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ON SUPEREXTENSIONS AND HYPERSPACES

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0. INTRODUCTION

The *superextension* $\lambda(X)$ (or λX) of a topological space X has been introduced by DE GROOT in [2]. Although its construction parallels the construction of Wallman compactifications, its properties are firmly distinct, and, in general, $\lambda(X)$ is a much nicer space. For instance, $\lambda(X)$ is a metric AR if (and only if) X is a metric continuum (cf. van MILL [4] or van de VEL [12]); $\lambda(X)$ is a C^{∞} and LC^{∞} space if X satisfies certain weak assumptions, such as separability + path connectedness, or, σ-compactness + finite (homotopy) category (cf. van MILL & van de VEL [9]). Also $\lambda(X)$ has the fixed point property if X is a connected normal T_1-space (cf. van de VEL [12]).

In all of these results, the *hyperspace* H(X) of a space X has been of invaluable help. The present paper is concerned with the relationship between the two kinds of topological extensions: λ, H. We shall first prove that $\lambda(X)$ is a subspace of H(H(X)) for compact X (cf. Section 2). The proof of this nontrivial fact depends on the use of "compact" subbases, which were studied in van MILL & van de VEL [8]. With these techniques, we are able to derive more results at the time, e.g. that a certain "transversality" map in H(H(X)) is continuous and that its fixed point set is exactly $\lambda(X)$. Also, we prove that a certain "convex closure operator" in H(H(X)) is continuous. Finally, we use subbase convexity theory again to derive a retraction property of $\lambda(X)$ in H(H(X)).

In view of the above facts, superextension theory can be looked upon as a kind of hyperspace theory. Both theories have also met with a same conjecture: H(X), or $\lambda(X)$, is a Hilbert cube for suitable X. Concerning

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$H(X)$, this conjecture has been settled in the affirmative by the work of CURTIS, SCHORI and WEST (cf. [1] and [11]). Concerning $\lambda(X)$, it has been proved by van MILL (cf. [4]) that $\lambda[0,1]$ is a Hilbert cube, and (recently) that λX is a Hilbert cube iff X is a nondegenerate metric continuum (cf. [7]). The proof of this result uses the above mentioned retraction property of $\lambda(X)$ in $H(H(X))$.

1. COMPACT SUBBASES IN HYPERSPACES

The *hyperspace* of a T_1 space X will be denoted by $H(X)$. If $A_1,\ldots,A_n$ are nonempty subsets of X, then we write

$$\langle A_1,\ldots,A_n\rangle = \{D \in H(X) \mid D \subset \bigcup_{i=1}^{n} A_i \text{ and } D \cap A_i \neq \emptyset \text{ for each } i = 1,\ldots,n\}.$$

With this notation, the family

$$\mathcal{H} = \mathcal{H}(X) = \{\langle C\rangle \mid C \in H(X)\} \cup \{\langle C,X\rangle \mid C \in H(X)\}$$

constitutes a closed subbase for $H(X)$.

If $\mathcal{S}$ is a closed subbase of X, then a nonempty subset C of X is called *$\mathcal{S}$-convex* if $C = \bigcap\mathcal{C}$ for some $\mathcal{C} \subset \mathcal{S}$. We let $H(X,\mathcal{S})$ denote the subspace of $H(X)$, consisting of all $\mathcal{S}$-convex sets of X. We say that the closed subbase $\mathcal{S}$ is *compact* if; (i) $H(X,\mathcal{S})$ is a normal T_1 family, and; (ii) the space $H(X,\mathcal{S})$ is compact.

Recall that a closed subbase $\mathcal{S}$ is *normal* if any two disjoint members of $\mathcal{S}$ can be separated by disjoint complements of members of $\mathcal{S}$, and that $\mathcal{S}$ is *T_1* if for each $S \in \mathcal{S}$ and $x \in X - S$ there is an $S' \in \mathcal{S}$ with $x \in S' \subset X - S$. See van MILL & van de VEL [8].

