Closed $G_δ$ subsets of supercompact Hausdorff spaces

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ABSTRACT

We give examples of compact Hausdorff spaces which are not embeddable as closed $G_δ$ subsets in a supercompact Hausdorff space.

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INTRODUCTION

A supercompact space is a space which has a binary subbase for its closed subsets, where a collection of subsets $\mathcal{S}$ of a set $X$ is called binary provided that for all $\mathcal{M} \subseteq \mathcal{S}$ with $\bigcap \mathcal{M} = \emptyset$ there are $M_0, M_1 \in \mathcal{M}$ with $M_0 \cap M_1 = \emptyset$. By Alexander's subbase lemma, every supercompact space is compact. The class of supercompact spaces was introduced by de Groot [9]. Many spaces are supercompact, for example all compact metric spaces, cf. Strok & Szymanski [14] (elementary proofs of this fact were recently found by van Douwen [6] and Mills [12]). The first examples of nonsupercompact compact Hausdorff spaces were found by Bell [1]. At the moment there is a variety of nonsupercompact compact Hausdorff spaces (cf. Bell [1], [2], van Douwen & van Mill [7], van Mill [11], Bell & van Mill [4]).

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Recently, Bell [3] showed that the one point compactification of the Cantor tree $\omega^2 \cup \omega^2$ (cf. Rudin [13]) can be embedded as a closed $G_\delta$ subset of a supercompact Hausdorff space. Since the one point compactification of the Cantor tree is not supercompact (cf. van Douwen & van Mill [7]) this yields an example of a nonsupercompact closed $G_\delta$ in a supercompact Hausdorff space. This suggests the question whether every compact Hausdorff space can be embedded as a $G_\delta$ subset in a supercompact Hausdorff space. The answer to this question is in the negative.

0.1. theorem: Let $X$ be a Hausdorff continuous image of a closed $G_\delta$ subset of a supercompact Hausdorff space, and let $K$ be a closed subset of $X$ such that $|K| > 2^\omega$. Then at least one point of $K$ is the limit of a nontrivial convergent sequence in $X$ (not necessarily in $K$).

This theorem is a consequence of a result in van Douwen & van Mill [7]. As a corollary, if $\beta X$ is a continuous image of a closed $G_\delta$ subset of a supercompact Hausdorff space then $X$ is pseudocompact. Also, under Martin's axiom (MA), every infinite Hausdorff continuous image of a closed $G_\delta$ subset of a supercompact Hausdorff space contains a nontrivial convergent sequence.

Since the one point compactification of the Cantor tree is a compactification of $\omega$ with the one point compactification of a discrete space as remainder, Bell's [3] result suggests the question whether every compactification of $\omega$ with the one point compactification of a discrete space as remainder can be embedded as a $G_\delta$ subset of a supercompact Hausdorff space. The answer to this question is in the negative. For every (faithfully indexed) almost disjoint family $\mathcal{M} = \{M_\alpha | \alpha \in \omega\}$ of infinite subsets of $\omega$ define $X_{\mathcal{M}}$ to be the space with underlying set the disjoint union of $\omega$ and $\omega$ and with topology generated by the collection

$$\{\{n\} \cup (M_\alpha \cup n) | \alpha \in \omega, n \in \omega\} \cup \{\{n\} | n \in \omega\}.$$

Notice that $X_{\mathcal{M}}$ is separable and that every subspace of $X_{\mathcal{M}}$ is locally compact and first countable. Also, the Cantor tree $\omega^2 \cup \omega^2$ is homeomorphic to some $X_{\mathcal{M}}$. We will prove the following theorem:

0.2. theorem: Let $\mathcal{M}$ be a maximal uncountable almost disjoint collection of infinite subsets of $\omega$. Then any compactification of $X_{\mathcal{M}}$ is not the continuous image of a closed $G_\delta$ subset of a supercompact Hausdorff space.

