SUPEREXTENSIONS WHICH ARE HILBERT CUBES

by

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Abstract

It is shown that each separable metric, not totally disconnected, topological space admits a superextension homeomorphic to the Hilbert cube. Moreover, for simple spaces, such as the closed unit interval or the $n$-spheres $S^n$, we give easily described subbases for which the corresponding superextension is homeomorphic to the Hilbert cube.

1. Introduction

In [6], DE GROOT defined a space $X$ to be supercompact provided that it possesses a binary closed subbase, i.e., a closed subbase $\mathcal{S}$ with the property that if $\mathcal{S}^c \subset \mathcal{S}$ and $\cap \mathcal{S}^c = \emptyset$ then there exist $S_0, S_1 \in \mathcal{S}$ such that $S_0 \cap S_1 = \emptyset$. Clearly, according to the lemma of ALEXANDER, every supercompact space is compact. The class of supercompact spaces contains the compact orderable spaces, compact tree-like spaces (BROUWER & SCHRIJVER [4], VAN MILL [10]) and compact metric spaces (STROK & SZYMAŃSKI [14]). Moreover, there are compact Hausdorff spaces which are not supercompact (BELL [2], VAN MILL [12]). There is a connection between supercompact spaces and graphs (see e.g., DE GROOT [7], BRULIJNING [5], SCHRIJVER [13]); moreover, supercompact spaces can be characterized by means of so-called interval structures (BROUWER & SCHRIJVER [4]).

Let $X$ be a $T_1$-space and $\mathcal{S}$ a closed $T_1$-subbase for $X$ (a closed subbase $\mathcal{S}$ for $X$ is called $T_1$ if for all $S \in \mathcal{S}$ and $x \in X$ with $x \notin S$, there exists an $S_0 \in \mathcal{S}$ with $x \in S_0$ and $S_0 \cap S = \emptyset$). The superextension $\lambda_\mathcal{S}(X)$ of $X$ relative the subbase $\mathcal{S}$ is the set of all maximal linked systems $\mathfrak{M} \subset \mathcal{S}$ (a subsystem of $\mathcal{S}$ is called linked if every two of its members meet; a maximal linked system or MLS is a linked system not properly contained in another linked system) topologized by taking $\{(\mathfrak{M} \in \lambda_\mathcal{S}(X) | S \in \mathfrak{M}) | S \in \mathcal{S}\}$ as a closed subbase. Clearly, this subbase is binary, hence $\lambda_\mathcal{S}(X)$ is supercompact, while moreover $X$ can be embedded in $\lambda_\mathcal{S}(X)$ by the natural embedding $i : X \rightarrow \lambda_\mathcal{S}(X)$ defined by $i(X) := \{S \in \mathcal{S} | x \in S\}$. VERBEEEK's monograph [15] is a good place to find the basic theorems about superextensions. In this paper we will show that for many spaces there are superextensions homeomorphic to the Hilbert cube $Q$; moreover for simple spaces such as the unit interval or the $n$-spheres $S_n$ we will present easily described subbases for which the corresponding super-


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extension is homeomorphic to $Q$. Here, a classical theorem of Keller [8],
which says that each infinite-dimensional compact convex subset of the separable
Hilbert space is homeomorphic to $Q$ (for a more up-to-date proof of this fact,
see also Bessaga & Pełczyński [3]), is of great help.

2. Some examples

In this section we will give some examples. If $X$ is an ordered space,
then the Dedekind completion of $X$ will be denoted by $\overline{X}$. Roughly speaking,
$\overline{X}$ can be obtained from $X$ by filling up every gap. We define $\overline{X}$ to be that
ordered space which can be obtained from $X$ by filling up every gap with two
points, except for possible endgaps, which we supply with one point. The compact space $\overline{X}$
thus obtained, clearly contains $X$ as a dense subspace. Define

$$\mathcal{G}_1 = \{A \subseteq X | \exists x \in X : A = (\leftarrow, x] \text{ or } A = [x, \rightarrow)\}$$

and

$$\mathcal{F}_1 = \{A \subseteq X | A \text{ is a closed half-interval}\}$$

(as usual, a half-interval is a subset $A \subseteq X$ such that either for all $a, b \in X$:
if $b \leq a \in A$ then $b \in A$, or for all $a, b \in X$: if $b \geq a \in A$ then $b \in A$) and

$$\mathcal{F}_2 = \{A \subseteq X | \exists A_0, A_1 \in \mathcal{F}_1 : A = A_0 \cup A_1 \text{ or } A = A_0 \cap A_1\},$$

respectively.

