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SUPEREXTENSIONS WHICH ARE HILBERT CUBES

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Abstract

It is shown that each separable metric, not totally disconnected, topological space admits a superextension homeomorphic to the Hilbert cube. Moreover, for simple spaces, such as the closed unit interval or the n-spheres S_n, we give easily described subbases for which the corresponding superextension is homeomorphic to the Hilbert cube.

1. Introduction

In [6], DE GROOT defined a space X to be *supercompact* provided that it possesses a binary closed subbase, i.e., a closed subbase $\mathcal{S}$ with the property that if $\mathcal{S}' \subset \mathcal{S}$ and $\bigcap \mathcal{S}' = \emptyset$ then there exist $S_0, S_1 \in \mathcal{S}'$ such that $S_0 \cap S_1 = \emptyset$. Clearly, according to the lemma of ALEXANDER, every supercompact space is compact. The class of supercompact spaces contains the compact orderable spaces, compact tree-like spaces (BROUWER & SCHRIJVER [4], VAN MILL [10]) and compact metric spaces (STROK & SZYMAŃSKI [14]). Moreover, there are compact Hausdorff spaces which are not supercompact (BELL [2], VAN MILL [12]). There is a connection between supercompact spaces and graphs (see e.g., DE GROOT [7], BRUIJNING [5], SCHRIJVER [13]); moreover, supercompact spaces can be characterized by means of so-called interval structures (BROUWER & SCHRIJVER [4]).

Let X be a T_1-space and $\mathcal{S}$ a closed T_1-subbase for X (a closed subbase $\mathcal{S}$ for X is called T_1 if for all $S \in \mathcal{S}$ and $x \in X$ with $x \notin S$, there exists an $S_0 \in \mathcal{S}$ with $x \in S_0$ and $S_0 \cap S = \emptyset$). The *superextension* $\lambda_{\mathcal{S}}(X)$ *of* X *relative the subbase* $\mathcal{S}$ is the set of all maximal linked systems $\mathcal{M} \subset \mathcal{S}$ (a subsystem of $\mathcal{S}$ is called *linked* if every two of its members meet; a *maximal linked system* or *mls* is a linked system not properly contained in another linked system) topologized by taking $\{\{\mathcal{M} \in \lambda_{\mathcal{S}}(X) \mid S \in \mathcal{M}\} \mid S \in \mathcal{S}\}$ as a closed subbase. Clearly, this subbase is binary, hence $\lambda_S(X)$ is supercompact, while moreover X can be embedded in $\lambda_{\mathcal{S}}(X)$ by the natural embedding $i : X \to \lambda_{\mathcal{S}}(X)$ defined by $i(X) := \{S \in \mathcal{S} \mid x \in S\}$. VERBEEK's monograph [15] is a gocd place to find the basic theorems about superextensions. In this paper we will show that for many spaces there are superextensions homeomorphic to the Hilbert cube Q; moreover for simple spaces such as the unit interval or the n-spheres S_n we will present easily described subbases for which the corresponding super-

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extension is homeomorphic to Q. Here, a classical theorem of KELLER [8], which says that *each infinite-dimensional compact convex subset of the separable Hilbert space is homeomorphic to* Q (for a more up-to-date proof of this fact, see also BESSAGA & PEŁCZYŃSKI [3]), is of great help.

2. Some examples

In this section we will give some examples. If X is an ordered space, then the Dedekind completion of X will be denoted by $\overline{X}$. Roughly speaking, $\overline{X}$ can be obtained from X by filling up every gap. We define $\overline{\overline{X}}$ to be that ordered space wich can be obtained from X by filling up every gap with two points, except for possible endgaps, which we supply with one point. The compact space $\overline{\overline{X}}$ thus obtained, clearly contains X as a dense subspace. Define

$$\mathcal{G}_1 = \{A \subset X \mid \exists x \in X : A = (\leftarrow, x] \text{ or } A = [x, \rightarrow)\}$$

and

$$\mathfrak{T}_1 = \{A \subset X \mid A \text{ is a closed half-interval}\}$$

(as usual, a half-interval is a subset $A \subset X$ such that either for all $a, b \in X$: if $b \leq a \in A$ then $b \in A$, or for all a, $b \in X$: if $b \geq a \in A$ then $b \in A$) and

$$\mathfrak{T}_2 = \{A \subset X \mid \exists A_0, A_1 \in \mathfrak{T}_1 : A = A_0 \cup A_1 \text{ or } A = A_0 \cap A_1\},$$

respectively.

Notice that $\mathcal{G}_1$ equals $\mathfrak{T}_1$ in case X is compact or connected. It is easy to see that $\lambda_{\mathcal{G}_1}(X) \cong \overline{X}$ and that $\lambda_{\mathfrak{T}_1}(X) \cong \overline{\overline{X}}$.

