SUBSPACES OF BASICALLY DISCONNECTED SPACES OR QUOTIENTS OF COUNTABLY COMPLETE BOOLEAN ALGEBRAS

BY

ERIC K. VAN DOUWEN[1] AND JAN VAN MILL[2]

ABSTRACT. Under $MA + 2^\omega = \omega_2$ there is a (compact) strongly zero-dimensional F-space of weight 2^ω which cannot be embedded in any basically disconnected space.

Dually, under $MA + 2^\omega = \omega_2$ there is a weakly countably complete (or almost σ-complete, or countable separation property) Boolean algebra of cardinality 2^ω which is not a homomorphic image of any countably complete Boolean algebra.

The key to our construction is the observation that if X is a subspace of a basically disconnected space and $\beta\omega \subseteq X$ then $\beta\omega$ is a retract of X.

Dually, if B is a homomorphic image of a countably complete Boolean algebra, and if h is a homomorphism from B onto $\mathcal{P}(\omega)$, the field of subsets of ω, then there is an embedding $e: \mathcal{P}(\omega) \to B$ such that $h \circ e = \mathrm{id}_{\mathcal{P}(\omega)}$.

1. **Introduction.** In order not to double the length of this paper we will not dualize each statement; the basic translations can be found in the abstract and in §2. The Boolean algebraic reader may assume that all spaces mentioned in this introduction are compact (see §2) and read "zero-dimensional" for "strongly zero-dimensional".

It is well known that $\beta\omega$ is extremally disconnected, but that its closed subspace $\beta\omega - \omega$ is not even basically disconnected [GJ, 6W.3]. However, $\beta\omega - \omega$ is a zero-dimensional F-space. Since, more generally, the property of being a normal F-space is closed hereditary, this suggests consideration of the following statement.

FE: Every strongly zero-dimensional F-space can be embedded in an Extremally disconnected space.

It is convenient to factor *FE* as *FB + BE*, where

FB: Every strongly zero-dimensional F-space can be embedded in a Basically disconnected space.

BE: Every Basically disconnected space can be embedded in an Extremally disconnected space.

The earliest reference we are aware of where one of these statements is considered is [L]. Here Louveau attributes the question (or conjecture, Louveau is not clear) of whether *FE* holds to Choquet, and proves that under CH the restriction of

FE to compact spaces of weight $\leqslant c$ holds, i.e. CH implies

(*): every compact zero-dimensional *F*-space of weight $\leqslant c$ can be embedded in an extremally disconnected space.

The question of whether *BE* holds was raised by Koppelberg, [K] (in Boolean algebraic language), who was apparently unaware of Louveau's paper. She proved that the restriction of *BE* to compact spaces of weight $\leqslant c$ holds under CH.

The question of whether *FB* holds was raised by van Douwen, Monk and Rubin, [vDMR], who also repeated the question of whether *BE* holds.

In this paper we prove that *FB* is not a theorem of ZFC, hence neither is *FE*. We have no information about *BE*.

EXAMPLE (MA + $c = \omega_2$). *There is a strongly zero-dimensional F-space V which cannot be embedded in a basically disconnected space, such that βV has weight c.*

This example also shows that (*) is not a theorem of ZFC. In view of the fact that several results about *F*-spaces which had been proved under CH actually are equivalent to CH, cf. [vD₁], [vDvM], we conjecture that (*) is equivalent to CH. However, we need more than $\neg$CH for our construction since βV "really" has weight 2^{ω_1}.

Our example is based on the following.

LEMMA. *If X can be embedded in a basically disconnected space, then X has the property that every subspace of X which is homeomorphic to $\beta\omega$ is a retract of X.*

We build *V* with $\beta\omega \subseteq V$. In order to ensure that $\beta\omega$ is not a retract of *V* we badly need $\neg$CH, and we do not know how to avoid it. We use MA + $c = \omega_2$ to make *V* a strongly zero-dimensional *F*-space. We could have used less, but we do not bother because it does not help to get a real example, and one needs more than $\neg$CH to make βV a counterexample to (*).