Notice that $\mathcal{G}_1$ equals $\mathcal{F}_1$ in case $X$ is compact or connected. It is easy to
see that $\lambda_{\mathcal{G}_1}(X) \cong \overline{X}$ and that $\lambda_{\mathcal{F}_1}(X) \cong \overline{X}$.

What about $\lambda_{\mathcal{F}_2}(X)$?

Example (i). If $X = I$, then $\lambda_{\mathcal{G}_1}(X) = \lambda_{\mathcal{F}_1}(X) \cong I$. On the other hand
$\lambda_{\mathcal{F}_2}(X)$ is homeomorphic to the Hilbert cube $Q$ (see Section 4).

Example (ii). If $X = Q$, then $\lambda_{\mathcal{G}_1}(X) \cong I$ and $\lambda_{\mathcal{F}_2}(X)$ is a non-metrizable
separable compact ordered space, which has much in common with the well-
known Alexandroff double of the closed unit interval. In this case, $\lambda_{\mathcal{F}_2}(X)$ is
a compact totally disconnected perfect space of weight $2^{2^n}$. (The total discon-
ectedness of $\lambda_{\mathcal{F}_2}(X)$ follows from the following observation: for every $T_0,
T_1 \in \mathcal{F}_2$ with $T_0 \cap T_1 = \emptyset$ there exists a $T'_0 \in \mathcal{F}_2$ such that $T_0 \subseteq T'_0$ and
$T'_0 \cap T_1 = \emptyset$ and $X \setminus T'_0 \in \mathcal{F}_2$. For every finite linked system $\{X \setminus T_i | T_i \in \mathcal{F}_2,
i \in \{1, 2, \ldots, n\}\}$ it is easy to construct two distinct mls's $\mathcal{L}_0$ and $\mathcal{L}_1$ belonging
to $\bigcap_{i=1}^n \{\mathcal{M} \in \lambda_{\mathcal{F}_2}(X) | T_i \notin \mathcal{M}\}$ showing that $\lambda_{\mathcal{F}_2}(X)$ is perfect. Finally $\lambda_{\mathcal{F}_2}$ can
be embedded in $\lambda_{\mathcal{F}_1}(X)$; hence weight $(\lambda_{\mathcal{F}_2}(X)) = 2^{2^n}$.
Example (iii). If $X = \mathbb{R} \setminus \mathbb{Q}$, then $\lambda_{\mathbb{Q}}(X) \simeq I$, while $\lambda_{\mathbb{Q}_1}(X) \simeq \lambda_{\mathbb{Q}_2}(X) \simeq C$, the Cantor discontinuum, for it is easy to see that $\lambda_{\mathbb{Q}_1}(X)$ and $\lambda_{\mathbb{Q}_2}(X)$ both are totally disconnected compact metric perfect spaces.

Finally define

$$\mathcal{G}_2 = \{ A \subset X | \exists A_0, A_1 \in \mathcal{G}_1: A = A_0 \cup A_1 \text{ or } A = A_0 \cap A_1 \}.$$

Notice that $\mathcal{G}_2$ equals $T_2$ in case $X$ is compact or connected.

Example (i). If $X = I$, then $\lambda_{\mathbb{Q}_1}(X) \simeq Q$ (Section 4).

Example (ii). If $X = \mathbb{Q}$, then $\lambda_{\mathbb{Q}_1}(X) \simeq Q$.

Example (iii). If $X = \mathbb{R} \setminus \mathbb{Q}$, then $\lambda_{\mathbb{Q}_2}(X) \simeq Q$.