What about $\lambda_{\mathfrak{T}_2}(X)$?

EXAMPLE (i). If $X = I$, then $\lambda_{\mathcal{G}_1}(X) = \lambda_{\mathfrak{T}_1}(X) \cong I$. On the other hand $\lambda_{\mathfrak{T}_2}(X)$ is homeomorphic to the Hilbert cube Q (see Section 4).

EXAMPLE (ii). If $X = \mathbf{Q}$, then $\lambda_{\mathcal{G}_1}(X) \cong I$ and $\lambda_{\mathfrak{T}_1}(X)$ is a non-metrizable separable compact ordered space, which has much in common with the well-known Alexandroff double of the closed unit interval. In this case, $\lambda_{\mathfrak{T}_2}(X)$ is a compact totally disconnected perfect space of weight $2^{\aleph_0}$. (The total disconnectedness of $\lambda_{\mathfrak{T}_2}(X)$ follows from the following observation: for every T_0, $T_1 \in \mathfrak{T}_2$ with $T_0 \cap T_1 = \emptyset$ there exists a $T_0' \in \mathfrak{T}_2$ such that $T_0 \subset T_0'$ and $T_0' \cap T_1 = \emptyset$ and $X \setminus T_0' \in \mathfrak{T}_2$. For every finite linked system $\{X \setminus T_i \mid T_i \in \mathfrak{T}_2, i \in \{1, 2, \ldots, n\}\}$ it is easy to construct two distinct mls's $\mathfrak{L}_0$ and $\mathfrak{L}_1$ belonging to $\bigcap_{i=1}^{n} \{\mathfrak{M} \in \lambda_{\mathfrak{T}_2}(X) \mid T_i \notin \mathfrak{M}\}$ showing that $\lambda_{\mathfrak{T}_2}(X)$ is perfect. Finally $\lambda_{\mathfrak{T}_1}$ can be embedded in $\lambda_{\mathfrak{T}_2}(X)$; hence weight $(\lambda_{\mathfrak{T}_2}(X)) = 2^{\aleph_0}$.

EXAMPLE (iii). If $X = \mathbf{R}\setminus\mathbf{Q}$, then $\lambda_{\mathcal{G}_1}(X) \simeq I$, while $\lambda_{\mathcal{S}_1}(X) \simeq \lambda_{\mathcal{S}_2}(X) \simeq C$, the Cantor discontinuum, for it is easy to see that $\lambda_{\mathcal{S}_1}(X)$ and $\lambda_{\mathcal{S}_2}(X)$ both are totally disconnected compact metric perfect spaces.

Finally define

$$\mathcal{G}_2 = \{A \subset X \mid \exists A_0,\ A_1 \in \mathcal{G}_1 \colon A = A_0 \cup A_1 \text{ or } A = A_0 \cap A_1\}.$$

Notice that $\mathcal{G}_2$ equals T_2 in case X is compact or connected.

EXAMPLE (i). If $X = I$, then $\lambda_{\mathcal{G}_2}(X) \simeq Q$ (Section 4).

EXAMPLE (ii). If $X = \mathbf{Q}$, then $\lambda_{\mathcal{G}_2}(X) \simeq Q$.

EXAMPLE (iii). If $X = \mathbf{R}\setminus\mathbf{Q}$, then $\lambda_{\mathcal{G}_2}(X) \simeq Q$.

The fact that $\lambda_{\mathcal{G}_2}(\mathbf{Q}) \simeq \lambda_{\mathcal{G}_2}(\mathbf{R}\setminus\mathbf{Q}) \simeq Q$ can be derived from the result $\lambda_{\mathcal{G}_2}(I) \simeq Q$. To see this, define

$$\mathcal{G}_2' = \{A \subset I \mid A \in \mathcal{G}_2 \text{ and } A \text{ has rational endpoints}\}$$

and

$$\mathcal{G}_2'' = \{A \subset I \mid A \in \mathcal{G}_2 \text{ and } A \text{ has irrational endpoints}\}.$$

By Theorem 5 and Theorem 7 of [11] (cf. Theorem 3.1 below), it follows that

$$\lambda_{\mathcal{G}_2}(I) \simeq \lambda_{\mathcal{G}_2'}(I) \simeq \lambda_{\mathcal{G}_2}(\mathbf{Q})$$

and

$$\lambda_{\mathcal{G}_2}(I) \simeq \lambda_{\mathcal{G}_2''}(I) \simeq \lambda_{\mathcal{G}_2}(\mathbf{R}\setminus\mathbf{Q}).$$