1. theorem 0.1; proof and consequences

1.1. proof of theorem 0.1: Indeed, let $Y$ be a supercompact Hausdorff space, let $X$ and $K$ be as in Theorem 0.1 and let $Z$ be a closed $G_\delta$ in $Y$ which is mapped by $f$ onto $X$. Write $Z = \bigcap_{n \in \omega} U_n$, where the $U_n$'s are open subsets of $Y$. It is easily verified that a space has a binary
subbase if and only if it has a binary subbase closed under arbitrary intersections. Let \( S \) be a binary subbase for \( Y \) which is closed under arbitrary intersections. For each \( n \in \omega \) let \( J_n \) be a finite subcollection of \( S \) such that \( Z \subseteq \cup J_n \subseteq U_n \). For each \( z \in Z \) and \( n \in \omega \) take \( F_n(z) \in J_n \) containing \( z \). In addition, for each \( z \in Z \) define \( F(z) := \cap_{n \in \omega} F_n(z) \). Then \( F(z) \in S \) for each \( z \in Z \), hence \( F(z) \) is supercompact, \( \bigcup_{z \in Z} F(z) = Z \) and the collection \( \{ F(z) \mid z \in Z \} \) has cardinality at most \( 2^\omega \). Since \( |K| > 2^\omega \) there is a \( z \in Z \) and a countably infinite subset \( E \subseteq K \) such that \( E \subseteq f(F(z)) \). By a theorem in van Douwen & van Mill [7] it follows that at least one cluster point of \( E \) is the limit of a nontrivial convergent sequence in \( f(F(z)) \). This completes the proof.

1.2. Corollary: Suppose that \( \beta X \) is a continuous image of a closed \( G_\delta \) subset of a supercompact Hausdorff space. Then \( X \) is pseudocompact.

Proof: Assume that \( X \) is not pseudocompact. Then we may assume that \( \omega \subseteq X \) and that \( \omega \) is \( C \)-embedded in \( X \) (cf. Gillman & Jerison [8]). Then \( \beta \omega \cap X = X - X \) and since \( |\beta \omega - \omega| = 2^\omega \) (cf. Gillman & Jerison [8]) by Theorem 0.1 there is an \( x \in \beta \omega - \omega \) which is the limit of a nontrivial convergent sequence in \( \beta X \). It is easily seen that this is impossible.

Recall that Martin's axiom \((MA)\) states that no compact \( ccc \) Hausdorff space is the union of less than \( 2^\omega \) nowhere dense sets (cf. Martin & Solovay [10]). It is known (cf. Booth [5]) that \( MA \) implies \( P(2^\omega) \), i.e. the statement that for every collection \( \mathcal{A} \) of fewer than \( 2^\omega \) subsets of \( \omega \) such that each finite subcollection of \( \mathcal{A} \) has infinite intersection there is an infinite \( F \subseteq \omega \) such that \( F - A \) is finite for all \( A \in \mathcal{A} \). It is easily seen that \( P(2^\omega) \) implies that \( \beta \omega - \omega \) is not the union of \( 2^\omega \) nowhere dense sets. This implies that, under \( P(2^\omega) \), every compactification \( \gamma \omega \) of \( \omega \) with the property that no sequence in \( \omega \) converges to \( \omega \) has cardinality greater than \( 2^\omega \). For let \( \gamma \omega \) be such a compactification of \( \omega \) and let \( f: \beta \omega \to \gamma \omega \) be the unique continuous surjection which extends the identity on \( \omega \). Now the fact that no sequence in \( \omega \) converges implies that \( f^{-1}(x) \) is nowhere dense in \( \beta \omega - \omega \) for all \( x \in \gamma \omega - \omega \). Hence \( P(2^\omega) \) implies that \( |\gamma \omega - \omega| > 2^\omega \).

1.3. Corollary \((P(2^\omega))\): Let \( X \) be a Hausdorff continuous image of a closed \( G_\delta \) subset of a supercompact Hausdorff space. If \( X \) is infinite then \( X \) contains a nontrivial convergent sequence.

Proof: If \( |X| > 2^\omega \) then this follows from Theorem 0.1. On the other hand, if \( |X| < 2^\omega \) then this follows from \( P(2^\omega) \).

1.4. Question: Is Corollary 1.3 true in ZFC?
2. PROOF OF THEOREM 0.2

Recall that a family of subsets $\mathcal{A}$ of $\omega$ is called almost disjoint provided that $A \cap B$ is finite for all distinct $A, B \in \mathcal{A}$. It is known that there is an almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of cardinality $2^\omega$ (cf. Gillman & Jerison [8]). We need the following lemma.