The fact that $\lambda_{\mathbb{Q}_2}(\mathbb{Q}) \simeq \lambda_{\mathbb{Q}_2}(\mathbb{R} \setminus \mathbb{Q}) \simeq Q$ can be derived from the result $\lambda_{\mathbb{Q}_2}(I) \simeq Q$. To see this, define

$$\mathcal{G}_3 = \{ A \subset I | A \in \mathcal{G}_2 \text{ and } A \text{ has rational endpoints} \}$$

and

$$\mathcal{G}_2^* = \{ A \subset I | A \in \mathcal{G}_2 \text{ and } A \text{ has irrational endpoints} \}.$$

By Theorem 5 and Theorem 7 of [11] (cf. Theorem 3.1 below), it follows that

$$\lambda_{\mathbb{Q}_2}(I) \simeq \lambda_{\mathbb{Q}_2}(I) \simeq \lambda_{\mathbb{Q}_2}(\mathbb{Q})$$

and

$$\lambda_{\mathbb{Q}_2}(I) \simeq \lambda_{\mathbb{Q}_2}(I) \simeq \lambda_{\mathbb{Q}_2}(\mathbb{R} \setminus \mathbb{Q}).$$

3. Superextensions which are Hilbert cubes

In this section we will show that for each separable metric, not totally disconnected topological space $X$, there exists a normal closed $T_1$-subbase $\mathcal{S}$ such that $\lambda_\mathcal{S}(X)$ is homeomorphic to the Hilbert cube $Q$. First we will give some preliminary definitions and recapitulate some well-known results from the literature, which are needed in the remainder of this section. A closed subset $B$ of $Q$ is called a $Z$-set ([1]) if for any non-empty homotopically trivial open subset $O$ of $Q$, the set $O \setminus B$ is again non-empty and homotopically trivial. Examples of $Z$-sets are compact subsets of $(0, 1)^\mathbb{N}$ and closed subsets of $Q$ which project onto a point in infinitely many coordinates. In fact, $Z$-sets can be characterized by the property that for every $Z$-set $B$ there exists an autohomeomorphism $\Phi$ of $Q$ which maps $B$ onto a set which projects onto a point in infinitely many coordinates ([1]). Obviously the property of being a $Z$-set is a topological invariant. Moreover, it is easy to show that a closed countable union of $Z$-sets is again a $Z$-set (cf. Kroonenberg [9]). The importance of $Z$-sets is illustrated by the following theorem due to Anderson [1].
THEOREM. Any homeomorphism between two Z-sets in \( Q \) can be extended to an autohomeomorphism of \( Q \).

We will apply this theorem to show that every separable metric, not totally disconnected topological space \( X \) can be embedded in \( Q \) in such a way that \( Q \) has the structure of a superextension of \( X \), i.e., every point of \( Q \) represents an mls in a suitable closed subbase for \( X \). The canonical binary subbase for \( Q \) is

\[
\mathcal{S} = \{ A \subset Q | A = \Pi^{-1}_n [0, x] \text{ or } A = \Pi^{-1}_n [x, 1], \text{ with } n \in \mathbb{N} \text{ and } x \in I \}
\]

and consequently, if we embed \( X \) in \( Q \) in such a way that for every two elements \( T_0, T_1 \in \mathcal{S} \) with \( T_0 \cap T_1 \neq \emptyset \) we have that \( T_0 \cap T_1 \cap X \neq \emptyset \), then \( Q \) is a superextension of \( X \); this is a consequence of the following theorem ([11], Theorem 5).

**Theorem 3.1.** Let \( X \) be a subspace of the topological \( T_1 \)-space \( Y \). Then \( Y \) is homeomorphic to a superextension of \( X \) if and only if \( Y \) possesses a binary closed subbase \( \mathcal{S} \) such that for all \( T_0, T_1 \in \mathcal{S} \) with \( T_0 \cap T_1 \neq \emptyset \) we have that \( T_0 \cap T_1 \cap X \neq \emptyset \).

In particular, in Theorem 3.1 \( Y \cong \lambda_{\mathcal{S}}(X) \), where \( \mathcal{S} \cap X = \{ T \cap X | T \in \mathcal{S} \} \).

**Theorem 3.2.** For every separable metric, not totally disconnected topological space \( X \) there exists a normal closed \( T_1 \)-subbase \( \mathcal{S} \) such that \( \lambda_{\mathcal{S}}(X) \) is homeomorphic to the Hilbert cube \( Q \).