3. Superextensions which are Hilbert cubes

In this section we will show that for each separable metric, not totally disconnected topological space X, there exists a normal closed T_1-subbase $\mathcal{S}$ such that $\lambda_{\mathcal{S}}(X)$ is homeomorphic to the Hilbert cube Q. First we will give some preliminary definitions and recapitulate some well-known results from the literature, which are needed in the remainder of this section. A closed subset B of Q is called a Z-*set* ([1]) if for any non-empty homotopically trivial open subset O of Q, the set $O\setminus B$ is again non-empty and homotopically trivial. Examples of Z-sets are compact subsets of $(0, 1)^\infty$ and closed subsets of Q which project onto a point in infinitely many coordinates. In fact, Z-sets can be characterized by the property that for every Z-set B there exists an autohomeomorphism Φ of Q which maps B onto a set which projects onto a point in infinitely many coordinates ([1]). Obviously the property of being a Z-set is a topological invariant. Moreover, it is easy to show that a closed countable union of Z-sets is again a Z-set (cf. KROONENBERG [9]). The importance of Z-sets is illustrated by the following theorem due to ANDERSON [1].

THEOREM. *Any homeomorphism between two Z-sets in Q can be extended to an autohomeomorphism of Q.*

We will apply this theorem to show that every separable metric, not totally disconnected topological space X can be embedded in Q in such a way that Q has the structure of a superextension of X, i.e., every point of Q represents an mls in a suitable closed subbase for X. The canonical binary subbase for Q is

$$\mathfrak{T} = \{A \subset Q | A = \Pi_n^{-1}[0, x] \text{ or } A = \Pi_n^{-1}[x, 1], \text{ with } n \in \mathbf{N} \text{ and } x \in I\}$$

and consequently, if we embed X in Q in such a way that for every two elements $T_0, T_1 \in \mathfrak{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$, then Q is a superextension of X; this is a consequence of the following theorem ([11], Theorem 5).

THEOREM 3.1. *Let X be a subspace of the topological T_1-space Y. Then Y is homeomorphic to a superextension of X if and only if Y possesses a binary closed subbase $\mathfrak{T}$ such that for all T_0, $T_1 \in \mathfrak{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$.*

In particular, in Theorem 3.1 $Y \simeq \lambda_{\mathfrak{T} \cap X}(X)$, where $\mathfrak{T} \cap X = \{T \cap X | T \in \mathfrak{T}\}$.

THEOREM 3.2. *For every separable metric, not totally disconnected topological space X there exists a normal closed T_1-subbase $\mathfrak{S}$ such that $\lambda_S(X)$ is homeomorphic to the Hilbert cube Q.*

PROOF. Assume that X is embedded in $Q(= I^{\mathbf{N}})$ and let C be a non-trivial component of X. Choose a convergent sequence B in C. Furthermore, define a sequence $\{y_n\}_{n=0}^{\infty}$ in Q by

$$(y_n)_i = \begin{cases} 1 & \text{if } \; i \neq n \\ 0 & \text{if } \; i = n, \end{cases}$$

for $i = 1, 2, \ldots,$.

It is clear that

$$\lim_{n \to \infty} y_n = y_0 .$$

Moreover define $z \in Q$ by $z_i = 0$ $(i = 1, 2, \ldots,)$. Then

$$E = \{y_n | n \in \mathbf{N}\} \cup \{z\} \cup \{y_0\}$$

is a convergent sequence and therefore is homeomorphic to B. Since B and E both are closed countable unions of Z-sets in Q. they themselves are Z-sets. Choose a homeomorphism $\Phi : B \to E$ and extend this homeomorphism to an autohomeomorphism of Q. This procedure shows that we may assume that

X is embedded in Q in such a way that $E \subset C$. Let $T_0, T_1 \in \mathfrak{T}$ such that $T_0 \cap T_1 \neq \emptyset$, where $\mathfrak{T}$ is the canonical binary closed subbase for Q. We need only consider the following 4 cases:

Case 1: $T_0 = \Pi_{n_0}^{-1}[0, x]$; $T_1 = \Pi_{n_0}^{-1}[y, 1]$ $(x \geq y)$. Since $z \in T_0$ and $y_0 \in T_1$ and C is connected, it follows that $\emptyset \neq T_0 \cap T_1 \cap C \subset T_0 \cap T_1 \cap X$.

Case 2: $T_0 = \Pi_{n_0}^{-1}[0, x]$; $T_1 = \Pi_{n_1}^{-1}[y, 1]$ $(n_0 \neq n_1)$. Then $y_{n_0} \in T_0 \cap \cap T_1 \cap X$.

Case 3: $T_0 = \Pi_{n_0}^{-1}[0, x]$; $T_1 = \Pi_{n_1}^{-1}[0, y]$. Then $z \in T_0 \cap T_1 \cap X$.