<u>THEOREM 1.1</u>. *Let X be compact T_1, and let $\mathcal{S}$ be a closed normal T_1 subbase of X which is closed under formation of intersections. Then the following assertions are equivalent:*

(a) *$\mathcal{S}$ is a compact subbase;*

(b) *the $\mathcal{S}$-convex closure operator $I_{\mathcal{S}}: H(X) \to H(X,\mathcal{S})$ which sends $C \in H(X)$ onto $I_{\mathcal{S}}(C) = \bigcap\{S \mid C \subset S \in \mathcal{S}\}$, is continuous;*

(c) *the space $H(X,\mathcal{S})$ admits a closed normal T_1 subbase, consisting of all sets of type $\langle C\rangle \cap H(X,\mathcal{S})$ or $\langle C,X\rangle \cap H(X,\mathcal{S})$, where $C \in H(X,\mathcal{S})$.*

See [15], Theorem 2.6.

We now present a characterization of convexity in H(X), relative to its canonical subbase $\mathcal{H} = \mathcal{H}(X)$. This result will be used to prove our basical result that $\mathcal{H}$ is actually a compact subbase for compact X.

Let $\mathcal{A} \subset H(X)$ be closed and nonempty, and let $B \in H(X)$. If B meets all members of $\mathcal{A}$, then we call B *a transversal set of* $\mathcal{A}$. We let $\perp(\mathcal{A})$ denote the collection of all transversal sets of $\mathcal{A}$. With this notation, one can easily check the following formula on the convex closure operator $I_{\mathcal{H}}$, related to the subbase $\mathcal{H}$ of H(X):

$$I_{\mathcal{H}}(\mathcal{A}) = \cap\{\langle B,X\rangle \mid B \in \perp(\mathcal{A})\} \cap \langle \cup\mathcal{A}\rangle$$

THEOREM 1.2. *Let X be compact Hausdorff, and let $\mathcal{A} \subset H(X)$ be closed and nonempty. Then the following assertions are equivalent:*

(i) *$\mathcal{A}$ is $\mathcal{H}$-convex;*

(ii) *if $B \in H(X)$ and if $A \subset B \subset \cup\mathcal{A}$ for some $A \in \mathcal{A}$, then $B \in \mathcal{A}$.*

PROOF. Let $\mathcal{A}$ be $\mathcal{H}$-convex, let $B \in H(X)$, and assume that $A \subset B \subset \cup\mathcal{A}$ for some $A \in \mathcal{A}$. For each $C \in \perp(\mathcal{A})$, we have that $C \cap A \neq \emptyset$, and hence that $C \cap B \neq \emptyset$. Also, $B \in \langle \cup\mathcal{A}\rangle$, whence $B \in I_{\mathcal{H}}(\mathcal{A}) = \mathcal{A}$ by the above formula.

Assume next that $\mathcal{A}$ satisfies condition (ii), and that there is a $B \in I_{\mathcal{H}}(\mathcal{A}) - \mathcal{A}$. Then $B \subset \cup\mathcal{A}$, and by (ii), $\langle B\rangle \cap \mathcal{A} = \emptyset$. $\mathcal{A}$ being closed and $\langle B\rangle$ being compact, there is an open set $\mathcal{O} \supset \langle B\rangle$ of H(X) of type

$$\bigcup_{k=1}^{m} \langle O_1^k,\ldots,O_p^k\rangle, \qquad O_l^k \text{ open in } X,$$

which does not meet $\mathcal{A}$. For each $b \in B$ we put

$$O_b = \cap\{O_l^k \mid b \in O_l^k,\ k = 1,\ldots,m,\ l = 1,\ldots,p\}.$$

In this way, we obtain but a finite number of different open sets of X, say $O_1,\ldots,O_n$. Writing $I = \{1,\ldots,n\}$, we show that

$$\langle B\rangle \subset \cup\{\langle O_j \mid j \in J\rangle \mid \emptyset \neq J \subset I\} \subset \mathcal{O} \tag{*}$$

In fact,

for some $b_1,\dots,b_r \in B$. Hence there is a $k \in \{1,\dots,m\}$ such that

$$\{b_1,\dots,b_r\} \in \langle 0_1^k,\dots,0_p^k\rangle.$$

Therefore, each 0_j is contained in some 0_l^k, and each 0_l^k contains some 0_j, whence

$$\langle 0_j \mid j \in J\rangle \subset \langle 0_1^k,\dots,0_p^k\rangle.$$

The other half of (*) is obviuous, using $B \subset \bigcup_{i=1}^n 0_i$.