REMARK (ADDED IN PROOF). Because there seems to be a misunderstanding about the role of the hypothesis $c = \omega_2$ we mention the *V* can be constructed under MA + $c = \kappa^+$ with κ any regular uncountable cardinal.

2. Preliminaries. As usual, an ordinal is the set of smaller ordinals and a cardinal is an initial ordinal.

Some of the definitions below are equivalent to the original definitions only in case of normal spaces but we will not bother. (This explains why in certain statements we have to hypothesize normality.) All our spaces are T_1. *Clopen* means closed and open. A space is called *zero-dimensional* if its clopen sets are a base, *strongly zero-dimensional* if any two disjoint closed sets are included in disjoint clopen sets. Recall that if *X* is normal then *X* is strongly zero-dimensional iff βX is (strongly) zero-dimensional.

A space is called

   extremally disconnected / basically disconnected / an F-space

if any two disjoint open sets

   without further condition / one of which is an F_σ / both of which are F_σ

have disjoint closures. Note that a normal space X has one of these three properties iff βX has it. (Normality is not needed for extremal disconnectedness.) Also note that the method of proof that a Lindelöf space is normal can be used to show that a normal space is an F-space (if and) only if any two F_σ-subsets F and G with $\bar{F} \cap G = F \cap \bar{G} = \varnothing$ have disjoint closures, (see e.g. [vD$_2$] for the easy proof). This implies that the property of being a normal F-space is closed hereditary. We also point out to the Boolean algebraic reader that although a normal basically disconnected space is strongly zero-dimensional, the analogous statement for F-spaces is false: Gillman and Henriksen give an example of a compact connected F-space in [GH, 2.8].

A BA (= Boolean algebra) will be identified with its universe. A BA B is called

complete / countably complete / weakly countably complete

if for any two subsets P and Q such that $p \wedge q = 0$ for $p \in P$, $q \in Q$

without further condition/with $|P| = \omega$ or $|Q| = \omega$/*with* $|P| = |Q| = \omega$

there is an $s \in B$ which *separates* P and Q, i.e. $p \leqslant s$ for $p \in P$ and $q \leqslant s'$ for $q \in Q$. (Obviously this is equivalent to the usual definition of complete and countably complete.) It is well known and easy to prove that if X is a zero-dimensional compact space, then X is

extremally disconnected/basically disconnected/an F-space

if and only if the clopen algebra of X is

complete/countably complete/weakly countably complete.

It is important to realize that in fact this is true if X is strongly zero-dimensional, whether compact or not. This explains why we do not have to make our example V compact, once we have V we also have a compact example: just replace V by βV.

3. Proof of the Lemma.

TOPOLOGICAL PROOF. We may assume X itself is basically disconnected, hence that X is compact (for βX is basically disconnected since X is). Choose pairwise disjoint clopen B_n's with $n \in B_n$ for $n \in \omega$. Then $r = \bigcup_n B_n \times \{n\}$ is a retraction $\bigcup_n B_n \to \omega$. Let $A = (\bigcup_n B_n)^-$. Then A is open and $A = \beta(\bigcup_n B_n)$ since $\bigcup_n B_n$ is an open F_σ. Hence $\beta r \cup (X - A) \times \{0\}$ is a retraction $X \to \beta\omega$.