Case 4: $T_0 = \Pi_{n_0}^{-1}[x, 1]$; $T_1 = \Pi_{n_1}^{-n}[y, 1]$. Then $y_0 \in T_0 \cap T_1 \cap X$.

This completes the proof of the theorem.

4. A superextension of the closed unit interval

In the present section we will prove that $\lambda_{\mathcal{G}_2}(I)$ is homeomorphic to the Hilbert cube, where $\mathcal{G}_2 = \{[x, y] \mid x, y \in I\} \cup \{[0, x] \cup [y, 1] \mid x, y \in I\}$. For this purpose we introduce

$$\mathfrak{F} = \{f : I \to I \mid f(0) = 0 \text{ and if } x, y \in I \text{ and } x \leq y \text{ then } 0 \leq f(y) - f(x) \leq \leq y - x\}.$$

Hence each $f \in \mathfrak{F}$ is continuous and monotone non-decreasing. On $\mathfrak{F}$ we define a topology by considering $\mathfrak{F}$ as a subspace of $C[I, I]$ with the point-open topology. We obtain the same topology on $\mathfrak{F}$ by ordering $\mathfrak{F}$ partially as follows:

$$f \leq g \text{ iff for each } x \in I : f(x) \leq g(x), \; (f, g \in \mathfrak{F}),$$

and then taking as a closed subbase for $\mathfrak{F}$ the collection of all subsets of the form $\{f \in \mathfrak{F} \mid f \leq f_0\}$ or $\{f \in \mathfrak{F} \mid f \geq f_0\}$, where f_θ runs through $\mathfrak{F}$. We first prove that $\mathfrak{F} \cong Q$ and next that $\lambda_{\mathcal{G}_2}(I) \cong \mathfrak{F}$; we conclude that $\lambda_{\mathcal{G}_2}(I) \cong Q$.

Notice that by KELLER's theorem each compact metrizable convex infinite-dimensional subspace X of I^I is homeomorphic to the Hilbert cube Q, since, by the fact that X is metrizable, X can be embedded as a convex subspace of I^∞; finally I^∞ can be affinely embedded in l_2. This observation will be used in the proof of Theorem 4.1 and Theorem 5.1.

THEOREM 4.1. $\mathfrak{F} \cong Q$.

PROOF. We show that $\mathfrak{F}$ is a compact, infinite-dimensional, convex subspace of I^I, with countable base; hence, by KELLER's theorem, $\mathfrak{F}$ is homeomorphic to the Hilbert cube Q.

$\mathscr{F}$ is clearly a convex subspace of I^I; it is also clear that $(\mathscr{F}, \leq)$, as defined above, is a complete lattice, whence $\mathscr{F}$ is compact. $\mathscr{F}$ has a countable subbase, since the collection of all subsets of the forms $\{f \in \mathscr{F} \mid f(x) \leq y\}$ and $\{f \in \mathscr{F} \mid f(x) \leq y\}$ where $x, y \in \mathbf{Q} \cap I$, forms a countable closed subbase for $\mathscr{F}$.

Finally, $\mathscr{F}$ is infinite-dimensional, because Q can be embedded in $\mathscr{F}$. For, let $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in I^{\mathbf{N}}$. Let $G(\mathbf{a})$ be the smallest function f in $\mathscr{F}$ (in the ordering $\leq$ of $\mathscr{F}$) such that for each $i = 1, 2, 3, \ldots$ the following holds:

$$f\left(\frac{3}{2^{i+1}}\right) \geq \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}}\, a_i\,.$$

It can be seen easily that G defines a topological embedding of Q in $\mathscr{F}$.

THEOREM 4.2. $\lambda_{\mathscr{G}_2}(I) \simeq \mathscr{F}$.

PROOF. Define a function $K\colon \lambda_{\mathscr{G}_2}(I) \to I$ by :

$$K(\mathscr{M}) = \inf\{x \in I \mid [0, x] \in \mathscr{M}\}, \quad (\mathscr{M} \in \lambda_{\mathscr{G}_2}(I)),$$

and a function $H\colon \lambda_{\mathscr{G}_2}(I) \to \mathscr{F}$ by:

$$H(\mathscr{M})\,(i) = \inf\{x \in I \mid [0, x] \cup [y, 1] \in \mathscr{M},\ x + y = K(\mathscr{M}) + i\},$$
$$(i \in I,\ \mathscr{M} \in \lambda_{\mathscr{G}_2}(I))\,.$$

