Closed G_δ subsets of supercompact Hausdorff spaces

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Communicated by Prof. W. T. van Est at the meeting of June 17, 1978

ABSTRACT

We give examples of compact Hausdorff spaces which are not embeddable as closed G_δ subsets in a supercompact Hausdorff space.

Key words and phrases: supercompact, linked system, Cantor tree, density.
AMS(MOS) subject classification (1970): 54D35.

INTRODUCTION

A *supercompact* space is a space which has a binary subbase for its closed subsets, where a collection of subsets $\mathscr{S}$ of a set X is called *binary* provided that for all $\mathscr{M} \subset \mathscr{S}$ with $\cap \mathscr{M} = \emptyset$ there are $M_0, M_1 \in \mathscr{M}$ with $M_0 \cap M_1 = \emptyset$. By Alexander's subbase lemma, every supercompact space is compact. The class of supercompact spaces was introduced by de Groot [9]. Many spaces are supercompact, for example all compact metric spaces, cf. Strok & Szymanski [14] (elementary proofs of this fact were recently found by van Douwen [6] and Mills [12]). The first examples of nonsupercompact compact Hausdorff spaces were found by Bell [1]. At the moment there is a variety of nonsupercompact compact Hausdorff spaces (cf. Bell [1], [2], van Douwen & van Mill [7], van Mill [11], Bell & van Mill [4]).

* The first author is supported ty the Netherlands Organization for the Advancement of Pure Research (Z.W.O.); Juliana van Stolberglaan 148, 's-Gravenhage, the Netherlands.

Recently, Bell [3] showed that the one point compactification of the Cantor tree ${}^{\omega}_{\circ}2 \cup {}^{\omega}2$ (cf. Rudin [13]) can be embedded as a closed G_δ subset of a supercompact Hausdorff space. Since the one point compactification of the Cantor tree is not supercompact (cf. van Douwen & van Mill [7]) this yields an example of a nonsupercompact closed G_δ in a supercompact Hausdorff space. This suggests the question whether every compact Hausdorff space can be embedded as a G_δ subset in a supercompact Hausdorff space. The answer to this question is in the negative.

0.1. THEOREM: *Let X be a Hausdorff continuous image of a closed G_δ subset of a supercompact Hausdorff space, and let K be a closed subset of X such that $|K| > 2^{\omega}$. Then at least one point of K is the limit of a nontrivial convergent sequence in X (not necessarily in K).*

This theorem is a consequence of a result in van Douwen & van Mill [7]. As a corollary, if βX is a continuous image of a closed G_δ subset of a supercompact Hausdorff space then X is pseudocompact. Also, under Martins axiom (MA), every infinite Hausdorff continuous image of a closed G_δ subset of a supercompact Hausdorff space contains a nontrivial convergent sequence.

Since the one point compactification of the Cantor tree is a compactification of ω with the one point compactification of a discrete space as remainder, Bell's [3] result suggests the question whether every compactification of ω with the one point compactification of a discrete space as remainder can be embedded as a G_δ subset of a supercompact Hausdorff space. The answer to this question is in the negative. For every (faithfully indexed) almost disjoint family $\mathscr{M} = \{M_\alpha | \alpha \in \varkappa\}$ of infinite subsets of ω define $X_{\mathscr{M}}$ to be the space with underlying set the disjoint union of $\varkappa$ and ω and with topology generated by the collection

$$\{\{\alpha\} \cup (M_\alpha - n) | \alpha \in \varkappa, n \in \omega\} \cup \{\{n\} | n \in \omega\}.$$

Notice that $X_{\mathscr{M}}$ is separable and that every subspace of $X_{\mathscr{M}}$ is locally compact and first countable. Also, the Cantor tree ${}^{\omega}_{\circ}2 \cup {}^{\omega}2$ is homeomorphic to some $X_{\mathscr{M}}$. We will prove the following theorem:

0.2. THEOREM: *Let $\mathscr{M}$ be a maximal uncountable almost disjoint collection of infinite subsets of ω. Then any compactification of $X_{\mathscr{M}}$ is not the continuous image of a closed G_δ subset of a supercompact Hausdorff space.*

