \bigcup_{i=1}^{n-1} \mathcal{S}_{i} \) (notice that \( g_{n-1}(\mathcal{M}_{2}) = g_{n-1}(\mathcal{M}_{1}) \) and consequently without loss of generality \( M_{0} \in \mathcal{S}_{0} \). However, \( \mathcal{S}_{1} \) is also supernormal and therefore we may assume that \( M_{0} \in \mathcal{S}_{1} \). It now follows that \( f_{c}(\mathcal{M}_{0}) \neq f_{c}(\mathcal{M}_{1}) \), since \( \mathcal{N} \) is supernormal (Corollary 3). Therefore \( g_{n-1}^{-1}(\mathcal{N}) \) and \( f_{c}(\mathcal{M}_{0}) \) are homomorphic. However this shows that \( g_{n-1}(\mathcal{M}_{2}) \) is a point or is homeomorphic to an interval, since all points of \( f_{c}(\mathcal{M}_{0}) \) must be elements of a horizontal line through a point of the embedded copy of \( I \), a point which is completely determined by \( \mathcal{N} \), and since \( g_{n-1}^{-1}(\mathcal{N}) \) is connected (Lemma 10). ■

Theorem 7. The superextension of the closed interval is homeomorphic to the Hilbert cube.

Proof. As a consequence of Chapman's theorem it follows from Lemma 11 that all bonding maps in the inverse limit system for \( I \) are near-homeomorphisms. All superextensions in the inverse system are Hilbert cubes (Proposition 4) and therefore Lemma 9 and Brown's approximation theorem give the result \( I \cong Q \). ■

5. \( \lambda_{\lambda_{i}}(I) \) is a \( Q \)-manifold.

Proof. Choose \( x \in \lambda_{\lambda_{i}}(I) \) and let \( \mathcal{N}_{0}, \mathcal{N}_{1} \in g^{-1}(\mathcal{N}) \).

Suppose \( x \in \lambda_{\lambda_{i}}(I) \) and let \( \mathcal{M}_{0}, \mathcal{M}_{1} \in g^{-1}(\mathcal{N}) \) and therefore there exist \( N_{0}, N_{1} \in \bigcup_{i=1}^{n-1} \mathcal{S}_{i} \) such that \( N_{0} \in \mathcal{M}_{0}, N_{1} \in \mathcal{N} \) and \( N_{0} \cap N_{1} = \emptyset \). However \( \bigcup_{i=1}^{n-1} \mathcal{S}_{i} \) is supernormal and, therefore \( \mathcal{N}_{0} \in \mathcal{M}_{0}, \mathcal{N}_{1} \in \mathcal{N} \) and \( \mathcal{N}_{0} \cap \mathcal{N}_{1} = \emptyset \). However \( \bigcup_{i=1}^{n-1} \mathcal{S}_{i} \) is supernormal and, therefore \( \mathcal{N} \in \mathcal{M}_{i}(i \in \{0,1\}) \) and \( g^{-1}(\mathcal{M}_{2}) \subseteq \mathcal{M}_{i} \) (Corollary 3). This proves that \( N_{0} \in \mathcal{M}_{0} \) and \( N_{1} \in \mathcal{M}_{1} \) and therefore

\[
\bigcup_{i=1}^{n-1} \lambda_{\lambda_{i}}(\mathcal{M}_{0}, \mathcal{M}_{1}) \subseteq N_{i},
\]

which is a contradiction since \( \mathcal{M}_{2} \in \bigcup_{i=1}^{n-1} \lambda_{\lambda_{i}}(\mathcal{M}_{0}, \mathcal{M}_{1}) \). ■
in $\lambda_{x}(I)$ such that $x \in O \subset U \subset B(x)$ and $O$ is homeomorphic to an open subset of $Q$.