**Proof.** Assume that \( X \) is embedded in \( Q(= I^\mathbb{N}) \) and let \( C \) be a non-trivial component of \( X \). Choose a convergent sequence \( B \) in \( C \). Furthermore, define a sequence \( \{ y_n \}_{n=0}^\infty \) in \( Q \) by

\[
(y_n)_i = \begin{cases} 
1 & \text{if } i \neq n \\
0 & \text{if } i = n,
\end{cases}
\]

for \( i = 1, 2, \ldots, \).

It is clear that

\[
\lim_{n \to \infty} y_n = y_0.
\]

Moreover define \( z \in Q \) by \( z_i = 0 \) (\( i = 1, 2, \ldots, \)). Then

\[
E = \{ y_n | n \in \mathbb{N} \} \cup \{ z \} \cup \{ y_0 \}
\]

is a convergent sequence and therefore is homeomorphic to \( B \). Since \( B \) and \( E \) both are closed countable unions of Z-sets in \( Q \), they themselves are Z-sets. Choose a homeomorphism \( \Phi : B \to E \) and extend this homeomorphism to an autohomeomorphism of \( Q \). This procedure shows that we may assume that
$X$ is embedded in $Q$ in such a way that $E \subset C$. Let $T_0, T_1 \in \mathcal{F}$ such that $T_0 \cap T_1 \neq \emptyset$, where $\mathcal{F}$ is the canonical binary closed subbase for $Q$. We need only consider the following 4 cases:

**Case 1:** $T_0 = \Pi_{n_0}^{-1}[0, x]$; $T_1 = \Pi_{n_1}^{-1}[y, 1]$ ($x \geq y$). Since $z \in T_0$ and $y \in T_1$ and $C$ is connected, it follows that $\emptyset \neq T_0 \cap T_1 \cap C \subset T_0 \cap T_1 \cap X$.

**Case 2:** $T_0 = \Pi_{n_0}^{-1}[0, x]$; $T_1 = \Pi_{n_1}^{-1}[y, 1]$ ($n_0 \neq n_1$). Then $y_{n_0} \in T_0 \cap T_1 \cap X$.

**Case 3:** $T_0 = \Pi_{n_0}^{-1}[0, x]$; $T_1 = \Pi_{n_1}^{-1}[0, y]$. Then $z \in T_0 \cap T_1 \cap X$.

**Case 4:** $T_0 = \Pi_{n_0}^{-1}[x, 1]$; $T_1 = \Pi_{n_1}^{-1}[y, 1]$. Then $y_{0} \in T_0 \cap T_1 \cap X$.

This completes the proof of the theorem.

4. A superextension of the closed unit interval

In the present section we will prove that $\lambda_{\mathcal{G}_2}(I)$ is homeomorphic to the Hilbert cube, where $\mathcal{G}_2 = \{(x, y) \mid x, y \in I\} \cup \{(0, x) \cup \{y, 1\} \mid x, y \in I\}$. For this purpose we introduce

$$\mathcal{F} = \{f : I \to I \mid f(0) = 0 \text{ and if } x, y \in I \text{ and } x \leq y \text{ then } 0 \leq f(y) - f(x) \leq y - x\}.$$ 

Hence each $f \in \mathcal{F}$ is continuous and monotone non-decreasing. On $\mathcal{F}$ we define a topology by considering $\mathcal{F}$ as a subspace of $C[I, I]$ with the point-open topology. We obtain the same topology on $\mathcal{F}$ by ordering $\mathcal{F}$ partially as follows:

$$f \leq g \text{ iff for each } x \in I : f(x) \leq g(x), \ (f, g \in \mathcal{F}),$$

and then taking as a closed subbase for $\mathcal{F}$ the collection of all subsets of the form $\{f \in \mathcal{F} \mid f \leq f_0\}$ or $\{f \in \mathcal{F} \mid f \geq f_0\}$, where $f_0$ runs through $\mathcal{F}$. We first prove that $\mathcal{F} \cong Q$ and next that $\lambda_{\mathcal{G}_2}(I) \cong \mathcal{F}$; we conclude that $\lambda_{\mathcal{G}_2}(I) \cong Q$.

Notice that by KELLER's theorem each compact metrizable convex infinite-dimensional subspace $X$ of $I^I$ is homeomorphic to the Hilbert cube $Q$, since, by the fact that $X$ is metrizable, $X$ can be embedded as a convex subspace of $I^\infty$; finally $I^\infty$ can be affinely embedded in $l_2$. This observation will be used in the proof of Theorem 4.1 and Theorem 5.1.