BOOLEAN ALGEBRAIC PROOF. Let B be a countably complete BA, let C be a BA, let $\mathscr{P}(\omega)$ be the field of subsets of ω, and let

$$g: B \to C \quad \text{and} \quad f: C \to \mathscr{P}(\omega)$$

be homomorphisms onto. Choose disjoint b_n's in B with $f(g(n)) = n$ for $n \in \omega$, and let $a = \bigvee_n b_n$. Then we can define an embedding $e: \mathscr{P}(\omega) \to C$ such that $f \circ e = \mathrm{id}_{\mathscr{P}(\omega)}$ by

$$e(Y) = \begin{cases} g\left(\bigvee_{n \in Y} b_n\right) & \text{if } 0 \notin Y, \\ g\left(\left(\bigvee_{n \in Y} b_n\right) \vee a'\right) & \text{if } 0 \in Y. \end{cases}$$

4. The example.

We want to use the Lemma to guarantee that our example cannot be embedded into a basically disconnected space. So we aim at adding a set I to $\beta\omega$ and at topologizing $V = \beta\omega \cup I$ (with $I \cap \beta\omega = \varnothing$) in such a way that

(1) $\beta\omega$ is not a retract of V.

We achieve (1) by a brute force application of $\neg CH$, as the next fact makes clear. (Note that (2), (3) and (4) imply $\neg CH$ since (4) forces $|I| \geqslant \omega_1$.)

FACT 1. *Assume $\beta\omega - \omega$ has a subset C such that*

(2) *there is an open family $\mathcal{U} = \{ U_x : x \in C \}$ in $\beta\omega - \omega$ such that $x \in U_x$ for $x \in C$ and $U_x \cap U_y = \varnothing$ for distinct $x, y \in C$,*

(3) $|I| < |C|$ *and*

(4) *if G is a G_δ-set in V with $C \subseteq G$, then $C \subseteq \overline{G \cap I}$.*

Then $\beta\omega$ is not a retract of V.

□   Suppose there is a retraction $r : V \to \beta\omega$. Then $G = r^{\leftarrow}(\beta\omega - \omega)$ is a G_δ-set in V with $C \subseteq G$. So if $J = G \cap I$ then $C \subseteq \bar{J}$, therefore $C \, (= r^{\rightarrow}C) \subseteq (r^{\rightarrow}J)^{-}$. But $r^{\rightarrow}J \subseteq r^{\rightarrow}G = \beta\omega - \omega$, hence $(r^{\rightarrow}J) \cap U_x \neq \varnothing$ for $x \in C$. This is impossible since $U_x \cap U_y = \varnothing$ for distinct $x, y \in C$ and $|r^{\rightarrow}J| \leqslant |I| < |C|$. □

Let I be a set of cardinality ω_1, and let C be a subset of $\beta\omega - \omega$ of cardinality ω_2, to be specified later. We plan to choose, for each $x \in C$, an uncountable subset F_x of I such that

(5) $|F_x \cap F_y| \leqslant \omega$ for distinct $x, y \in C$.

Again we specify our choice later. Let $T = I \cup C$ and topologize T as follows: points of I are isolated, a basic neighborhood of $x \in C$ has the form

$$\{x\} \cup (F_x - K) \quad \text{with } |K| \leqslant \omega.$$

This is a valid neighborhood assignment because of (5). Evidently T is a (regular) P-space. Topologize V by pasting T and $\beta\omega$ together, or, formally, $U \subseteq V$ is open in V if $U \cap \beta\omega$ is open in $\beta\omega$ and $U \cap T$ is open in T. It is clear that (3) and (4) hold. The natural way to make (2) hold is to start with a disjoint open family $\mathcal{U}$ in $\beta\omega - \omega$ with $|\mathcal{U}| = \omega_2$, choose $c(U) \in U$ for $U \in \mathcal{U}$ and let $C = \{c(U) : U \in \mathcal{U}\}$. Unfortunately it is not clear that V is going to be normal, or even Hausdorff, unless we could assert that T is normal. However, we have the following fact.

FACT 2. [$\neg CH$]: *No P-space which has a dense subset of cardinality ω_1 and a closed discrete subset of cardinality ω_2 is normal.*

We prove this fact, which we obviously do not need for the construction of V, in the Appendix.

Note that both Facts 1 and 2 depend on the way we added I to $\beta\omega$, and that apparently one cannot get Fact 1 without also getting Fact 2. We overcome this difficulty by assuming MA.