Let us first analyze $B(x)$. Consider $F = \{0, 1\}^{1,2,3,\ldots}$ and for each $\sigma = (\sigma_i) \in F$ define

$$X(\sigma) = \bigcup_{i=0}^{n} \pi_{i}^{-1}[D^{n} \cap \pi_{i}^{-1}[\Pi_{i}^{-1}([a_i, a_i + 1] \cap \lambda_{x}(I)) \cap \lambda_{x}(I)].$$

It then is clear that

$$\bigcup_{\sigma \in F} X(\sigma) = B(x).$$

A. For each $\sigma \in F$ the set $X(\sigma)$ is closed and convex in $\lambda_{x}(I)$. Assume to the contrary that for some $\sigma \in F$ the set $X(\sigma)$ were not convex. Then there exist $y, z \in X(\sigma)$ and $x, \beta \in R$ with $x > 0$ and $\beta > 0$ and $x + \beta = 1$ such that $ay + \beta z \notin X(\sigma)$.

We claim that

$$\bigcup_{i \in I} (a_i + \beta z_i) \cup \bigcup_{i \in I} (a_i + \beta z_i)$$

is not linked, for else it would follow that $ay + \beta z \notin \lambda_{x}(I)$, and as $(a_i + \beta z_i) = a_i + \beta z_i$; for each $i$, it is easily seen that also $ay + \beta z \notin X(\sigma)$. Therefore there exist two indices $i_0, j_0$ such that $(a_i + \beta z_i)_{i_0}$ is not linked and consequently there exists an $M$ in $(a_i + \beta z_i)_{i_0}$ and an $N$ in $(a_i + \beta z_i)_{j_0}$ such that $M \cap N$ is open. Now, let $i_0$ and $j_0$ be both elements of $E = \{0, 1, 2, 3, \ldots, n\}$, then, using the same technique as in Lemma 6, this leads to a contradiction, for we have chosen the intervals $[a_i, b_i]$ (i.e. $(q + 1, q + 2, \ldots, n)$) in such a way that $\Pi_{i}^{-1}([a, b])$ has an isolated point for every subinterval $[a_i, b_i]$. Therefore, let us assume that $i_0 \in \{2, 3, \ldots, q\}$. Since straight horizontal lines through $(a_i + \beta z_i)_{i_0}$ and $(a_i + \beta z_i)_{j_0}$ must intersect the embedded copies of $I$ in the same point, the situation sketched in Figure 10 is the only possibility (except for an interchange of the indices $i_0$ and $j_0$, which induces a similar situation).

(ii) In Figure 10 we have drawn the points $y_i, x_i, y_{i_0}, x_{j_0}, z_0$, and $x_{j_0}$ in such a way that $\Pi_{x_{j_0}} < \Pi_{y_{i_0}} < \Pi_{x_{j_0}}$, and $\Pi_{y_{i_0}} < \Pi_{x_{j_0}} < \Pi_{x_{j_0}}$. This is not the only possible configuration. More generally, we may assume that either $(\Pi_{y_{i_0}} \cup \Pi_{z_0}) \leq \Pi_{x_{j_0}} < \Pi_{y_{i_0}} \cup \Pi_{z_0}$ and $\Pi_{y_{i_0}} \cup \Pi_{z_0} < \Pi_{x_{j_0}} < \Pi_{y_{i_0}} \cup \Pi_{z_0}$ (these two cases are similar), for in all other cases it is easy to see that $(\Pi_{y_{i_0}} \cup \Pi_{z_0}) \leq \Pi_{x_{j_0}}$ is linked. The lack of generality in our diagram will cause no trouble, as will appear from the proof.

We distinguish two subcases:

(a) $\Pi_{y_{i_0}} \leq \Pi_{x_{j_0}} < \Pi_{y_{i_0}} \cup \Pi_{z_0}$

Since $M \leq \Pi_{x_{j_0}} \leq \Pi_{y_{i_0}} \cup \Pi_{z_0}$, if $I \subseteq \Pi_{x_{j_0}} \leq \Pi_{y_{i_0}} \cup \Pi_{z_0}$ since $\Pi_{y_{i_0}} \cap \Pi_{x_{j_0}} \cap I$ has no isolated points and since $\Pi_{x_{j_0}} \cup \Pi_{z_0} \subseteq \Pi_{y_{i_0}} \cup \Pi_{z_0}$ (these two cases are similar), then it is easy to see that $\Pi_{y_{i_0}} \cup \Pi_{z_0}$ is linked.

