TOPOLOGICAL STRUCTURES II, MATHEMATICAL CENTRE TRACTS 115 (1979) 167-168

A SIMPLE OBSERVATION CONCERNING THE EXISTENCE OF NON-LIMIT POINTS IN SMALL COMPACT F—SPACES

Jan van Mill

All spaces are completely regular and for all undefined terms we refer to [CN].

FRANKIEWICZ [F] has shown that under MA (a consequence of CH) in each compact extremally disconnected space X of weight 2^{ω} there is a point $x \in X$ which is not a limit point of any countable discrete subset of X. For related results see [K_1], [K_2], [vM].

The aim of this note is to point out that under the stronger hypothesis CH a stronger result can quite easily be derived; apparently, this proof has been overlooked.

THEOREM (CH): *Let X be a compact F-space of weight 2^{ω}. Then there is an $x \in X$ such that $x \notin \bar{D}$ for each countable discrete $D \subset X - \{x\}$.*

PROOF. Striving for a contradiction, we assume that each point of X is a limit point of some countable discrete set. Let $\{U_n : n \in \omega\}$ be a family of nonempty pairwise disjoint open F_σ's of X. Define $Y = \bigcap_{n<\omega} \bar{Y}_n$ and let $\mathcal{B}$ be the collection of all nonempty open F_σ's of Y. The family

$$\mathcal{E} = \{\partial B : B \in \mathcal{B}\}$$

has clearly cardinality 2^{ω}. List $\mathcal{E}$ as $\{E_\alpha : \alpha < \omega_1\}$ (by CH) and let $\{F_\alpha : \alpha < \omega_1\}$ list the boundaries of the nonempty closed G_δ's of $\bar{Y} - Y$. Since each nonempty G_δ in $Y^* = \bar{Y} - Y$ has nonempty interior ([FG, 3.1]) by a straightforward induction ([R]) we can construct for each $\alpha < \omega_1$ a nonempty open set V_α of Y^* such that

- if $\alpha < \kappa$ then $\bar{V}_\kappa \subset V_\alpha$;
- $V_\alpha \cap (\bar{E}_\alpha \cup F_\alpha) = \emptyset$

(observe that $\bar{E}_\alpha \cap Y^*$ is nowhere dense in Y^*, cf. [WO, 2.11]). Take $x \in \bigcap_{\alpha<\omega_1} \bar{V}_\alpha$. By assumption $x \in \bar{D}$ for some countable discrete $D \subset X - \{x\}$. Since x is a P-point of Y^*, we may assume that $D \cap Y^* = \emptyset$. Moreover, since X is an F-space and D is countable, we may assume that $D \cap (X - \bar{Y}) = \emptyset$. List D as $\{d_n : n < \omega\}$. By assumption, each point of D is a limit point of some countable discrete set in X. Since $D \subset Y$ and since Y contains a dense open F_σ of X, each point of D is a limit point of some countable discrete set in Y. Hence, there exists a family $\mathcal{U} = \{U_n^m : n,m < \omega\}$ of nonempty open F_σ's of Y such that

- $d_n \in (\bigcup\{U_n^m : m < \omega\})^- - \bigcup\{U_n^m : m < \omega\}$;
- if $k \neq n$ then $\bigcup\{U_k^m : m < \omega\} \cap \bigcup\{U_n^m : m < \omega\} = \emptyset$.

Put $B = \bigcup_{n<\omega} \bigcup_{m<\omega} U_n^m$. Then $D \subset \partial B$, and hence, by construction, $x \notin \bar{D}$; a contradiction. □

REFERENCES

[CN] COMFORT, W.W. & S. NEGREPONTIS, *The theory of ultrafilters*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band 211, Springer (1974).

[FG] FINE, N.J. & L. GILLMAN, *Extension of continuous functions in* $\beta\mathbb{N}$, Bull. Amer. Math. Soc. 66 (1960) 376-381.

[F] FRANKIEWICZ, R., *Non-accessible points in extremally disconnected compact spaces*, (to appear in Fund. Math.).

[K_1] KUNEN, K., *Some points in* βN, Math. Proc. Cambridge Philosophical Society, 80 (1976) 385-398.

[K_2] KUNEN, K., *Weak P-points in* N^*, (to appear).

[vM] MILL, J. van, *Weak P-points in compact F-spaces*, (to appear).

[R] RUDIN, W., *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. 23 (1956) 409-419.

[WO] WOODS, R.G., *Co-absolutes of remainders of Stone-Čech compactifications*, Pacific J. Math. (1971) 545-560.