Under $MA + c = \omega_2$ there is a point $p \in \beta\omega - \omega$ which has a clopen neighborhood base $\langle P_\xi : \xi < \omega_2 \rangle$ in $\beta\omega$ such that

(6) if $\xi < \eta < \omega_2$ then $P_\eta - \omega \subset P_\xi - \omega$.

($\subset$ denotes *proper* inclusion.) Choose $c_\xi \in (P_\xi - P_{\xi+1}) - \omega$ for $\xi < \omega_2$ and let $C = \{c_\xi : \xi < \omega_2\}$. Then clearly (2) holds. Also

(7) if U is a neighborhood of p then $|C - U| \leqslant \omega_1$.

It is precisely this fact which makes our construction work. For even though T is not normal we do have

FACT 3. *We can construct T in such a way that for each $A \subseteq C$ with $|A| \leqslant \omega_1$ there is a clopen $S \subseteq T$ with $S \cap C = A$.*

□ We modify a construction of [vD_3]. Let

$$I = \omega_1 \times \omega_1, \qquad \Phi = \{f \subseteq \omega_1 \times \omega_1 : f \text{ is a function from } \omega_1 \text{ to } \omega_1\}.$$

Define a partial order $<^*$ on Φ by

$$f <^* g \quad \text{if } |\{\xi < \omega_1 : f(\xi) \geqslant g(\xi)\}| \leqslant \omega.$$

With an easy induction, pick $f_\xi \in \Phi$ for $\xi < \omega_2$ such that

(8) if $\xi < \eta$ then $f_\xi <^* f_\eta$.

For $x = c_\xi \in C$ we let $F_x = f_\xi$. Clearly (5) holds. To see that Fact 3 holds, let $A \subseteq \omega_2$ have cardinality $\leqslant \omega_1$. Let $\alpha = \sup(A)$. Then $\alpha < \omega_2$. Put

$$W = \{\langle \xi, \eta \rangle \in I : \eta \leqslant f_\alpha(\xi)\} \cup \{f_\xi : \xi < \alpha\}.$$

If $\gamma \leqslant \alpha$ then $|f_\gamma - W| \leqslant \omega$, and if $\gamma > \alpha$ then $|f_\gamma \cap W| \leqslant \omega$, because of (8), hence W is clopen. Since $\{f_\xi : \xi \in A\} \subseteq W$ and $|\alpha| \leqslant \omega_1$, and since T is a P-space, we can easily find, using (5) (or, equivalently, (7)), disjoint open $W', W'' \subseteq W$ such that for all $\xi \leqslant \alpha$, if $\xi \in A$ then $f_\xi \in W'$ and if $\xi \notin A$ then $f_\xi \in W''$. Then $S = W'$ is as required. (This way to use $<^*$, in a slightly different form, is due to Proctor [P].) □

We now are ready to complete the proof that V is as required.

FACT 4. *V is strongly zero-dimensional.*

□ Let F and G be disjoint closed subsets of V. Since $\beta\omega$ is strongly zero-dimensional there is a clopen $H \subseteq \beta\omega$ with $F \cap \beta\omega \subseteq H$ and $G \cap \beta\omega \subseteq \beta\omega - H$. Assume for a moment

(9) there is a clopen $K \subseteq V$ with $K \in \beta\omega = H$.

Then $L = (K \cup F) - G$ is a clopen set in V which includes F and is disjoint from G. L is open since $K \cup F = K \cup (F \cap I)$ and L is closed since $L = (K \cup F) - (G \cap I)$ (recall that I consists of isolated points).

