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WALLMAN COMPACTIFICATIONS AND THE CONTINUUM HYPOTHESIS

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0. INTRODUCTION

All spaces are completely regular. In [10] UL'JANOV constructed a variety of compactifications which are not Wallman compactifications. In addition, combining these results with those of BANDT [1] he showed the following interesting theorem:

(*) CH is equivalent to the statement that every compactification of a separable space is a Wallman compactification.

Consequently, by applying constructions of ŠAPIRO [7] or STEINER & STEINER [9] it follows that under $\neg$CH there is a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ which is not a Wallman compactification. Since under CH every compactification of $\mathbb{N}$ is a Wallman compactification (by (*)) we even have that:

(**) CH is equivalent to the statement that every compactification of $\mathbb{N}$ is a Wallman compactification.

Also, there is a theorem of HAGER [4] which states:

(***) Every compactification of a pseudo-compact space is a Wallman compactification.

At first glance, (**) and (***) do not give us any information concerning non pseudo-compact spaces.

In this note we will show that (**) and (***) imply the following two theorems.

<u>THEOREM 1</u>. CH *is equivalent to the statement that there is a non pseudo-compact space all compactifications of which are Wallman compactifications.*

<u>THEOREM 2</u>. [$\neg$CH]. *For any space* X *the following assertions are equivalent*

(i) X *is pseudo-compact*

(ii) *All compactifications of* X *are Wallman compactifications.*

These two theorems imply that there is no honest (= not requiring additional set theoretic axioms) example of a non pseudo-compact space every compactification of which is a Wallman compactification.

1. THE THEOREMS

Recall that a Wallman compactification of a space X is a compactification γX which has a closed base $\mathcal{B}$ satisfying the following two conditions:
(a) $\mathcal{B}$ is closed under finite intersections and finite unions.
(b) for all $B \in \mathcal{B}$ we have that $B = c\ell_{\gamma X}(B \cap X)$.
(this can easily be derived from a theorem in STEINER [8]).

All our results follow from the following proposition, which is of independent interest.

PROPOSITION 1.1. *Let* X *be any space every compactification of which is a Wallman compactification. Let* B *be a closed subspace of* X. *If one of the following conditions is satisfied,*
(i) X *is normal*
(ii) B *is a* C*-embedded copy of* $\mathbb{N}$ *in* X,
then every compactification of B *is a Wallman-compactification.*

PROOF. Let γB be any compactification of B.

Note that the closure operator in βX has the following two properties:
(a) In both cases B is C^*-embedded in X, so $c\ell_{\beta X}B = \beta B$,
(b) If T is a closed subset of X such that $T \cap B = \Phi$, then $c\ell_{\beta X}B \cap c\ell_{\beta X}T = \Phi$.
- When X is normal, this is clear;
- When B is a C-embedded copy of $\mathbb{N}$, it follows from [3] (GILLMAN & JERISON, page 51 3L).

Let $f: \beta B \to \gamma B$ be the unique map which extends the idenity on B. Define

$$Z := \gamma B \cup (\beta X - \beta B)$$

Let $\xi: \beta X \to Z$ be defined by

$$\begin{cases} \xi(x) = x & (x \in \beta X \setminus \beta B) \\ \xi(x) = f(x) & (x \in \beta B) \end{cases}$$

It is clear that Z supplied with the quotient-topology is a (Hausdorff)-compactification of X, say $Z = \gamma_0 X$, such that $c\ell_Z B = \gamma B$. By assumption, Z is a Wallman compactification of X. Let $\mathcal{T}$ be a closed base for Z such that $\mathcal{T}$ is closed under finite unions and finite intersections, while in addition $c\ell_Z(T \cap X) = T$ for all $T \in \mathcal{T}$. Define

$$\mathcal{F} = \{T \cap \gamma B \mid T \in \mathcal{T}\}.$$

It is clear that $\mathcal{F}$ is a closed base for γB which is closed under finite unions and finite intersections. We claim that:

$$c\ell_{\gamma B}(F \cap B) = F$$

for all $F \in \mathcal{F}$, which suffices to prove the proposition.

Indeed, take $F \in \mathcal{F}$, say $F = T \cap \gamma B$ and assume there is a point x such that $x \in F - c\ell_{\gamma B}(F \cap B)$. Since $\mathcal{T}$ is a closed base for $\gamma_0 X$ we may take $T_0 \in \mathcal{T}$ such that $x \in T_0$ and $T_0 \cap c\ell_{\gamma B}(F \cap B) = \Phi$. Define $T_1 = T \cap T_0$.

Then

$$T_1 \cap B = T \cap T_0 \cap B = T_0 \cap F \cap B = \Phi.$$

So, (b) implies that $c\ell_{\beta X}(T_1 \cap X) \cap c\ell_{\beta X}B = c\ell_{\beta X}(T_1 \cap X) \cap \beta B = \Phi$. Therefore, $c\ell_Z(T_1 \cap X) \cap \gamma B \subset \xi[c\ell_{\beta x}(T_1 \cap X)] \cap \xi[\beta B] = \Phi$. But this is a contradiction, since

$$x \in F \cap T_0 \cap \gamma B = T \cap T_0 \cap \gamma B = T_1 \cap \gamma B = c\ell_Z(T_1 \cap X) \cap \gamma B.$$

We conclude that γB is a Wallman compactification. □

From this proposition the two theorems are immediately clear.

2. REMARKS

Recall that a compactification γX of X is called a GA compactification provided that there is a closed subbase $\mathcal{T}$ for γX such that:

(α) for each $x \in \gamma X$ and $T \in \mathcal{T}$ such that $x \notin T$ there is a $T_0 \in \mathcal{T}$ with $x \in T_0$ and $T_0 \cap T = \Phi$.

(β) for all disjoint $T_0, T_1 \in \mathcal{T}$ there is a finite cover $\mathcal{M}$ of γX by elements of $\mathcal{T}$ such that each $M \in \mathcal{M}$ meets at most one of T_0 and T_1.

(γ) for all $T_0, T_1 \in \mathcal{T}$ with $T_0 \cap T_1 \neq \Phi$ we have that $T_0 \cap T \cap X \neq \Phi$.

(cf. van MILL [5]). In [5] and [6] it has been shown that if γX is a compactification of X of weight at most 2^{ω} then γX is a GA compactification. As remarked in the introduction, $\neg$CH implies that there is a compactification of $\mathbb{N}$ which is not a Wallman compactification. Hence there is a consistent example of a GA compactification which is not a Wallman compactification. Whether there is a real example of such a compactification is unknown. In addition, it is unknown whether every compactification is a GA compactification.

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