**Theorem 4.1.** $\mathcal{F} \cong Q$.

**Proof.** We show that $\mathcal{F}$ is a compact, infinite-dimensional, convex subspace of $I^I$, with countable base; hence, by KELLER's theorem, $\mathcal{F}$ is homeomorphic to the Hilbert cube $Q$. 

2
is clearly a convex subspace of $I^1$; it is also clear that $(\mathcal{F}, \leq)$, as defined above, is a complete lattice, whence $\mathcal{F}$ is compact. $\mathcal{F}$ has a countable subbase, since the collection of all subsets of the forms \( \{ f \in \mathcal{F} | f(x) \leq y \} \) and \( \{ f \in \mathcal{F} | f(x) \leq y \} \) where $x, y \in \mathcal{Q} \cap I$, forms a countable closed subbase for $\mathcal{F}$.

Finally, $\mathcal{F}$ is infinite-dimensional, because $Q$ can be embedded in $\mathcal{F}$. For, let $a = (a_1, a_2, a_3, \ldots) \in I^\mathbb{N}$. Let $G(a)$ be the smallest function $f$ in $\mathcal{F}$ (in the ordering $\leq$ of $\mathcal{F}$) such that for each $i = 1, 2, 3, \ldots$ the following holds:

$$f \left( \frac{3}{2^{i+1}} \right) \geq \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} a_i .$$

It can be seen easily that $G$ defines a topological embedding of $Q$ in $\mathcal{F}$.

**Theorem 4.2.** $\lambda_{\mathcal{Q}}(I) \hookrightarrow \mathcal{F}$.

**Proof.** Define a function $K: \lambda_{\mathcal{Q}}(I) \to I$ by:

$$K(\mathcal{M}) = \inf \{ x \in I | [0, x] \in \mathcal{M}, (\mathcal{M} \in \lambda_{\mathcal{Q}}(I)),$$

and a function $H: \lambda_{\mathcal{Q}}(I) \to \mathcal{F}$ by:

$$H(\mathcal{M})(i) = \inf \{ x \in I | [0, x] \cup [y, 1] \in \mathcal{M}, x + y = K(\mathcal{M}) + i \},$$

$$H(\mathcal{M})(i) \in \mathcal{M} \in \lambda_{\mathcal{Q}}(I).$$

We prove that $H$ is an homeomorphism between $\lambda_{\mathcal{Q}}(I)$ and $\mathcal{F}$.

First we observe that:

$$K(\mathcal{M}) \leq x \text{ iff } [0, x] \in \mathcal{M};$$

$$K(\mathcal{M}) \geq x \text{ iff } [x, 1] \in \mathcal{M};$$

$$K(\mathcal{M}) = x \text{ iff } [0, x] \in \mathcal{M} \text{ and } [x, 1] \in \mathcal{M};$$

$$H(\mathcal{M})(i) \leq x \text{ iff } [0, x] \cup [K(\mathcal{M}) + i - x, 1] \in \mathcal{M};$$

$$H(\mathcal{M})(i) \geq x \text{ iff } [x, K(\mathcal{M}) + i - x] \in \mathcal{M};$$

$$H(\mathcal{M})(i) = x \text{ iff } [0, x] \cup [K(\mathcal{M}) + i - x, 1] \in \mathcal{M} \text{ and }$$

$$[x, K(\mathcal{M}) + i - x] \in \mathcal{M};$$

these facts follows easily from the fact that $\mathcal{M}$ is a maximal linked system in $\mathcal{Q}_2$. Also we have $K(\mathcal{M}) = H(\mathcal{M})(1)$.