We prove that H is an homeomorphism between $\lambda_{\mathscr{G}_2}(I)$ and $\mathscr{F}$.
First we observe that:

$$K(\mathscr{M}) \leq x \text{ iff } [0, x] \in \mathscr{M};$$

$$K(\mathscr{M}) \geq x \text{ iff } [x, 1] \in \mathscr{M};$$

$$K(\mathscr{M}) = x \text{ iff } [0, x] \in \mathscr{M} \text{ and } [x, 1] \in \mathscr{M};$$

$$H(\mathscr{M})\,(i) \leq x \text{ iff } [0, x] \cup [K(\mathscr{M}) + i - x, 1] \in \mathscr{M};$$

$$H(\mathscr{M})\,(i) \geq x \text{ iff } [x, K(\mathscr{M}) + i - x] \in \mathscr{M};$$

$$H(\mathscr{M})\,(i) = x \text{ iff } [0, x] \cup [K(\mathscr{M}) + i - x, 1] \in \mathscr{M} \text{ and }$$
$$[x, K(\mathscr{M}) + i - x] \in \mathscr{M};$$

these facts follows easily from the fact that $\mathscr{M}$ is a maximal linked system in $\mathscr{G}_2$. Also we have $K(\mathscr{M}) = H(\mathscr{M})(1)$.

Next we show that $H(\mathscr{M}) \in \mathscr{F}$, for each maximal linked system $\mathscr{M}$. In fact (i) $H(\mathscr{M})(0) = 0$, for $[0, 0] \cup [K(\mathscr{M}), 1] \in \mathscr{M}$ and $[0, K(\mathscr{M})] \in \mathscr{M}$; (ii) if $i \leq j$, $H(\mathscr{M})(i) = x$, $H(\mathscr{M})(j) = y$, then $x \leq y$, for $[x, K(\mathscr{M}) + j - x] \supset$ $\subset [x, K(\mathscr{M}) + i - x] \in \mathscr{M}$, hence $[x, K(\mathscr{M}) + j - x] \in \mathscr{M}$ and $y = H(\mathscr{M})(j) \geq x$;

also $y - x \leq j - i$, for $[y - j + i, K(\mathfrak{M}) + i - (y - j + i) \supset [y, K(\mathfrak{M}) + j - y] \in \mathfrak{M}$, hence $x = H(\mathfrak{M})(i) \geq y - j + i$.

H is a one-to-one function, for suppose $\mathfrak{M}_1, \mathfrak{M}_2 \in \lambda_{\mathcal{G}_2}(I)$, $\mathfrak{M}_1 \neq \mathfrak{M}_2$ and $H(\mathfrak{M}_1) = H(\mathfrak{M}_2)$. Let $a = K(\mathfrak{M}_1) = H(\mathfrak{M}_1)(1) = H(\mathfrak{M}_2)(1) = K(\mathfrak{M}_2)$, i.e., $[0, a] \in \mathfrak{M}_1 \cap \mathfrak{M}_2$ and $[a, 1] \in \mathfrak{M}_1 \cap \mathfrak{M}_2$. Since $\mathfrak{M}_1 \neq \mathfrak{M}_2$ we may suppose that there are x' and y' such that $[0, x]' \cup [y', 1] \in \mathfrak{M}_1 \setminus \mathfrak{M}_2$. Since $[0, a] \in \mathfrak{M}_2$ and $[a, 1] \in \mathfrak{M}_2$, we have $x' < a < y'$. Let $i = x' + y' - a \in [x', y'] \subset I$. Then since $[0, x'] \cup [a + i - x', 1] = [0, x'] \cup [y', 1] \in \mathfrak{M}_1 \setminus \mathfrak{M}_2$, we find that $H(\mathfrak{M}_1)(i) \leq x' < H(\mathfrak{M}_2)(i)$ and this is a contradiction. H is also a surjection. Take $f \in \mathfrak{F}$ and let:

$$\mathfrak{L} = \{[f(i), f(1) + i - f(i)] \mid i \in I\} \cup \{[0, f(i)] \cup [f(1) + i - f(i), 1] \mid i \in I\}.$$

Then by definition of $\mathfrak{F}$, it is easy to see that $\mathfrak{L}$ is a linked system in $\mathcal{G}_2$. $\mathfrak{L}$ is contained in some maximal linked system $\mathfrak{M}$ of $\mathcal{G}_2$, and for this $\mathfrak{M}$ it holds that $K(\mathfrak{M}) = f(1)$ while for each $i \in I{:}H(\mathfrak{M})(i) = f(i)$; i.e., $H(\mathfrak{M}) = f$. Finally we prove that H is continuous. Let $i, x \in I$. Then

$$\{\mathfrak{M} \in \lambda_{\mathcal{G}_2}(I) \mid H(\mathfrak{M})(i) \leq x\} = \bigcap_{y \in I} \{\mathfrak{M} \in \lambda_{\mathcal{G}_2}(I) \mid [0, x] \cup [y, 1] \in \mathfrak{M} \text{ or } [0, x + y - i] \in \mathfrak{M}\},$$