1. THEOREM 0.1; PROOF AND CONSEQUENCES

1.1. PROOF OF THEOREM 0.1: Indeed, let Y be a supercompact Hausdorff space, let X and K be as in Theorem 0.1 and let Z be a closed G_δ in Y which is mapped by f onto X. Write $Z = \bigcap_{n \in \omega} U_n$, where the U_n's are open subsets of Y. It is easily verified that a space has a binary subbase if and only if it has a binary subbase closed under arbitrary intersections. Let $\mathscr{S}$ be a binary subbase for Y which is closed under arbitrary intersections. For each $n \in \omega$ let $\mathscr{J}_n$ be a finite subcollection of $\mathscr{S}$ such that $Z \subset \bigcup \mathscr{J}_n \subset U_n$. For each $z \in Z$ and $n \in \omega$ take $F_n(z) \in \mathscr{J}_n$ containing z. In addition, for each $z \in Z$ define $F(z) := \bigcap_{n \in \omega} F_n(z)$. Then $F(z) \in \mathscr{S}$ for each $z \in Z$, hence $F(z)$ is supercompact, $\bigcup_{z \in Z} F(z) = Z$ and the collection $\{F(z) | z \in Z\}$ has cardinality at most 2^ω. Since $|K| > 2^\omega$ there is a $z \in Z$ and a countably infinite subset $E \subset K$ such that $E \subset f[F(z)]$. By a theorem in van Douwen & van Mill [7] it follows that at least one cluster point of E is the limit of a nontrivial convergent sequence in $f[F(z)]$. This completes the proof. □

1.2. COROLLARY: *Suppose that βX is a continuous image of a closed G_δ subset of a supercompact Hausdorff space. Then X is pseudocompact.*

PROOF: Assume that X is not pseudocompact. Then we may assume that $\omega \subset X$ and that ω is C-embedded in X (cf. Gillman & Jerison [8]). Then $\beta\omega - \omega \subset \beta X - X$ and since $|\beta\omega - \omega| = 2^{2^\omega}$ (cf. Gillman & Jerison [8]) by Theorem 0.1 there is an $x \in \beta\omega - \omega$ which is the limit of a nontrivial convergent sequence in βX. It is easily seen that this is impossible. □

Recall that Martin's axiom (MA) states that no compact *ccc* Hausdorff space is the union of less than 2^ω nowhere dense sets (cf. Martin & Solovay [10]). It is known (cf. Booth [5]) that MA implies $P(2^\omega)$, i.e. the statement that for every collection $\mathscr{A}$ of fewer than 2^ω subsets of ω such that each finite subcollection of $\mathscr{A}$ has infinite intersection there is an infinite $F \subset \omega$ such that $F - A$ is finite for all $A \in \mathscr{A}$. It is easily seen that $P(2^\omega)$ implies that $\beta\omega - \omega$ is not the union of 2^ω nowhere dense sets. This implies that, under $P(2^\omega)$, every compactification $\gamma\omega$ of ω with the property that no sequence in ω converges has cardinality greater than 2^ω. For let $\gamma\omega$ be such a compactification of ω and let $f: \beta\omega \to \gamma\omega$ be the unique continuous surjection which extends the identity on ω. Now the fact that no sequence in ω converges implies that $f^{-1}(x)$ is nowhere dense in $\beta\omega - \omega$ for all $x \in \gamma\omega - \omega$. Hence $P(2^\omega)$ implies that $|\gamma\omega - \omega| > 2^\omega$.