Next we show that $H(\mathcal{M}) \in \mathcal{F}$, for each maximal linked system $\mathcal{M}$. In fact (i) $H(\mathcal{M})(0) = 0$, for $[0, 0] \cup [K(\mathcal{M}), 1] \in \mathcal{M}$ and $[0, K(\mathcal{M})] \in \mathcal{M}$; (ii) if $i \leq j, H(\mathcal{M})(i) = x, H(\mathcal{M})(j) = y$, then $x \leq y$, for $[x, K(\mathcal{M}) + j - x] \supset \subset [x, K(\mathcal{M}) + i - x] \in \mathcal{M}$, hence $[x, K(\mathcal{M}) + j - x] \in \mathcal{M}$ and $y = H(\mathcal{M})(j) \geq x;
also \( y - x \leq j - i \), for \([y - j + i, K(\mathbb{M}) + i - (y - j + i)] \supset [y, K(\mathbb{M}) + j - y] \sim \mathbb{M} \), hence \( x = H(\mathbb{M})(i) \geq y - j + i \).

\( H \) is a one-to-one function, for suppose \( \mathbb{M}_1, \mathbb{M}_2 \in \lambda_{\mathcal{G}}(I) \), \( \mathbb{M}_1 \neq \mathbb{M}_2 \) and \( H(\mathbb{M}_1) = H(\mathbb{M}_2) \). Let \( a = K(\mathbb{M}_1) = H(\mathbb{M}_1)(1) = H(\mathbb{M}_2)(1) = K(\mathbb{M}_2) \), i.e., \([0, a] \in \mathbb{M}_1 \cap \mathbb{M}_2 \) and \([a, 1] \in \mathbb{M}_1 \cap \mathbb{M}_2 \). Since \( \mathbb{M}_1 \neq \mathbb{M}_2 \) we may suppose that there are \( x' \) and \( y' \) such that \([0, x'] \cup [y', 1] \in \mathbb{M}_1 \setminus \mathbb{M}_2 \). Since \([0, a] \in \mathbb{M}_2 \) and \([a, 1] \in \mathbb{M}_2 \), we have \( x' < a < y' \). Let \( i = x' + y' - a \in [x', y'] \subset I \).

Then since \([0, x'] \cup [a + i - x', 1] = [0, x'] \cup [y', 1] \subseteq \mathbb{M}_1 \setminus \mathbb{M}_2 \), we find that \( H(\mathbb{M}_1)(i) \leq x' < H(\mathbb{M}_2)(i) \) and this is a contradiction. \( H \) is also a surjection. Take \( f \in \mathcal{S} \) and let:

\[ \mathcal{L} = \{ [f(i), f(1) + i - f(i)] \mid i \in I \} \cup \{ [0, f(i)] \cup [f(1) + i - f(i), 1] \mid i \in I \} \]

Then by definition of \( \mathcal{S} \), it is easy to see that \( \mathcal{L} \) is a linked system in \( \mathcal{G}_2 \). \( \mathcal{L} \) is contained in some maximal linked system \( \mathbb{M} \) of \( \mathcal{G}_2 \), and for this \( \mathbb{M} \) it holds that \( K(\mathbb{M}) = f(1) \) while for each \( i \in I : H(\mathbb{M})(i) = f(i) \); i.e., \( H(\mathbb{M}) = f \). Finally we prove that \( H \) is continuous. Let \( i, x \in I \). Then

\[ \{ \mathbb{M} \in \lambda_{\mathcal{G}}(I) \mid H(\mathbb{M})(i) \leq x \} = \bigcap_{y \in I} \{ \mathbb{M} \in \lambda_{\mathcal{G}}(I) \mid [0, x] \cup [y, 1] \subseteq \mathbb{M} \} \quad \text{or} \quad [0, x + y - i] \subseteq \mathbb{M} \}

and hence this set is closed. For, let \( \mathbb{M} \in \lambda_{\mathcal{G}}(I) \) such that \( H(\mathbb{M})(i) \leq x \); this last inequality means that \([0, x] \cup [K(\mathbb{M}) + i - x, 1] \subseteq \mathbb{M} \). If \( y \geq K(\mathbb{M}) + i - x \), then \([0, y + x - i] \supset [0, K(\mathbb{M})] \subseteq \mathbb{M} \); if \( y \leq K(\mathbb{M}) + i - x \) then \([0, x] \cup [y, 1] \supset [0, x] \cup [K(\mathbb{M}) + i - x, 1] \subseteq \mathbb{M} \).