and hence this set is closed. For, let $\mathfrak{M} \in \lambda_{\mathcal{G}_2}(I)$ such that $H(\mathfrak{M})(i) \leq x$; this last inequality means that $[0, x] \cup [K(\mathfrak{M}) + i - x, 1] \in \mathfrak{M}$. If $y \geq K(\mathfrak{M}) + i - x$, then $[0, y + x - i] \supset [0, K(\mathfrak{M})] \in \mathfrak{M}$; if $y \leq K(\mathfrak{M}) + i - x$ then $[0, x] \cup [y, 1] \supset [0, x] \cup [K(\mathfrak{M}) + i - x, 1] \in \mathfrak{M}$.

Conversely, suppose that

$$[0, x] \cup [y, 1] \in \mathfrak{M} \text{ or } [0, x + y - i] \in \mathfrak{M}.$$

for each $y \in I$, then also $[0, x + y - i] \notin \mathfrak{M}$ for each $y < K(\mathfrak{M}) + i - x$; hence $[0, x] \cup [y, 1] \in \mathfrak{M}$; we conclude that $[0, x] \cup [K(\mathfrak{M}) + i - x, 1] \in \mathfrak{M}$, i.e., $H(\mathfrak{M})(i) \leq x$.

In the same way one proves:

$$\{\mathfrak{M} \in \lambda_{\mathcal{G}_2}(I) \mid H(\mathfrak{M})(i) \geq x\} = \bigcap_{y \in I} \{\mathfrak{M} \in \lambda_{\mathcal{G}_2}(I) \mid [x, y] \in \mathfrak{M} \text{ or } [x + y - i, 1] \in \mathfrak{M}\},$$

and hence is closed.

As a consequence of these two theorems we have, as announced,

THEOREM 4.3. $\lambda_{\mathcal{G}_2}(I) \simeq Q$.

5. A superextension of the n-sphere

In this final section we show that the superextension of the n-sphere S^n with respect to the collection of all closed massive n-balls in S^n is homeomorphic with the Hilbert-cube. As usual, the n-sphere S^n is the space

$$\left\{(x_0, x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i^2 = 1\right\}$$

and the closed massive n-ball with centre $\mathbf{x} \in S^n$ and radius $\varepsilon \geq 0$ is the set

$$B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in S^n \mid d(\mathbf{x}, \mathbf{y}) \leq \varepsilon\}.$$

Writing $\mathscr{B}$ for the collection of all closed massive n-balls in S^n, we will prove that, if $n \geq 1$, $\lambda_{\mathscr{B}}(S^n) \simeq Q$. Obviously $\lambda_{\mathscr{B}}(S^1)$ is the superextension of the circle with respect to the set of closed intervals. For the definition of $\mathscr{B}$ it does not matter whether the euclidian metric of $\mathbf{R}^{n+1}$ or the sphere metric of S^n (in this case the distance between $\mathbf{x}$ and $\mathbf{y}$ in S^n is $\arccos \sum_{i=0}^{n} x_i y_i$, i.e., the minimum length of a curve between $\mathbf{x}$ and $\mathbf{y}$ on S^n) is used. However, in the proof of the theorem we need the latter metric and we call this metric d. Furthermore we define, for each point $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in S^n$, the antipode $\bar{\mathbf{x}}$ of $\mathbf{x}$ by $\bar{\mathbf{x}} = (-x_0, -x_1, \ldots, -x_n)$.

THEOREM 5.1. *If* $n \geq 1$, $\lambda_{\mathscr{B}}(S^n)$ *is homeomorphic to the Hilbert-cube* Q.

PROOF. In fact we show that $\lambda_{\mathscr{B}}(S^n)$ is compact and infinite-dimensional and has a countable base and that $\lambda_{\mathscr{B}}(S^n)$ can be embedded as a convex subspace in $\mathbf{R}^{S^n}$; hence, by KELLER's theorem, $\lambda_{\mathscr{B}}(S^n)$ is homeomorphic to Q. Clearly, $\lambda_{\mathscr{B}}(S^n)$ is compact.