2.1. LEMMA: Let $\{A_\alpha | \alpha \in \kappa\}$ be an uncountable (faithfully indexed) maximal almost disjoint family of infinite subsets of $\omega$. If $\{P_n : \omega \to m_n\}$ is a sequence of partitions of $\omega$ into finitely many sets, then there is an $f \in \omega^\omega$ such that

$$|\bigcap_{n \in \omega} \{\alpha | A_\alpha \cap \bigcap_{i \in \kappa} P_i^{-1}(f(i)) = \omega\}| > \omega_1.$$

PROOF: We choose $f(n) \in m_n$ by induction so that

(1) for every finite $F \subseteq \kappa$ we have that

$$|\bigcap_{i \in \kappa} P_i^{-1}(f(i)) - \bigcup_{j \in F} A_j| = \omega.$$

Indeed, suppose that $\{f(i) | i \in n\}$ have been defined such that (1) is satisfied. If $n = 0$, then define $f(0)$ to be an arbitrary element of $m_0$ such that for every finite $F \subseteq \kappa$ we have that $|P_0^{-1}(f(0)) - \bigcup_{j \in F} A_j| = \omega$. It is clear that this is possible since $m_0$ is finite and $\kappa$ is infinite. If $n \neq 0$ then define

$$M_{n-1} := \bigcap_{i \in \kappa} P_i^{-1}(f(i))$$

and notice that

$$\mathcal{A}' = \{A_\alpha \cap M_{n-1} | A_\alpha \cap M_{n-1} = \omega\}$$

is an uncountable maximal almost disjoint family of infinite subsets of $M_{n-1}$. Since $P_n \upharpoonright M_{n-1}$ is a partition of $M_{n-1}$ and since $M_{n-1}$ is infinite by induction hypothesis there is an $m \in m_n$ such that

$$|(P_n \upharpoonright M_{n-1})^{-1}(m) - \bigcup \mathcal{J}| = \omega$$

for every finite subcollection $\mathcal{J} \subseteq \mathcal{A}'$. Now define $f(n) := m$; then it is clear that (1) is satisfied.

Suppose that there are only countably many $\alpha$, say $\{\alpha_m | m \in \omega\}$, such that for all $n, m \in \omega$ we have that $|A_{\alpha_m} \cap \bigcap_{i \in \kappa} P_i^{-1}(f(i))| = \omega$. Then we may pick, by (1), distinct $p_n \in \omega$ such that

$$p_n \in \bigcap_{i \in \kappa} P_i^{-1}(f(i)) - \bigcup_{j \in \kappa} A_{\alpha_j}, \quad (n \in \omega).$$

Define $A := \{p_n | n \in \omega\}$.

There are two cases: suppose first that $A \in \{A_\alpha | \alpha \in \kappa\}$. Then, since $|A \cap \bigcap_{i \in \kappa} P_i^{-1}(f(i))| = \omega$ for all $n \in \omega$ we have that $A = A_{\alpha_m}$ for some $m$, which is impossible by definition of the $p_n$’s. Therefore $A \notin \{A_\alpha | \alpha \in \kappa\}$. By maximality we can find a $\beta \in \kappa$ such that $|A_\beta \cap A| = \omega$. Since

$$|A - \bigcap_{i \in \kappa} P_i^{-1}(f(i))| < \omega$$

for all $n \in \omega$.

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we conclude that
\[ |A_\beta \cap \bigcap_{i \in \mathbb{N}} P_i^{-1}(f(i))| = \omega \text{ for all } n \in \omega, \]
so \( \beta = x_m \) for some \( m \). But since \( |A \cap A_{\alpha_n}| < \omega \) for all \( n \in \omega \) we have a contradiction.

We now can prove the main result in this section.

2.2. PROOF OF THEOREM 0.2: List \( \mathcal{M} \) as \( \{M_\alpha| \alpha \in \mathbb{X}\} \). Assume that \( Y \) is a supercompact Hausdorff space, that \( Z \subset Y \) is a closed \( G_\delta \) and that \( g: Z \to \gamma X_\mathcal{M} \) is a continuous surjection from \( Z \) onto the compactification \( \gamma X_\mathcal{M} \) of \( X_\mathcal{M} \). Let \( \mathcal{S} \) be a binary subbase for \( Y \) which is closed under arbitrary intersections. Let \( \{U_n| n \in \omega\} \) be a sequence of open subsets of \( Y \) whose intersection is \( Z \). Since \( U_n - g^{-1}(n) \) is a neighborhood of \( Z - g^{-1}(n) \) and since \( Z - g^{-1}(n) \) is closed in \( Y \), we can find \( S_0^n, \ldots, S_{m_n-2}^n \in \mathcal{S} \) such that \( U_n - g^{-1}(n) \supseteq S_0^n \cup \cdots \cup S_{m_n-2}^n \supseteq Z - g^{-1}(n) \). For each \( n \in \omega \) pick \( d_n \in Z \) such that \( g(d_n) = n \). Define \( D := \{d_n| n \in \omega\} \). Take \( P_n: \omega \to m_n \) to be a partition refining \( \{S_j^n \cap D| j \in m_n - 1\} \cup \{d(i)| i \in n\} \), in such a way that \( P_n^{-1}(j) \subseteq S_j^n \cap D \) for each \( j \in m_n - 1 \) and \( P_n^{-1}(\{m_n - 1\}) = \{d(i)| i \in n\} \). For each \( \alpha \in \mathbb{X} \) let \( A_\alpha := \{d(n)| n \in M_\alpha\} \). Now pick \( j \) as in Lemma 2.1. We then have, by the compactness of \( Z \), that
\[ g(\bigcap_{n \in \omega} S_j^n) \supseteq \bigcap_{n \in \omega} \{x| A_\alpha \cap \bigcap_{i \in \mathbb{N}} P_i^{-1}(f(i))| = \omega\}. \]
Let \( S := \bigcap_{n \in \omega} S_j^n \). Notice that \( S \subset Z - g^{-1}(\omega) \) and in addition that \( S \) is uncountable by Lemma 2.1.