1.3. COROLLARY ($P(2^\omega)$): *Let X be a Hausdorff continuous image of a closed G_δ subset of a supercompact Hausdorff space. If X is infinite then X contains a nontrivial convergent sequence.*

PROOF: If $|X| > 2^\omega$ then this follows from Theorem 0.1. On the other hand, if $|X| \leqslant 2^\omega$ then this follows from $P(2^\omega)$. □

1.4. QUESTION: *Is Corollary 1.3 true in ZFC?*

2. PROOF OF THEOREM 0.2

Recall that a family of subsets $\mathscr{A}$ of ω is called *almost disjoint* provided that $A \cap B$ is finite for all distinct $A, B \in \mathscr{A}$. It is known that there is an almost disjoint family $\mathscr{A} \subset \mathscr{P}(\omega)$ of cardinality 2^{ω} (cf. Gillman & Jerison [8]). We need the following lemma.

2.1. LEMMA: *Let $\{A_\alpha | \alpha \in \varkappa\}$ be an uncountable (faithfully indexed) maximal almost disjoint family of infinite subsets of ω. If $\{P_n : \omega \to m_n\}$ is a sequence of partitions of ω into finitely many sets, then there is an $f \in {}^{\omega}\omega$ such that*

$$| \bigcap_{n \in \omega} \{\alpha | \, |A_\alpha \cap \bigcap_{i \in n} P_i^{-1}(f(i))| = \omega\}| \geqslant \omega_1.$$

PROOF: We choose $f(n) \in m_n$ by induction so that

(1) for every finite $F \subset \varkappa$ we have that
$| \bigcap_{i \in n} P_i^{-1}(f(i)) - \bigcup_{j \in F} A_j| = \omega.$

Indeed, suppose that $\{f(i) | i \in n\}$ have been defined such that (1) is satisfied. If $n = 0$, then define $f(0)$ to be an arbitrary element of m_0 such that for every finite $F \subset \varkappa$ we have that $|P_0^{-1}(f(0)) - \bigcup_{j \in F} A_j| = \omega$. It is clear that this is possible since m_0 is finite and $\varkappa$ is infinite. If $n \neq 0$ then define

$$M_{n-1} := \bigcap_{i \in n-1} P_i^{-1}(f(i))$$

and notice that

$$\mathscr{A}' = \{A_\alpha \cap M_{n-1} | \, |A_\alpha \cap M_{n-1}| = \omega\}$$

is an uncountable maximal almost disjoint family of infinite subsets of M_{n-1}. Since $P_n \restriction M_{n-1}$ is a partition of M_{n-1} and since M_{n-1} is infinite by induction hypothesis there is an $m \in m_n$ such that

$$|(P_n \restriction M_{n-1})^{-1}(m) - \cup \mathscr{J}| = \omega$$

for every finite subcollection $\mathscr{J} \subset \mathscr{A}'$. Now define $f(n) := m$; then it is clear that (1) is satisfied.

Suppose that there are only countably many α, say $\{\alpha_m | m \in \omega\}$, such that for all $n, m \in \omega$ we have that $|A_{\alpha_m} \cap \bigcap_{i \in n} P_i^{-1}(f(i))| = \omega$. Then we may pick, by (1), distinct $p_n \in \omega$ such that

$$p_n \in \bigcap_{i \in n} P^{-1}(f(i)) - \bigcup_{j \in n} A_{\alpha_j} \qquad (n \in \omega).$$

Define $A := \{p_n | n \in \omega\}$.

There are two cases: suppose first that $A \in \{A_\alpha | \alpha \in \varkappa\}$. Then, since $|A \cap \bigcap_{i \in n} P_i^{-1}(f(i))| = \omega$ for all $n \in \omega$ we have that $A = A_{\alpha_m}$ for some m, which is impossible by definition of the p_n's. Therefore $A \notin \{A_\alpha | \alpha \in \varkappa\}$. By maximality we can find a $\beta \in \varkappa$ such that $|A_\beta \cap A| = \omega$. Since

$$|A - \bigcap_{i \in n} P_i^{-1}(f(i))| < \omega \text{ for all } n \in \omega$$

we conclude that

$$|A_\beta \cap \bigcap_{i\in n} P_i^{-1}(f(i))| = \omega \text{ for all } n \in \omega,$$

so $\beta = \alpha_m$ for some m. But since $|A \cap A_{\alpha_n}| < \omega$ for all $n \in \omega$ we have a contradiction. □

We now can prove the main result in this section.