Let $A \in \mathcal{A}$. If A does not meet $\bigcap_{i=1}^n X - 0_i$, then $A \subset \bigcup_{i=1}^n 0_i$, and hence $A \in \langle 0_j \mid j \in J\rangle$, where $J = \{i \mid A \cap 0_i \neq \emptyset\}$, contradicting that $\mathcal{A} \cap \mathcal{O} = \emptyset$. Hence $\bigcap_{i\in I} X - 0_i$ is a transversal set of $\mathcal{A}$ which does not meet B. This contradicts the fact that B is in $I_{\mathcal{H}}(\mathcal{A})$. □

As a direct consequence of this theorem, it follows that $\perp(\mathcal{A})$ is $\mathcal{H}$-convex for each nonempty closed $\mathcal{A} \subset H(X)$.

<u>THEOREM 1.3</u>. *Let* X *be compact Hausdorff. Then* $\mathcal{H} = \mathcal{H}(X)$ *is a compact subbase of* $H(X)$.

<u>PROOF</u>. Let $\mathcal{A} \in HH(X)$ be nonconvex. Then by the previous theorem, there exists a $B \in H(X)$ and an $A_0 \in \mathcal{A}$ such that

$$A_0 \subset B \subset \cup\mathcal{A}; \qquad B \notin \mathcal{A}.$$

Let $\mathcal{O}$, $\mathcal{P}$ be disjoint open sets of $H(X)$ such that $B \in \mathcal{P}$, $\mathcal{A} \subset \mathcal{O}$. Then

$$B \in \langle 0_1,\dots,0_n\rangle \subset \mathcal{P}$$

for some open sets $0_1,\dots,0_n$ of X. We assume that, among the latter, $0_1,\dots,0_p$ $(p \le n)$ are *all* sets meeting A_0. Notice that $p < n$, and that $A_0 \in \langle 0_1,\dots,0_p\rangle$. For each k with $p < k \le n$, we choose $b_k \in B \cap 0_k$. As $B \subset \cup\mathcal{A}$, there is an $A_k \in \mathcal{A}$ with $b_k \in A_k$, and hence $A_k \cap 0_k \neq \emptyset$. Therefore,

$$\mathcal{V} = \langle\mathcal{O}\rangle \cap \langle\langle 0_1,\dots,0_p\rangle, \langle 0_{p+1},X\rangle,\dots,\langle 0_n,X\rangle,H(x)\rangle$$

is a neighbourhood of $\mathcal{A}$ in $HH(X)$, no member of which is $\mathcal{H}$-convex. In fact, if $\mathcal{A}' \in \mathcal{V}$, then there exist $A_0', A_{p+1}',\dots,A_n' \in \mathcal{A}'$ such that

$$A'_0 \in \langle O_1,\dots,O_p\rangle; \quad A'_k \in \langle O_k, X\rangle \qquad \text{for } p < k \le n.$$

Choose $a'_k \in A'_k \cap O_k$ for each $p < k \le n$, and let $B' = A'_0 \cup \{a'_{p+1},\dots,a'_n\}$. Then

$$A'_0 \subset B' \subset \cup\mathcal{A}'; \quad B' \in \langle O_1,\dots,O_n\rangle \subset \mathcal{P}; \quad \mathcal{A}' \subset \mathcal{O},$$

whence $B' \notin \mathcal{A}'$, and $\mathcal{A}'$ is not $\mathcal{H}$-convex.

This shows that the space $H(H(X),\mathcal{H})$ is compact, being a closed subspace of the compact space $HH(X)$ (cf. MICHAEL [3]), and it remains to be verified that the family $H(H(X),\mathcal{H})$ is normal and T_1:

Let $\mathcal{A}$, $\mathcal{B} \subset H(X)$ be disjoint $\mathcal{H}$-convex sets, say

$$\mathcal{A} = \cap\{\langle C,X\rangle \mid C \in \perp(\mathcal{A})\} \cap \langle A\rangle \qquad (A = \cup\mathcal{A}),$$

$$\mathcal{B} = \cap\{\langle D,X\rangle \mid D \in \perp(\mathcal{B})\} \cap \langle B\rangle \qquad (B = \cup\mathcal{B}).$$