For each \( \alpha \in \mathbb{X} \) the set \( g^{-1}(M_\alpha \cup \{x\}) \) is open and closed in \( Z \). Hence we may take an open set \( V_\alpha \subset Y(\alpha \in \mathbb{X}) \) such that
\[ \text{cl}_Y (V_\alpha) \cap Z = V_\alpha \cap Z = g^{-1}(M_\alpha \cup \{x\}). \]
Notice that for distinct \( \alpha, \beta \in \mathbb{X} \) we have that \( V_\alpha \cup V_\beta \subset g^{-1}(\omega) \cup (Y - Z) \).
Set \( H = \bigcap_{n \in \omega} \{\alpha| A_\alpha \cap \bigcap_{i \in \mathbb{N}} P_i^{-1}(f(i))| = \omega\} \). For each \( \alpha \in H \) let \( \mathcal{J}_\alpha \) be a finite subcollection of \( \mathcal{S} \) such that \( g^{-1}(M_\alpha \cup \{x\}) \subset \bigcup \mathcal{J}_\alpha \subset V_\alpha \). Since \( \mathcal{J}_\alpha \) is finite we may take \( S_\alpha \in \mathcal{J}_\alpha \) such that \( |A_\alpha \cap \bigcap_{i \in \mathbb{N}} P_i^{-1}(f(i)) \cap S_\alpha| = \omega \) for all \( n \in \omega \). Since \( D \) is countable and \( H \) is uncountable there exist distinct \( \alpha, \beta \in H \) such that \( S_\alpha \cap S_\beta \neq \emptyset \). It is clear that
\[ S_\alpha \cap S_\beta \cap S = S_\alpha \cap S_\beta \cap \bigcap_{n \in \omega} S_j^n \subset V_\alpha \cap V_\beta \cap (Z - g^{-1}(\omega)) = \emptyset. \]
Therefore, since \( \mathcal{S} \) is binary and since \( S_\alpha \cap S_\beta \neq \emptyset \), we may assume, without loss of generality, that there is an \( n_0 \in \omega \) such that \( S_\alpha \cap S_j^{n_0} = \emptyset \).
However, since \( P_j^{-1}(f(n_0)) \subseteq S_j^{n_0} \), and since \( |A_\alpha \cap \bigcap_{i \in n_0} P_i^{-1}(f(i)) \cap S_\alpha| = \omega \) this is a contradiction.

3. DENSITY OF CLOSED \( G_\delta \)'S IN SUPERCOMPACT HAUSDORFF SPACES

In this section we show that if \( Z \) is a closed \( G_\delta \) in a supercompact Hausdorff space \( X \) then \( d(Z) \leq 2^\omega d(X) \).
Recall that the density \(d(X)\) of a topological space \(X\) is the least cardinal \(\kappa\) for which there is a dense subset of cardinality \(\kappa\).

If \(\mathcal{S}\) is a binary subbase for \(X\) then for all \(A \subset X\) we define \(I(A) \subset X\) by
\[
I(A) := \cap \{ S \in \mathcal{S} | A \subset S \}.
\]

Notice that \(\text{cl}_X (A) \subset I(A)\), since each element of \(\mathcal{S}\) is closed, that \(I(I(A)) = I(A)\) and that \(I(A) \subset I(B)\) if \(A \subset B \subset X\). The following lemma was proved in van Douwen & van Mill [7]. For the sake of completeness we will give its proof here also.

3.1. Lemma: Let \(\mathcal{S}\) be a binary subbase for the supercompact Hausdorff space \(X\). Let \(p \in X\). If \(U\) is a neighborhood of \(p\) and if \(A\) is a subset of \(X\) with \(p \in \text{cl}_X (A)\), then there is a subset \(B \subset A\) with \(p \in \text{cl}_X (B)\) and \(I(B) \subset U\).