2.2. PROOF OF THEOREM 0.2: List $\mathscr{M}$ as $\{M_\alpha | \alpha \in \varkappa\}$. Assume that Y is a supercompact Hausdorff space, that $Z \subset Y$ is a closed G_δ and that $g: Z \to \gamma X_{\mathscr{M}}$ is a continuous surjection from Z onto the compactification $\gamma X_{\mathscr{M}}$ of $X_{\mathscr{M}}$. Let $\mathscr{S}$ be a binary subbase for Y which is closed under arbitrary intersections. Let $\{U_n | n \in \omega\}$ be a sequence of open subsets of Y whose intersection is Z. Since $U_n - g^{-1}(n)$ is a neighborhood of $Z - g^{-1}(n)$ and since $Z - g^{-1}(n)$ is closed in Y, we can find $S_0^n, \ldots, S_{m_n-2}^n \in \mathscr{S}$ such that $U_n - g^{-1}(n) \supset S_0^n \cup \ldots \cup S_{m_n-2}^n \supset Z - g^{-1}(n)$. For each $n \in \omega$ pick $d_n \in Z$ such that $g(d_n) = n$. Define $D := \{d_n | n \in \omega\}$. Take $P_n: \omega \to m_n$ to be a partition refining $\{S_j^n \cap D | j \in m_n - 1\} \cup \{d(i) | i \in n\}$, in such a way that $P_n^{-1}(j) \subset S_j^n \cap D$ for each $j \in m_n - 1$ and $P_n^{-1}(\{m_n - 1\}) = \{d(i) | i \in n\}$. For each $\alpha \in \varkappa$ let $A_\alpha := \{d(n) | n \in M_\alpha\}$. Now pick f as in Lemma 2.1. We then have, by the compactness of Z, that

$$g(\bigcap_{n\in\omega} S_{f(n)}^n) \supset \bigcap_{n\in\omega} \{\alpha|\ |A_\alpha \cap \bigcap_{i\in n} P_i^{-1}(f(i))| = \omega\}.$$

Let $S := \bigcap_{n\in\omega} S_{f(n)}^n$. Notice that $S \subset Z - g^{-1}(\omega)$ and in addition that S is uncountable by Lemma 2.1.

For each $\alpha \in \varkappa$ the set $g^{-1}(M_\alpha \cup \{\alpha\})$ is open and closed in Z. Hence we may take an open set $V_\alpha \subset Y (\alpha \in \varkappa)$ such that

$$\mathrm{cl}_Y (V_\alpha) \cap Z = V_\alpha \cap Z = g^{-1}(M_\alpha \cup \{\alpha\}).$$

Notice that for distinct $\alpha, \beta \in \varkappa$ we have that $V_\alpha \cup V_\beta \subset g^{-1}(\omega) \cup (Y - Z)$. Set $H = \bigcap_{n\in\omega} \{\alpha|\ |A_\alpha \cap \bigcap_{i\in n} P_i^{-1}(f(i))| = \omega\}$. For each $\alpha \in H$ let $\mathscr{J}_\alpha$ be a finite subcollection of $\mathscr{S}$ such that $g^{-1}(M_\alpha \cup \{\alpha\}) \subset \cup \mathscr{J}_\alpha \subset V_\alpha$. Since $\mathscr{J}_\alpha$ is finite we may take $S_\alpha \in \mathscr{J}_\alpha$ such that $|A_\alpha \cap \bigcap_{i\in n} P_i^{-1}(f(i)) \cap S_\alpha| = \omega$ for all $n \in \omega$. Since D is countable and H is uncountable there exist distinct $\alpha, \beta \in H$ such that $S_\alpha \cap S_\beta \neq \emptyset$. It is clear that

$$S_\alpha \cap S_\beta \cap S = S_\alpha \cap S_\beta \cap \bigcap_{n\in\omega} S_{f(n)}^n \subset V_\alpha \cap V_\beta \cap (Z - g^{-1}(\omega)) = \emptyset.$$