Then $A \cap B$ cannot meet all members of $\perp(\mathcal{A}) \cup \perp(\mathcal{B})$, for otherwise $A \cap B \in \mathcal{A} \cap \mathcal{B}$. So e.g. $A \cap B \cap C = \emptyset$, where $C \in \perp(\mathcal{A})$. X being normal, there exist closed sets K, L in X with

$$A \cap C \subset K - L; \quad B \subset L - K; \quad K \cup L = X.$$

Hence,

$$\mathcal{A} \subset \langle A\rangle \cap \langle C,X\rangle \subset \langle A \cap C, X\rangle \subset \langle K,X\rangle$$

$$\mathcal{B} \subset \langle B\rangle \subset \langle L\rangle,$$

whereas $\mathcal{A} \cap \langle L\rangle = \emptyset$, $\mathcal{B} \cap \langle K,X\rangle = \emptyset$, and $\langle L\rangle \cup \langle K,X\rangle = H(X)$. The T_1-property is obvious. □

Combining Theorems 1.1 and 1.3 yields:

<u>COROLLARY 1.4</u>. *Let X be compact Hausdorff. Then the convex closure operator*

$$I_{\mathcal{H}}: HH(X) \to H(H(X),\mathcal{H})$$

is continuous. □

A *linked system on a space* X is a collection $M \subset H(X)$ such that any two members of M have a nonempty intersection. Equivalently, $M \subset \perp(M)$. A linked system M on X is *maximal* (or, M is an *mls*) if it is not properly contained in another linked system on X. The reader can verify that M is an mls iff $M = \perp(M)$.

COROLLARY 1.5. *Let* X *be compact Hausdorff. Then the transversality map* $\perp: H(H(X)) \to H(H(X))$ *is continuous, and its fixed point set is exactly the collection* $\lambda(X)$ *of all mls's on* X.

PROOF. As we noted before, $\perp(\mathcal{A})$ is H-convex for each $\mathcal{A} \in HH(X)$. Hence, the map $\perp$ factors through the subspace $H(H(X),H)$ of $HH(X)$. To prove continuity of $\perp$, it now suffices to use the closed subbase of $H(H(X),H)$, consisting of all sets of type $\langle S\rangle$ or $\langle S,H(X)\rangle$, where $S \subset H(X)$ is H-convex (cf. Theorem 1.1(c)). For convenience, we write $f = \perp$, and we let

$$S = \bigcap\{\langle B,X\rangle \mid B \in \perp(S)\} \cap \langle C\rangle \qquad (S \neq \emptyset).$$

(i). *Computation of* $f^{-1}\langle S,H(X)\rangle$. Let $\mathcal{A} \in HH(X)$. Then $\mathcal{A} \in f^{-1}\langle S,H(X)\rangle$ iff $\perp(\mathcal{A}) \cap S \neq \emptyset$, iff for some $A \in \perp(\mathcal{A})$, $A \subset C$ and A meets all members of $\perp(S)$, iff $C \in \perp(\mathcal{A})$, iff $\mathcal{A} \subset \langle C,X\rangle$. Hence:

$$f^{-1}\langle S,H(X)\rangle = \langle\langle C,X\rangle\rangle.$$

(ii) *Computation of* $f^{-1}\langle S\rangle$. Assume first that $C \neq X$. Then $f^{-1}\langle S\rangle = \emptyset$, since for each $\mathcal{A} \in f^{-1}\langle S\rangle$, $X \in \perp(\mathcal{A}) \subset S \subset \langle C\rangle$, which is impossible. Assume now that $C = X$, and let $\perp(\mathcal{A}) \subset S$. Then

$$\forall B \in \perp(S)\,\exists A \in \mathcal{A}: A \subset B \tag{*}$$

In fact, assume to the contrary that for some $B \in \perp(S)$, $A \cap (X - B) \neq \emptyset$ for all $A \in \mathcal{A}$. Fix $a_A \in A - B$ for each $A \in \mathcal{A}$. X being regular, there exist disjoint open sets O_A, P_A of X with $a_A \in O_A$ and $B \subset P_A$. By the compactness of $\mathcal{A} \subset H(X)$, there exist $A_1,\ldots,A_n \in \mathcal{A}$ such that each $A \in \mathcal{A}$ meets one of $O_{A_1},\ldots,O_{A_n}$. Let $P = \bigcap_{i=1}^{n} P_{A_i}$. Then each $A \in \mathcal{A}$ meets the closed set $X - P$, whence $X - P \in \perp(\mathcal{A})$. However, $B \cap (X - P) = \emptyset$, contradicting that $\perp(\mathcal{A}) \subset S \subset \langle B,X\rangle$.