To prove that $\lambda_{\mathscr{B}}(S^n)$ has a countable base, let X be a countable dense subset of S^n. Define $\mathscr{B}_0 = \{B(\mathbf{x}, \varepsilon) \mid \mathbf{x} \in X,\ \varepsilon \in \mathbf{Q},\ \varepsilon \geq 0\}$. It is not difficult to see that $P: \lambda_{\mathscr{B}}(S^n) \to \lambda_{\mathscr{B}_0}(S^n)$, such that $P(\mathscr{M}) = \mathscr{M} \cap \mathscr{B}_0$ ($\mathscr{M} \in \lambda_{\mathscr{B}}(S^n)$) is a homeomorphism; hence, since $\lambda_{\mathscr{B}_0}(S^n)$ has a countable base, $\lambda_{\mathscr{B}}(S^n)$ also has a countable base. Next, $\lambda_{\mathscr{B}}(S^n)$ is infinite-dimensional, since $\lambda_{\mathscr{G}_2}(I)(\simeq Q)$ can be embedded in $\lambda_{\mathscr{B}}(S^n)$. For, let

$$Y = \{\mathbf{x} \in S^n \mid \mathbf{x} = (x_0, x_1, \ldots, x_n),\ x_1 \geq 0,\ x_2 = \ldots = x_n = 0\};$$

this subspace is homeomorphic to I. Let $\mathscr{G}_2$ be as defined in Section 3, i.e., $\mathscr{G}_2$ is the collection of all closed subsets Y' if Y such that Y' is connected or $Y \setminus Y'$ is connected. Define $T: \lambda_{\mathscr{G}_2}(Y) \to \lambda_{\mathscr{B}}(S^n)$ by $T(\mathscr{M}) = \{B \in \mathscr{B} \mid B \cap Y \in \mathscr{M}\}$ ($\mathscr{M} \in \lambda_{\mathscr{G}_2}(I)$). Again it is not difficult to prove that T is a topological embedding. Hence $\lambda_{\mathscr{G}_2}(I) \simeq Q$ can be embedded in $\lambda_{\mathscr{B}}(S^n)$, i.e., $\lambda_{\mathscr{B}}(S^n)$ is infinite-dimensional.

Finally we embed $\lambda_{\mathfrak{B}}(S^n)$ as a convex subspace in $\mathbf{R}^{S^n}$, by means of the function $U:\lambda_{\mathfrak{B}}(S^n) \to \mathbf{R}^{S^n}$, determined by:

$$U(\mathfrak{M})(\mathbf{x}) = \inf\{\varepsilon \geq 0 | B(\mathbf{x}, \varepsilon) \in \mathfrak{M}\}, \ (\mathfrak{M} \in \lambda_{\mathfrak{C}_2}(S^n), \mathbf{x} \in S^n).$$

The mapping U is continuous and one-to-one since $U(\mathfrak{M})(\mathbf{x}) \leq \varepsilon$ iff $B(\mathbf{x}, \varepsilon) \in \mathfrak{M}$, and $U(\mathfrak{M})(\mathbf{x}) \geq \varepsilon$ iff $B(\bar{\mathbf{x}}, \pi - \varepsilon) \in \mathfrak{M}$. And indeed, $U[\lambda_{\mathfrak{B}}(S^n)]$ is a convex subspace of $\mathbf{R}^{S^n}$. In order to show this, we need only prove: if $\mathfrak{M}_1, \mathfrak{M}_2 \in \lambda_{\mathfrak{B}}(S^n)$, then there exists an $\mathfrak{M} \in \lambda_{\mathfrak{B}}(S^n)$ such that $U(\mathfrak{M}) = \frac{1}{2} U(\mathfrak{M}_1) + \frac{1}{2} U(\mathfrak{M}_2)$ ($U[\lambda_{\mathfrak{B}}(S^n)]$ being compact and hence closed in $\mathbf{R}^{S^n}$). So take $\mathfrak{M}_1, \mathfrak{M}_2 \in \lambda_{\mathfrak{B}}(S^n)$ and let $\mathfrak{M}_3 = \{B(\mathbf{x}, \varepsilon) \,|\, \mathbf{x} \in S^n, \ \varepsilon \geq \frac{1}{2} U(\mathfrak{M}_1)(\mathbf{x}) + \frac{1}{2} U(\mathfrak{M}_2)(\mathbf{x})\}$. Then $\mathfrak{M}_3$ is a linked system, because if $B(\mathbf{x}, \varepsilon)$ and $B(\mathbf{y}, \delta) \in \mathfrak{M}_3$ $(\mathbf{x}, \mathbf{y} \in S^n, \varepsilon \geq \frac{1}{2} U(\mathfrak{M}_1)(\mathbf{x}) + \frac{1}{2} U(\mathfrak{M}_2)(\mathbf{x}), \ \delta \geq \frac{1}{2} U(\mathfrak{M}_1)(\mathbf{y}) + \frac{1}{2}(U(\mathfrak{M}_2)(\mathbf{y}))$, then:

$$d(\mathbf{x}, \mathbf{y}) \leq U(\mathfrak{M}_1)(\mathbf{x}) + U(\mathfrak{M}_1)(\mathbf{y}),$$