Therefore, since $\mathscr{S}$ is binary and since $S_\alpha \cap S_\beta \neq \emptyset$, we may assume, without loss of generality, that there is an $n_0 \in \omega$ such that $S_\alpha \cap S_{f(n_0)}^{n_0} = \emptyset$. However, since $P_{n_0}^{-1}(f(n_0)) \subset S_{f(n_0)}^{n_0}$ and since $|A_\alpha \cap \bigcap_{i\in n_0} P_i^{-1}(f(i)) \cap S_\alpha| = \omega$ this is a contradiction. □

3. DENSITY OF CLOSED G_δ'S IN SUPERCOMPACT HAUSDORFF SPACES

In this section we show that if Z is a closed G_δ in a supercompact Hausdorff space X then $d(Z) \leqslant 2^\omega d(X)$.

Recall that the density $d(X)$ of a topological space X is the least cardinal $\varkappa$ for which there is a dense subset of cardinality $\varkappa$.

If $\mathscr{S}$ is a binary subbase for X then for all $A \subset X$ we define $I(A) \subset X$ by

$$I(A) := \cap \{S \in \mathscr{S} | A \subset S\}.$$

Notice that $\mathrm{cl}_X(A) \subset I(A)$, since each element of $\mathscr{S}$ is closed, that $I(I(A)) = I(A)$ and that $I(A) \subset I(B)$ if $A \subset B \subset X$. The following lemma was proved in van Douwen & van Mill [7]. For the sake of completeness we will give its proof here also.

3.1. LEMMA: *Let $\mathscr{S}$ be a binary subbase for the supercompact Hausdorff space X. Let $p \in X$. If U is a neighborhood of p and if A is a subset of X with $p \in \mathrm{cl}_X(A)$, then there is a subset $B \subset A$ with $p \in \mathrm{cl}_X(B)$ and $I(B) \subset U$.*

PROOF: Since X is regular, p has a neighborhood V such that $p \in \mathrm{cl}_X(V) \subset U$. Let $\mathscr{I}$ denote the collection of finite intersections of elements from $\mathscr{S}$. Choose a finite $\mathscr{J} \subset \mathscr{I}$ such that $\mathrm{cl}_X(V) \subset \cup \mathscr{J} \subset U$. Now $\mathscr{J}$ is finite, and $A \cap V \subset \cup \mathscr{J}$, and $p \in \mathrm{cl}_X(A \cap V)$; hence there is an $S \in \mathscr{J}$ with $p \in \mathrm{cl}_X(A \cap V \cap S)$. Let $B := A \cap V \cap S$. Then $p \in \mathrm{cl}_X(B)$, and $B \subset A$, and $I(B) \subset S \subset \cup \mathscr{J} \subset U$. □

We now prove the main result in this section.

3.2. THEOREM: *Let $\mathscr{S}$ be a binary subbase for the Hausdorff space X. Then $d(S) \leqslant d(X)$ for all $S \in \mathscr{S}$.*

PROOF: Let D be a dense subset of X and choose $S \in \mathscr{S}$. For each $d \in D$ choose a point $e(d) \in \bigcap_{S \in \mathscr{S}} I(\{d, s\}) \cap S$. Notice that this is possible since $\mathscr{S}$ is binary. We claim that $E := \{e(d) | d \in D\}$ is dense in S. Indeed, take $x \in S$ and let U be any neighborhood of x. By Lemma 3.1 there is a subset $B \subset D$ such that x is in the closure of B and $I(B) \subset U$. Choose $d_0 \in D$ arbitrarily. Then

$e(d_0) \in \bigcap_{s \in S} I(\{d_0, s\}) \cap S \subset I(\{d_0, x\}) \cap S \subset I(B) \cap S \subset U \cap S$.