Conversely, if $\mathcal{A} \in HH(X)$ satisfies (*), then $\perp(\mathcal{A}) \subset S$. In fact, to each $B \in \perp(S)$ we can assign an $A \in \mathcal{A}$ with $A \subset B$. Hence, if $D \in \perp(\mathcal{A})$, then D meets each $A \in \mathcal{A}$, and hence it meets B, proving that

$$\perp(\mathcal{A}) \subset \bigcap\{\langle B,X\rangle \mid B \in \perp(\mathcal{S})\} = \mathcal{S}.$$

Using the formula (*), it now follows that

$$f^{-1}\langle\mathcal{S}\rangle = \bigcap\{\langle\langle B\rangle,H(X)\rangle \mid B \in \perp(\mathcal{S})\}.$$

In both cases (i) and (ii), we find that the inverse image is a closed set of HH(X). □

2. SUPEREXTENSIONS

For a T_1-space X, the collection $\lambda(X)$ of all maximal linked systems on X is given a topology, generated by the closed subbase

$$H(X)^+ = \{C^+ \mid C \in H(X)\},$$

where $C^+ = \{\mathcal{M} \in \lambda(X) \mid C \in \mathcal{M}\}$. With this topology, $\lambda(X)$ is called *the superextension of* X. See VERBEEK [13] or van MILL [6] for details. Notice that $\lambda(X)$ is compact.

The present section is mainly concerned with embedding and retraction properties of $\lambda(X)$ in HH(X).

THEOREM 2.1. *Let* X *be a compact Hausdorff space. Then* $\lambda(X)$ *is a subspace of* HH(X).

PROOF. As each $\mathcal{M} \in \lambda(X)$ is obviously a closed subfamily of H(X), and satisfies $\mathcal{M} = \perp(\mathcal{M})$, we find that $\mathcal{M}$ is $\mathcal{H}(X)$-convex and hence that $\lambda(X)$ is a *subset* of $H(H(X),\mathcal{H})$. We are again in a position to use the closed subbase of $H(H(X),\mathcal{H})$ mentioned before, to prove that the inclusion mapping $\lambda(X) \subset HH(X)$ is continuous. Let $\mathcal{S}$ be $\mathcal{H}$-convex, say

$$\mathcal{S} = \bigcap\{\langle B,X\rangle \mid B \in \perp(\mathcal{S})\} \cap \langle C\rangle.$$

(i) $\langle\mathcal{S},H(X)\rangle \cap \lambda(X) = C^+$:

In fact, as $\mathcal{S} \neq \emptyset$, we have that $C \cap B \neq \emptyset$ for each $B \in \perp(\mathcal{S})$. Therefore, an mls $\mathcal{M}$ is in $\lambda(X) \cap \langle\mathcal{S},H(X)\rangle$ iff $\mathcal{M} \cap \mathcal{S} \neq \emptyset$, iff $C \in \mathcal{M}$, iff $\mathcal{M} \in C^+$.

(ii) $\langle S\rangle \cap \lambda(X) = \emptyset$ if $C \neq X$ and $\langle S\rangle \cap \lambda(X) = \bigcap\{B^+ \mid B \in \bot(S)\}$ otherwise: If $C \neq X$, then no mls M can satisfy $M \subset S \subset \langle C\rangle$ since $X \in M$. Assuming $C = X$ we have $M \subset S$ iff for each $B \in \bot(S)$ and for each $M \in M$, $B \cap M \neq \emptyset$, iff $\bot(S) \subset M$, iff $M \in \bigcap\{B^+ \mid B \in \bot(S)\}$. □

Notice that the above computed traces on $\lambda(X)$ are convex (or empty) relative to the canonical subbase of $\lambda(X)$.

A remarkable fact is that for metric compacta there is a direct proof of the above theorem without intervenience of compact subbases. Instead, we use the following metrizability result of VERBEEK [13]: if d is a metric on a compact space X, then the formula

$$\bar{d}(M,N) = \inf\{r \mid \forall M \in M\colon B_r(M) \in N\}$$

(where $B_r(M) = \{x \mid d(x,M) \le r\}$) defines a metric on $\lambda(X)$, compatible with its original topology. We notice that if X is compact metric, say with metric d, then H(X) is metrized by the well-known Hausdorff metric, denoted by d_H.