Conversely, suppose that

\[ [0, x] \cup [y, 1] \subseteq \mathbb{M} \quad \text{or} \quad [0, x + y - i] \subseteq \mathbb{M} \]

for each \( y \in I \), then also \([0, x + y - i] \subseteq \mathbb{M} \) for each \( y < K(\mathbb{M}) + i - x \); hence \([0, x] \cup [y, 1] \subseteq \mathbb{M} \); we conclude that \([0, x] \cup [K(\mathbb{M}) + i - x, 1] \subseteq \mathbb{M} \), i.e., \( H(\mathbb{M})(i) \leq x \). In the same way one proves:

\[ \{ \mathbb{M} \in \lambda_{\mathcal{G}}(I) \mid H(\mathbb{M})(i) \geq x \} = \bigcap_{y \in I} \{ \mathbb{M} \in \lambda_{\mathcal{G}}(I) \mid [x, y] \subseteq \mathbb{M} \} \quad \text{or} \quad [x + y - i, 1] \subseteq \mathbb{M} \}

and hence is closed.

As a consequence of these two theorems we have, as announced,

**Theorem 4.3.** \( \lambda_{\mathcal{G}}(I) \backsimeq Q \).
5. A superextension of the $n$-sphere

In this final section we show that the superextension of the $n$-sphere $S^n$ with respect to the collection of all closed massive $n$-balls in $S^n$ is homeomorphic with the Hilbert-cube. As usual, the $n$-sphere $S^n$ is the space

$$\left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\right\}$$

and the closed massive $n$-ball with centre $x \in S^n$ and radius $\varepsilon \geq 0$ is the set

$$B(x, \varepsilon) = \{y \in S^n \mid d(x, y) \leq \varepsilon\}.$$ 

Writing $\mathcal{B}$ for the collection of all closed massive $n$-balls in $S^n$, we will prove that, if $n \geq 1$, $\lambda_{\mathcal{B}}(S^n) \cong Q$. Obviously $\lambda_{\mathcal{B}}(S^n)$ is the superextension of the circle with respect to the set of closed intervals. For the definition of $\mathcal{B}$ it does not matter whether the euclidian metric of $\mathbb{R}^{n+1}$ or the sphere metric of $S^n$ (in this case the distance between $x$ and $y$ in $S^n$ is $\arccos \sum_{i=0}^{n} x_i y_i$, i.e., the minimum length of a curve between $x$ and $y$ on $S^n$) is used. However, in the proof of the theorem we need the latter metric and we call this metric $d$. Furthermore we define, for each point $x = (x_0, x_1, \ldots, x_n) \in S^n$, the antipode $\overline{x}$ of $x$ by $\overline{x} = (-x_0, -x_1, \ldots, -x_n)$.

**Theorem 5.1.** If $n \geq 1$, $\lambda_{\mathcal{B}}(S^n)$ is homeomorphic to the Hilbert-cube $Q$.

**Proof.** In fact we show that $\lambda_{\mathcal{B}}(S^n)$ is compact and infinite-dimensional and has a countable base and that $\lambda_{\mathcal{B}}(S^n)$ can be embedded as a convex subspace in $\mathbb{R}^\infty$; hence, by KELLER's theorem, $\lambda_{\mathcal{B}}(S^n)$ is homeomorphic to $Q$. Clearly, $\lambda_{\mathcal{B}}(S^n)$ is compact.

To prove that $\lambda_{\mathcal{B}}(S^n)$ has a countable base, let $X$ be a countable dense subset of $S^n$. Define $\mathcal{B}_0 = \{B(x, \varepsilon) \mid x \in X, \varepsilon \in Q, \varepsilon \geq 0\}$. It is not difficult to see that $P : \lambda_{\mathcal{B}}(S^n) \rightarrow \lambda_{\mathcal{B}_0}(S^n)$, such that $P(\mathcal{B}_0) = 3 \cap \mathcal{B}$ ($3 \in \lambda_{\mathcal{B}}(S^n)$) is a homeomorphism; hence, since $\lambda_{\mathcal{B}_0}(S^n)$ has a countable base, $\lambda_{\mathcal{B}}(S^n)$ also has a countable base. Next, $\lambda_{\mathcal{B}}(S^n)$ is infinite-dimensional, since $\lambda_{\mathcal{B}_0}(I)(\cong Q)$ can be embedded in $\lambda_{\mathcal{B}}(S^n)$. For, let

$$Y = \{x \in S^n \mid x = (x_0, x_1, \ldots, x_n), x_1 \geq 0, x_2 = \ldots = x_n = 0\};$$