Proof: Since \(X\) is regular, \(p\) has a neighborhood \(V\) such that \(p \in \text{cl}_X (V) \subset U\). Let \(\mathcal{I}\) denote the collection of finite intersections of elements from \(\mathcal{S}\). Choose a finite \(\mathcal{J} \subset \mathcal{I}\) such that \(\text{cl}_X (V) \subset \cup \mathcal{J} \subset U\). Now \(\mathcal{J}\) is finite, and \(A \cap V \subset \cup \mathcal{J}\), and \(p \in \text{cl}_X (A \cap V)\); hence there is an \(S \in \mathcal{J}\) with \(p \in \text{cl}_X (A \cap V \cap S)\). Let \(B := A \cap V \cap S\). Then \(p \in \text{cl}_X (B)\), and \(B \subset A\), and \(I(B) \subset S \subset \cup \mathcal{J} \subset U\).

We now prove the main result in this section.

3.2. Theorem: Let \(\mathcal{S}\) be a binary subbase for the Hausdorff space \(X\). Then \(d(S) < d(X)\) for all \(S \in \mathcal{S}\).

Proof: Let \(D\) be a dense subset of \(X\) and choose \(S \in \mathcal{S}\). For each \(d \in D\) choose a point \(e(d) \in \cap_{S \in \mathcal{S}} I(\{d, s\}) \cap S\). Notice that this is possible since \(\mathcal{S}\) is binary. We claim that \(E := \{ e(d) | d \in D \}\) is dense in \(S\). Indeed, take \(x \in S\) and let \(U\) be any neighborhood of \(x\). By Lemma 3.1 there is a subset \(B \subset D\) such that \(x\) is in the closure of \(B\) and \(I(B) \subset U\). Choose \(d_0 \in D\) arbitrarily. Then \(e(d_0) \in \cap_{S \in S} I(\{d_0, s\}) \cap S \subset I(\{d_0, x\}) \cap S \subset I(B) \cap S \subset U \cap S\).

This completes the proof.

3.3. Corollary: Let \(Z\) be a closed \(G_\delta\) subset in a supercompact Hausdorff space \(X\). Then \(d(Z) < 2^\omega d(X)\).

Proof: Let \(\mathcal{S}\) be a binary subbase for \(X\) which is closed under arbitrary intersections. As in the proof of Theorem 0.1, \(Z\) is the union of a family of at most \(2^\omega\) subsets of \(\mathcal{S}\). Hence Theorem 3.2 implies that \(d(Z) < 2^\omega d(X)\).
The results derived in this note suggest many questions. As noted in the introduction Bell [3] has shown that a closed $G_δ$ subset of a supercompact Hausdorff space need not be supercompact. This suggests the following question.

4.1. QUESTION: Suppose that $Z$ is a closed $G_δ$ in a supercompact Hausdorff space $X$. Is $\text{cmpn}(Z)$ finite?

(Recall that for compact Hausdorff spaces $X$, $\text{cmpn}(X)$ is the least integer $k$ for which there is a closed subbase $\mathcal{G}$ for $X$ such that if $M \subseteq \mathcal{G}$ with $\bigcap M = \emptyset$ then there is a subset of $M$ of cardinality $k$ which has an empty intersection; $\text{cmpn}(X) = \infty$ if such an integer does not exist (cf. Bell & van Mill [4]). It is known, cf. [4], that for every $k \geq 1$ there is a compact Hausdorff space $X_k$ for which $\text{cmpn}(X_k) = k$; in addition $\text{cmpn}(\beta\omega) = \infty$). Related to this question is the following one:

4.2. QUESTION: Suppose that $\beta X$ is a continuous image of a closed $G_δ$ of a compact Hausdorff space $Y$ with $\text{cmpn}(Y) < \infty$. Is $X$ pseudocompact?

4.3. QUESTION: Let $X$ be an infinite compact Hausdorff space for which $\text{cmpn}(X) < \infty$. Does $X$ contain a copy of $\omega$ which is not $C^*$-embedded in $X$? a nontrivial convergent sequence?

In section 2 we gave an example of a compact Hausdorff space $X$ which is the union of three metrizable subspaces and which is not embeddable as a $G_δ$ subset in a supercompact Hausdorff space. This suggests the following question.

4.4. QUESTION: Let $X$ be a compact Hausdorff space which is the union of two metrizable subspaces. Can $X$ be embedded as a $G_δ$ subset in a supercompact Hausdorff space?

REFERENCES