We now prove the following result, adding some information to Theorem 2.1:

<u>THEOREM 2.2</u>. *Let* (X,d) *be a compact metric space. Then the inclusion mapping*

$$(\lambda(X),\bar{d}) \to (HH(X),(d_H)_H)$$

is an isometry.

<u>PROOF</u>. Let $M,N \in \lambda(X)$ and let $\bar{d}(M,N) = r$. Hence, if $N \in N$, then $B_r(N) \in M$ and consequently, $d_H(N,M) \le r$. Similarly, $d_H(M,N) \le r$ for each $M \in M$, showing that $(d_H)_H(M,N) \le r$.
Let $s = (d_H)_H(M,N)$. For each $M \in M$ we can then find an $N \in N$ such that $d_H(M,N) \le s$, whence $N \subset B_s(M)$ and $B_s(M) \in N$. Therefore, $\bar{d}(M,N) \le s$. □

More information on the above (metric) embedding is presented in the next result.

Let $L(X) \subset HH(X)$ denote the subspace of all closed linked systems on X. Then $\lambda(X)$ is a subspace of $L(X)$. We now describe how to extend linked systems to maximal linked systems in a continuous way.

THEOREM 2.3. *Let* X *be a compact Hausdorff space. Then there is a continuous retraction*

$$h: L(X) \to \lambda(X)$$

extending each linked system to a maximal linked system. If X *is metrizable moreover, then* h *can be chosen such as to be a metric contraction.*

PROOF. Fix an $x \in X$. For each $L \in L(X)$ we put

$$h'(L) = L \cup \{M \mid x \in M \in H(X) \text{ and } L \cup \{M\} \text{ is linked}\} \qquad (*)$$

It has been proved in van MILL [5] that $h'(L)$ is a linked system which is contained in a unique maximal linked system, which we denote by $h(L)$. This gives a mapping $h: L(X) \to \lambda(X)$, and we show that h has all the desired properties:

If T is a closed subbase of a space Y, then we let $L(Y,T)$ denote the subspace of $H(H(Y,T))$, consisting of all closed linked systems $L \subset H(Y,T)$. With this notation, we have the following composition maps:

$$L(X) \xrightarrow{(\ )^+} L(\lambda(X),H(X)^+) \overset{\cap}{\to} H(\lambda(X),H(X)^+) \overset{p_x}{\to} \lambda(X) \qquad (**)$$

The first map, $(\ )^+$, sends $L \in L(X)$ $(= (L(X,H(X)))$ onto

$$L^+ = \{L^+ \mid L \in L\},$$

where $(\ )^+$ refers to the construction described at the beginning of this section. The second map is the *intersection operator*, sending $M \in L(\lambda(X),H(X)^+)$ onto $\cap M$. It is easy to verify that $\cap M \neq \emptyset$. The third map is a restriction of the so-called *nearest point mapping* of $\lambda(X)$,

$$p: \lambda(X) \times H(\lambda(X),H(X)^+) \to \lambda(X)$$

sending a pair (M,A) onto the unique point $N \in \lambda(X)$ with the property that

$$I\{M,N\} \cap A = \{N\}.$$

(cf. van MILL & van de VEL [8]). In (**), p_x denotes the map $p(x,-)$ (regarding $x \in X$ as a point of $\lambda(X)$), and it has been proved in van de VEL [12] that both constructions (*) and (**) coincide.

All mappings appearing in (**) are continuous, see van MILL & van de VEL [8]. Hence h is continuous

Assume now that X is metrizable, say with a metric d. Using the induced metrics on the superextension $\lambda(X)$ and on the various hyperspaces, we shall prove below that both $\cap$ and p_X are metric contractions. It remains to be verified that the first map, $(\ )^+$, is an isometry. But this is a straightforward consequence of the following elementary facts about $\lambda(X)$:

(i) $B_r(C)^+ = B_r(C^+)$ for each $C \in H(X)$ and $r \geq 0$;

(ii) $A \subset B$ iff $A^+ \subset B^+$ for each $A,B \in H(X)$.