this subspace is homeomorphic to $I$. Let $\mathcal{C}_2$ be as defined in Section 3, i.e., $\mathcal{C}_2$ is the collection of all closed subsets $Y'$ if $Y$ such that $Y'$ is connected or $Y \setminus Y'$ is connected. Define $T : \lambda_{\mathcal{B}_0}(Y) \rightarrow \lambda_{\mathcal{B}}(S^n)$ by $T(3) = \{B \in \mathcal{B} \mid B \cap Y \in 3\}$ ($3 \in \lambda_{\mathcal{B}}(I)$). Again it is not difficult to prove that $T$ is a topological embedding. Hence $\lambda_{\mathcal{B}_0}(I)(\cong Q)$ can be embedded in $\lambda_{\mathcal{B}}(S^n)$, i.e., $\lambda_{\mathcal{B}}(S^n)$ is infinite-dimensional.
Finally we embed \( \lambda (S^n) \) as a convex subspace in \( \mathbb{R}^{S^n} \), by means of the function \( U: \lambda (S^n) \rightarrow \mathbb{R}^{S^n} \), determined by:

\[
U(\mathcal{M})(x) = \inf \{ \varepsilon \geq 0 \mid B(x, \varepsilon) \in \mathcal{M} \}, \quad (\mathcal{M} \in \lambda (S^n), \ x \in S^n).
\]

The mapping \( U \) is continuous and one-to-one since \( U(\mathcal{M})(x) \leq \varepsilon \) iff \( B(x, \varepsilon) \in \mathcal{M} \), and \( U(\mathcal{M})(x) \geq \varepsilon \) iff \( B(x, \pi - \varepsilon) \in \mathcal{M} \). And indeed, \( U[\lambda (S^n)] \) is a convex subspace of \( \mathbb{R}^{S^n} \). In order to show this, we need only prove: if \( \mathcal{M}_1, \mathcal{M}_2 \in \lambda (S^n) \), then there exists an \( \mathcal{M} \in \lambda (S^n) \) such that \( U(\mathcal{M}) = \frac{1}{2} U(\mathcal{M}_1) + \frac{1}{2} U(\mathcal{M}_2) \) (\( U[\lambda (S^n)] \) being compact and hence closed in \( \mathbb{R}^{S^n} \)). So take \( \mathcal{M}_1, \mathcal{M}_2 \in \lambda (S^n) \) and let \( \mathcal{M}_3 = \{ B(x, \varepsilon) \mid x \in S^n, \ \varepsilon \geq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x) \} \). Then \( \mathcal{M}_3 \) is a linked system, because if \( B(x, \varepsilon) \) and \( B(y, \delta) \in \mathcal{M}_3 \) (\( x, y \in S^n, \ \varepsilon \geq \delta \geq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x) \)), then:

\[
d(x, y) \leq U(\mathcal{M}_1)(x) + U(\mathcal{M}_1)(y),
\]

and

\[
d(x, y) \leq U(\mathcal{M}_2)(x) + U(\mathcal{M}_2)(y);
\]

hence

\[
d(x, y) \leq \delta + \varepsilon,
\]

i.e.,

\[
B(x, \varepsilon) \cap B(y, \delta) \neq \emptyset.
\]

Let \( \mathcal{M}_3 \) be a maximal linked system containing \( \mathcal{M}_3 \) (in fact \( \mathcal{M}_3 \) is itself a maximal linked system). Then, clearly,

\[
U(\mathcal{M}_3)(x) \leq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x),
\]

and

\[
U(\mathcal{M}_3)(x) \leq \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x) \text{ for each } x \in S^n.
\]

But, since for each maximal linked system \( \mathcal{M} : U(\mathcal{M})(x) + U(\mathcal{M})(x) = \pi \), we have

\[
U(\mathcal{M}_3)(x) = \frac{1}{2} U(\mathcal{M}_1)(x) + \frac{1}{2} U(\mathcal{M}_2)(x) \text{ for each } x \in S^n.
\]

Thus

\[
U(\mathcal{M}_3) = \frac{1}{2} U(\mathcal{M}_1) + \frac{1}{2} U(\mathcal{M}_2).
\]
REFERENCES


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