(b) $\Pi_{y_{i_0}} \cup \Pi_{z_0} < \Pi_{x_{j_0}} < \Pi_{y_{i_0}} \cup \Pi_{z_0}$

As $N \subseteq \Pi_{x_{j_0}} \leq \Pi_{y_{i_0}} \cup \Pi_{z_0}$, we conclude that $\Pi_{x_{j_0}} \cap I \subseteq \Pi_{y_{i_0}} \cup \Pi_{z_0} \subseteq \Pi_{x_{j_0}} \cap I$, since $\Pi_{y_{i_0}} \cap \Pi_{z_0} \subseteq \Pi_{y_{i_0}} \cup \Pi_{z_0} \subseteq \Pi_{x_{j_0}} \cap I$. Therefore, if $\Pi_{x_{j_0}} \cap I$ contains no isolated point of $\Pi_{x_{j_0}} \leq \Pi_{y_{i_0}} \cup \Pi_{z_0}$, then this is a contradiction. If $\Pi_{x_{j_0}} \cap I$ contains an isolated point of $\Pi_{x_{j_0}} \leq \Pi_{y_{i_0}} \cup \Pi_{z_0}$, then $\Pi_{x_{j_0}} \cap I$ is not perfect, which is a contradiction. Now, since

$$\Pi_{x_{j_0}} \leq \Pi_{y_{i_0}} \cup \Pi_{z_0} \subseteq \Pi_{x_{j_0}} \cap I$$

it follows that also $\Pi_{y_{i_0}} \cup \Pi_{z_0}$, for if not, then $\Pi_{y_{i_0}} \cup \Pi_{z_0}$ is not linked. However, this implies that also $\Pi_{y_{i_0}} \cup \Pi_{z_0}$ is linked and consequently $\Pi_{y_{i_0}} \cup \Pi_{z_0}$ is a contradiction, since $\Pi_{y_{i_0}} \cup \Pi_{z_0}$ is linked.

It now follows that the neighborhood $B(x)$ of $x$ is a finite union of closed (and hence compact) convex subspaces. By a theorem of Quinn and Wong ([18], Theorem 3.4) it follows that $B(x)$ is a $Q$-manifold provided that $F$ the set $\bigcap \{X(\sigma) \mid \sigma \in F\}$ is either void or is homeomorphic to $Q$.

B. Let $F_0$ be a non-void subset of $F$. Then $\bigcap X(\sigma)$ is either void or is homeomorphic to $Q$.

Assume that $\bigcap X(\sigma)$ is non-void. It suffices to show that $\bigcap X(\sigma)$ is infinite dimensional for an infinite dimensional compact convex set of the Hilbert space is homeomorphic to $Q$ (Keller [13]). Choose $y \in \bigcap X(\sigma)$ and we again distinguish two subcases:

(a) For each $i \in \{2, 3, \ldots, n\}$ the point $\Pi_{y_i}$ is an element of $\{y_i, y_i + 1\}$.

Assume that $y$ is such that for every coordinate $y_i$ (i.e. $i \in \{0, 1, \ldots, n\}$) a straight horizontal line through $y_i$ does not intersect $I$ in 0 or 1. (This assumption is justified by the fact that if $y = i(0)$ or $i(1)$, then $\bigcap X(\sigma)$ is the intersection of a finite number
of sets, each of which intersects \( \{ f \} \) in a neighborhood of \( y \). This intersection, say \( f \), must be the same point for every coordinate. Define
\[
\delta_0 = \min \left\{ |y_i - c_i| \mid i \in \{2, 3, ..., n\} \right\}, \\
\delta_1 = \min \left\{ |y_i - c_i| \mid i \in \{2, 3, ..., n\} \right\}
\]
and choose \( n_0 \in E \) such that
\[
\frac{1}{n_0} < \frac{1}{4} \min \{ \delta_0, \delta_1, 2, f, 1 - f \}.
\]
For all \( j \in E \), let \( I_j \) be defined as in Lemma 7.

It is easy to show, using the same technique as in Lemma 7, that for all \( j \in E \) with \( j \leq n_0 \) and for each point \( d \) of \( I_j \times \{ f, 1 - f \} \), we have that \( \bigcup_{i=2}^n y_i \cup d \) is linked (notice that indeed \( I_j \times \{ f, 1 - f \} \subseteq \lambda_{\phi}(I_j) \)).