We now prove the contraction property of $\cap$ and p_X cited above. In order to simplify the argument, we give a proof which is valid for all spaces with a normal binary subbase, i.e. a closed normal subbase S such that for each linked system $S' \subset S$ we have that $\cap S' \neq \emptyset$.

As was shown in [8], there is also a *nearest point map*

$$p\colon X \times H(X,S) \to X$$

for such a subbase, satisfying a similar property as in the $\lambda(X)$-case, namely: for each $x \in X$ and $C \in H(X,S)$, $I_S(x,p(x,C)) \cap C = \{p(x,C)\}$, and $p(x,C)$ is the unique point with this property.

In [10], a metric d on X (with a closed subbase S) has been called S-*convex* provided that for each $C \in H(X,S)$ and each $r \geq 0$, $B_r(C) \in H(X,S)$. It is shown in [10] that the above mentioned metric $\bar{d}$ on $\lambda(X)$ is $H(X)^+$-convex, and that each metrizable space with a normal binary subbase S admits an S-convex metric.

<u>LEMMA</u>. *Let S be a normal binary subbase for* X *and let* d *be an S-convex metric on* X. *Then the intersection operator* $\cap\colon L(X,S) \to H(X,S)$ *is a metric contraction with respect to the metrics on* $L(X,S)$ *and* $H(X,S)$ *which are induced by* d.

<u>PROOF</u>. We first show that for each (nonempty) linked system $A \in H(X,S)$ and for each $r \geq 0$ the equality

$$B_r(\cap A) = \cap\{B_r(A) \mid A \in A\} \qquad (*)$$

holds. The inclusion "$\subset$" being obvious, take a point x in the right hand side of (*). Then $B_r(x)$ meets each $A \in A$, and since $B_r(x)$ is S-convex, we find that

$$B_r(x) \cap \cap\mathcal{A} \neq \emptyset$$

by the binarity of $\mathcal{S}$. Hence $x \in B_r(\cap\mathcal{A})$.

Now take $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}(X,\mathcal{S})$ such that $(d_H)_H(\mathcal{L}_1,\mathcal{L}_2) \leq r$. Then

$$\forall L_1 \in \mathcal{L}_1 \exists L_2 \in \mathcal{L}_2: d_H(L_1,L_2) \leq r$$

$$\forall L_2 \in \mathcal{L}_2 \exists L_1 \in \mathcal{L}_1: d_H(L_2,L_1) \leq r$$

and hence it easily follows that $B_r(\cap\mathcal{L}_1) = \cap\{B_r(L_1) \mid L_1 \in \mathcal{L}_1\} \supset \cap\mathcal{L}_2$ by the formula (*). Similarly $B_r(\cap\mathcal{L}_2) \supset \cap\mathcal{L}_1$, which proves that $d_H(\cap\mathcal{L}_1,\cap\mathcal{L}_2) \leq r$. □

The formula (*) is also applied in the proof of the next result:

<u>LEMMA</u>. *Let* $\mathcal{S}$ *be a normal binary subbase for* X *and let* d *be an* $\mathcal{S}$*-convex metric on* X. *Then for each* $x \in X$ *the nearest point map*

$$p(x,-): H(X,\mathcal{S}) \to X$$

is a metric contraction.

<u>PROOF</u>. Let $A,B \in H(X,\mathcal{S})$ and assume that $d_H(A,B) \leq r$. Writing $x_A = p(x,A)$ and $x_B = p(x,B)$, we show that $d(x_A,x_B) \leq r$. Indeed, since $A \subset B_r(B)$,

$$\emptyset \neq B_r(x_A) \cap B \subset B_r(I_{\mathcal{S}}(x,x_A)) \cap B;$$

whence by the construction of p (cf. the above remarks), $x_B \in B_r(I_{\mathcal{S}}(x,x_A))$. On the other hand $B \subset B_r(A)$, and consequently

$$x_B \in B_r(A) \cap B_r(I_{\mathcal{S}}(x,x_A)) = B_r(A \cap I_{\mathcal{S}}(x,x_A)) = B_r(x_A),$$

using formula (*) and the construction of p. □

It has been proved in [10] that the nearest point map p is a metric contraction in the first variable too, and that p(x,A) is also metrically a nearest point of A with regard to x.

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