This completes the proof. □

3.3. COROLLARY: *Let Z be a closed G_δ subset in a supercompact Hausdorff space X. Then $d(Z) \leqslant 2^\omega d(X)$.*

PROOF: Let $\mathscr{S}$ be a binary subbase for X which is closed under arbitrary intersections. As in the proof of Theorem 0.1, Z is the union of a family of at most 2^ω subsets of $\mathscr{S}$. Hence Theorem 3.2 implies that $d(Z) \leqslant 2^\omega d(X)$. □

4. OPEN QUESTIONS

The results derived in this note suggest many questions. As noted in the introduction Bell [3] has shown that a closed G_δ subset of a supercompact Hausdorff space need not be supercompact. This suggests the following question.

4.1. QUESTION: *Suppose that Z is a closed G_δ in a supercompact Hausdorff space X. Is $cmpn(Z)$ finite?*

(Recall that for compact Hausdorff spaces X, $cmpn(X)$ is the least integer k for which there is a closed subbase $\mathscr{S}$ for X such that if $\mathscr{M} \subset \mathscr{S}$ with $\cap \mathscr{M} = \emptyset$ then there is a subset of $\mathscr{M}$ of cardinality k which has an empty intersection; $cmpn(X) = \infty$ if such an integer does not exist (cf. Bell & van Mill [4]). It is known, cf. [4], that for every $k \geqslant 1$ there is a compact Hausdorff space X_k for which $cmpn(X_k) = k$; in addition $cmpn(\beta\omega) = \infty$). Related to this question is the following one:

4.2. QUESTION: *Suppose that βX is a continuous image of a closed G_δ of a compact Hausdorff space Y with $cmpn(Y) < \infty$. Is X pseudocompact?*

4.3. QUESTION: *Let X be an infinite compact Hausdorff space for which $cmpn(X) < \infty$. Does X contain a copy of ω which is not C^*-embedded in X? a nontrivial convergent sequence?*

In section 2 we gave an example of a compact Hausdorff space X which is the union of three metrizable subspaces and which is not embeddable as a G_δ subset in a supercompact Hausdorff space. This suggests the following question.

4.4. QUESTION: *Let X be a compact Hausdorff space which is the union of two metrizable subspaces. Can X be embedded as a G_δ subset in a supercompact Hausdorff space?*

REFERENCES

1. Bell, M. G. – Not all compact Hausdorff spaces are supercompact, Gen. Top. Appl. 8, 151–155 (1978).
2. Bell, M. G. – A cellular constraint in supercompact Hausdorff spaces, Canadian J. Math. 30, 1144–1151 (1978).
3. Bell, M. G. – A first countable supercompact Hausdorff space with a closed G_δ non-supercompact subspace (to appear in Colloq. Math.).
4. Bell, M. G. & J. van Mill – The compactness number of a compact topological space (to appear in Fund. Math.).
5. Booth, D. – Ultrafilters on a countable set, Ann. Math. Logic 2, 1–24 (1970).
6. Douwen, E. K. van – Special bases for compact metrizable spaces, (to appear in Fund. Math.).

7. Douwen E. K. van & J. van Mill – Supercompact spaces, (to appear in Gen. Top. Appl.).
8. Gillman, L. & M. Jerison – Rings of continuous functions, Princetion, N. J. (1960).
9. Groot, J. de – Supercompactness and superextensions, in: Contributions to extension theory of topological structures, Symp. Berlin 1967, Deutscher Verlag Wiss., Berlin 89–90 (1969).
10. Martin, D. A. & R. M. Solovay – Internal Cohen extensions, Ann. Math. Logic 2, 143–178 (1970).
11. Mill, J. van – A countable space no compactification of which is supercompact, Bull. l'acad. Pol. Sci., **25** (1977) 1129–1132.
12. Mills, C. F. – A simpler proof that compact metric spaces are supercompact, (to appear in Proc. Amer. Math. Soc.).
13. Rudin, M. E. – Lectures on set theoretic topology, Regional Conf. Ser. in Math., no. 23, Am. Math. Soc., Providence, RI (1975).
14. Strok, M. & A. Szymański – Compact metric spaces have binary bases, Fund. Math. **89**, 81–91 (1975).