Now, by induction, for each \( k \in \{ m \in E \mid n_0 \leq m \} \), we will construct a point \( b_k \) of \( \lambda_{\phi}(I_j) \) with the following property: for every \( j \in E \) with \( j \leq n_0 \), there exists a (non-degenerate) subinterval \( I_{jk} \) of \( I_j \) such that, for every point \( d_{jk} \in \bigcup_{k=2}^n I_{jk} \times \{ f, 1 - f \} \), the system
\[
\bigcup_{i=2}^n y_i \cup h_j \cup \bigcup_{k=2}^n \bigcup_{d_{jk}} \bigcup_{k=2}^n d_{jk}
\]
is linked.

For each \( j \in E \) with \( j \leq n_0 \), let \( h_j \) be the middle of the interval \( I_j \times \{ f, 1 - f \} \). Then the linked system
\[
\bigcup_{i=2}^n y_i \cup \bigcup_{k=1}^n h_j \cup \bigcup_{d_{jk}} \bigcup_{k=2}^n d_{jk}
\]
is contained in at least one maximal linked system \( g_0 \in \lambda_{\phi}(I_j) \). Define \( h_{j-1} = (g_0)_{j-1} \). The intervals \( I_{jk} \) \((j \leq n_0)\) now can be found in the following way:

(i) \( I_{11}^1 = I_{I_1} \) if \( I_{I_1}x_1 \in I_{I_1} \).
(ii) \( I_{21}^2 = \left[ \{ y_i \mid y_i \in \lambda_{\phi}(I_j) \} \cap I_j \right] \) if \( I_{I_2}x_2 \in \left[ \{ y_i \mid y_i \in \lambda_{\phi}(I_j) \} \cap I_j \right] \).
(iii) \( I_{31}^3 = \left[ \{ y_i \mid y_i \in \lambda_{\phi}(I_j) \} \cap I_j \right] \) if \( I_{I_3}x_3 \in \left[ \{ y_i \mid y_i \in \lambda_{\phi}(I_j) \} \cap I_j \right] \).

Again it is easy to verify that the intervals \( I_{jk}^j \) \((j \leq n_0)\), defined in this way, satisfy our requirements.

Now it is obvious that \( \bigcap_{\sigma \in E_0} x(\sigma) \) contains a copy of \( \bigcap_{\sigma \in E_0} I_{\phi}(\sigma) \), which shows that \( \bigcap_{\sigma \in E_0} x(\sigma) \) is infinite-dimensional.

(b) There exists a coordinate \( i_0 \in \{ 2, 3, ..., n \} \) such that \( I_{i_0}x_{i_0} \notin \lambda_{\phi}(I_j) \).

We will construct a point \( g \in \bigcap_{\sigma \in E_0} x(\sigma) \) such that \( I_{i_0}x_{i_0} \notin \lambda_{\phi}(I_j) \) for all \( i \in \{ 2, 3, ..., n \} \). Then case (a) is applicable to show that \( \bigcap_{\sigma \in E_0} x(\sigma) \) is infinite-dimensional. Without loss of generality we may assume that
\[
\bigcap_{\sigma \in E_0} x(\sigma) = \bigcap_{i=2}^n \bigcup_{j=2}^n \bigcup_{i=1}^n \bigcup_{d_{jk}} I_{jk} \cap \lambda_{\phi}(I_j),
\]
where each \( S_i \subseteq \{ n \} \) is convex in \( \lambda_{\phi}(I_j) \) while, moreover, for each \( i \geq 1 \), we have \( S_i = \bigcap_{\sigma \in E_0} \bigcap_{\sigma \neq \sigma_i} \lambda_{\phi}(I_j) \) for some (non-degenerate) interval \( I_j \). As in (a), we may assume that a straight horizontal line through \( y_i \) does not intersect \( I \) in 0 or 1. Let this intersection by \( f \). Define \( V = \{ [2, 3, ..., n] \} \). Now, for every \( i \in V \) there exists a subinterval \( I_{i_0} \) of \( I_j \) such that \( I_{i_0}x_{i_0} \notin \lambda_{\phi}(I_j) \). Let \( \delta_3 \) denote the length of this interval ( \( i \in V \)). Let \( \delta_3 = \min \{ \delta_3 \mid f \in V \} \). Moreover define
\[
\varrho_0 = \min \{ |y_i - c_i| \mid i \in \{ 2, 3, ..., n \} \} \times V, j \in \{ 0, 1 \}
\]
and
\[
\varrho = \frac{1}{4} \min \{ \varrho_0, \delta_3 \}.
\]
Choose for each \( i \in V \) a point \( g_i \in \bigcap_{\sigma \in E_0} \lambda_{\phi}(I_j) \) such that \( |y_{i_0}x_{i_0} - \varrho g_i| = \varrho \).