and

$$d(\mathbf{x}, \mathbf{y}) \leq U(\mathfrak{M}_2)(\mathbf{x}) + U(\mathfrak{M}_2)(\mathbf{y});$$

hence

$$d(\mathbf{x}, \mathbf{y}) \leq \delta + \varepsilon,$$

i.e.,

$$B(\mathbf{x}, \varepsilon) \cap B(\mathbf{y}, \delta) \neq \emptyset.$$

Let $\overline{\mathfrak{M}}_3$ be a maximal linked system containing $\mathfrak{M}_3$ (in fact $\mathfrak{M}_3$ is itself a maximal linked system). Then, clearly,

$$U(\overline{\mathfrak{M}}_3)(\mathbf{x}) \leq \frac{1}{2} U(\mathfrak{M}_1)(\mathbf{x}) + \frac{1}{2} U(\mathfrak{M}_2)(\mathbf{x}),$$

and

$$U(\overline{\mathfrak{M}}_3)(\mathbf{x}) \leq \frac{1}{2} U(\mathfrak{M}_1)(\mathbf{x}) + \frac{1}{2} U(\mathfrak{M}_2)(\mathbf{x}) \text{ for each } \mathbf{x} \in S^n.$$

But, since for each maximal linked system $\mathfrak{M} : U(\mathfrak{M})(\mathbf{x}) + U(\mathfrak{M})(\bar{\mathbf{x}}) = \pi$, we have

$$U(\mathfrak{M}_3)(\mathbf{x}) = \frac{1}{2} U(\mathfrak{M}_1)(\mathbf{x}) + \frac{1}{2} U(\mathfrak{M}_2)(\mathbf{x}) \text{ for each } \mathbf{x} \in S^n.$$

Thus

$$U(\overline{\mathfrak{M}}_3) = \frac{1}{2} U(\mathfrak{M}_1) + \frac{1}{2} U(\mathfrak{M}_2).$$

REFERENCES

[1] R. D. ANDERSON, On topological infinite deficiency, *Michigan Math. J.* **14** (1967), 365–383. *MR* **35** # 4893

[2] M. BELL, Not all compact Hausdorff spaces are supercompact, *General Topology and Appl.* **8** (1978), 151–155.

[3] CZ. BESSAGA, and A. PEŁCZYŃSKI, *Selected topics in infinite-dimensional topology*, Monografie Matematyczne 58, PWN, Warszawa, 1975. *Zbl.* **304.** 57001

[4] A. E. BROUWER and A. SCHRIJVER, *A characterzitaion of supercompactness with an application to treelike spaces*, Report Mathematical Centre ZW 34/74, Amsterdam, 1974. *Zbl.* **292.** 54020

[5] J. BRUIJNING, Characterizations of I^n and I^∞ using the graph theoretical representation of J. DE GROOT, *Topological Structures (Proc. Sympos., Amsterdam*, 1973), Mathematical Centre Tracts 52, Amsterdam, 1974, 38–47.

[6] J. DE GROOT, Superextensions and supercompactness, *Proc. I. Intern. Symp. on Extension Theory of Topological Structures and its Applications*, VEB Deutscher Verlag Wiss., Berlin, 1967, 89–90.

[7] J. DE GROOT, Graph representations of topological spaces, *Topological Structures (Proc. Sympos., Amsterdam*, 1973), Mathematical Centre Tracts 52, Amsterdam, 1974, 29–37. *MR* **51** # 9012

[8] O. H. KELLER, Die Homoiomorphie der kompakten konvexen Mengen im Hilbertschen Raum, *Math. Ann.* **105** (1931), 748–758. *Zbl* **3**, 224

[9] N. S. KROONENBERG, *Pseudo-interiors of hyperspaces*, Dissertation, Louisana State University (1974).

[10] J. VAN MILL, *A topological characterization of products of compact tree-like spaces*, Rapport 36, Wiskundig Seminarium Vrije Universiteit, Amsterdam, 1975.

[11] J. VAN MILL, *On supercompactness and superextensions*, Rapport 37, Wiskundig Seminarium Vrije Universiteit, Amsterdam, 1975.

[12] J. VAN MILL, *A topological property of supdrcompact Hausdorff spaces*, Report Mathematical Centre ZW 66/76, Amsterdam, 1976.

[13] A. SCHRIJVER, *Graphs and supercompact spaces*, Report Mathematical Centre ZW 37/74, Amsterdam, 1974.

[14] M. STROK and A. SZYMAŃSKI, Compact metric spaces have binary bases, *Fund. Math.* **89** (1975), 81–91. *MR* **52** # 4232

[15] A. VERBEEK, *Superextensions of topological spaces*, Mathematical Centre Tracts 41, Amsterdam, 1972. *MR* **50** # 11157

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