Recall that \( A = \{ 2, 3, ..., n \} \). We will show that
\[
\mathcal{S} = \bigcup_{i \in \mathcal{V}} \bigcup_{i \in \mathcal{A} \times V} y_i
\]
is linked and consequently each miss \( g \in \lambda_{\phi}(I_j) \) which contains \( \mathcal{S} \) is a point of \( \bigcap_{\sigma \in E_0} x(\sigma) \) such that \( I_{i_0}x_{i_0} \notin \lambda_{\phi}(I_j) \) for all \( i \in \{ 2, 3, ..., n \} \). Assume that \( \mathcal{S} \) were not linked. We again distinguish two subcases:

Case 1. There exist two indices \( i_0, j_0 \in V \) such that \( g_{i_0} \cup g_{j_0} \) is not linked. Then choose \( M \in \mathcal{M}_0 \) and \( N \in \mathcal{N}_0 \) such that \( M \cap N = \emptyset \). There are two subcases:

(i) One of the sets \( M, N \) contains the corresponding projection of \( y_i \), say \( y_{i_0} \in M \).
Since \( N \subseteq \Pi_0^{-1}[0, \Pi_0 g_{y_0}] \cap I \) and since \( [\Pi_0 g_{y_0} - \Pi_0 g_{y_0}] = [\Pi_0 g_{y_0}, \Pi_0 g_{y_0}] \) it follows that \( \Pi_0^{-1}[0, \Pi_0 g_{y_0}] \cap I = \Pi_0^{-1}[0, \Pi_0 g_{y_0}] \cap I \). However, this is a contradiction since \( \Pi_0^{-1}[0, \Pi_0 g_{y_0}] \cap I = I \).

\[ \begin{array}{c}
\Pi_0^{-1}[0, \Pi_0 g_{y_0}] \\
\Pi_0 \\
\Pi_0^{-1}[0, \Pi_0 g_{y_0}] \\
\Pi_0 \\
\Pi_0^{-1}[0, \Pi_0 g_{y_0}] \\
\Pi_0 \\
\end{array} \]

Fig. 11

(ii) None of the sets \( M, N \) contains the corresponding projection of \( y \).

It now follows that for example \( M \subseteq \Pi_0^{-1}[\Pi_0 g_{y_0}, 1] \cap I \).

\[ \begin{array}{c}
\Pi_0^{-1}[0, \Pi_0 g_{y_0}] \\
\Pi_0 \\
\Pi_0^{-1}[0, \Pi_0 g_{y_0}] \\
\Pi_0 \\
\Pi_0^{-1}[0, \Pi_0 g_{y_0}] \\
\Pi_0 \\
\end{array} \]

Fig. 12

However, this is a contradiction since \( M \) contains a component of length at least \( \frac{\lambda}{2} \) while all components of \( \Pi_0^{-1}(\Pi_0 g_{y_0}, 1] \cap I \) have length less than or equal to \( \frac{\lambda}{2} \), since \( \Pi_0^{-1}[\Pi_0 g_{y_0}, 1] \cap I \) contains no isolated points and the same is true for each subinterval of \( H_{y_0} \).

Case 2. There exist indices \( i \in I \) and \( j < I \) such that \( g_{y_0} \cup y_{j+1} \) is not linked. This can be treated in the same way as Case 1(ii).

This completes the proof of the lemma.

Added in proof. The main result of this paper that \( H \cup Q \) can also be derived by using a recent characterization of the Hilbert cube due to H. Toruńczyk.

References

[10] — All Hilbert cube manifolds are triangulable (preprint).

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Accept par la Rédaction le 25. 10. 1976