It remains to prove (9). Without loss of generality $p \notin H$. Then $|H \cap C| \leqslant \omega_1$, hence by Fact 3 there is a clopen S in T with $S \cap C = H \cap C$. Then $K = H \cup S$ is clopen in V since $K \cap \beta\omega = H$ is clopen in $\beta\omega$ and $K \cap T = S$ is clopen in T. Also, as just observed, $K \cap \beta\omega = H$. This proves (9). □

FACT 5. *V is an F-space.*

□ Let F and G be disjoint open F_σ's. Let $x \in \beta\omega$ be arbitrary. Since $\beta\omega$ is an F-space, we have

$$(F \cap \beta\omega)^- \cap (G \cap \beta\omega)^- = \varnothing$$

so we may assume without loss of generality that $x \notin (F \cap \beta\omega)^-$. Let U be a clopen neighborhood in V which does not intersect $F \cap \beta\omega$ (use Fact 4). Then $F \cap U$ is an F_σ-subset of V with $F \cap U \subseteq I$ ($\subseteq T$). Since T is a P-space, it follows that $F \cap U$ is closed in T, consequently $F \cap U$ is closed in V since $F \cap U \cap \beta\omega = \varnothing$ is closed in $\beta\omega$. Hence $x \notin \bar{F}$.

Since x was arbitrary in $\beta\omega$, it follows that $\bar{F} \cap \bar{G} \cap \beta\omega = \varnothing$. As the points of I are isolated this implies $\bar{F} \cap \bar{G} = \varnothing$. □

FACT 6. *βV has weight 2^ω.*

□ Since V is strongly zero-dimensional and $2^{\omega_1} = 2^\omega$ under MA $+ \neg$CH it suffices to show that V has 2^{ω_1} clopen sets.

Since $\beta\omega$ has 2^ω clopen sets and $|V - \beta\omega| = \omega_1$, V has no more than 2^{ω_1} clopen sets.

Recall that $I = \omega_1 \times \omega_1$, and note that the way we topologize V means that $\{0\} \times \omega_1$ is a closed discrete subset of V consisting of isolated points. It follows that V has at least 2^{ω_1} clopen sets. $\square$

REMARK. Perhaps the reader feels that with a more careful construction of T we can ensure that V has 2^ω clopen sets without using $2^\omega = 2^{\omega_1}$, but this is not the case. Let $A \subseteq C$ have cardinality ω_1. There is a clopen set U in V with $U \cap C = A$ (and this property we really need). Since T is a P-space (this property we used in an essential way in the proof of Fact 5) we can find a clopen W_x in T with $x \in W_x$ for $x \in A$ such that $W_x \subseteq U$ for $x \in A$ and $W_x \cap W_y = \varnothing$ for distinct $x, y \in A$. Choose $D \subseteq \bigcup_{x \in A}(W_x - \{x\})$ such that D intersects each W_x in one point. Then D is a closed subset of V consisting of isolated points, and $|D| = \omega_1$.

Appendix. On normal P-spaces with big closed discrete sets. Fact 2 of §4 stated that the statement

(P) there is a normal P-space with density ω_1 and a closed discrete subset of cardinality ω_2

is false under $\neg$CH. (P) is similar to the statement

(Q) there is a separable normal space with an uncountable closed discrete subset.

It is a well-known result of Jones [J], (only if), and Heath [H], (if), that

(Q) if and only if $2^\omega = 2^{\omega_1}$.

We use similar tools to prove our next result.

LEMMA. (P) *if and only if* $2^\omega = \omega_1$ *and* $2^{\omega_1} = 2^{\omega_2}$.

$\square$ *Only if.* Let X be a P-space with a dense subset D with $|D| = \omega_1$ and a closed discrete subset C with $|C| = \omega_2$.

There are 2^{ω_2} continuous real-valued functions on C but only 2^{ω_1} continuous real-valued functions on D, hence on X, so if X is normal then $2^{\omega_1} = 2^{\omega_2}$ by the Tietze-Urysohn Extension Theorem.

Now suppose X is normal but $2^\omega > \omega_1$. Then there is an injection $f\colon C \to R$. Since f is continuous and X is normal, f can be extended to a continuous $\tilde{f}^{\leftarrow}\colon X \to R$. For each $x \in C$ the set $f^{\leftarrow}\{f(x)\}$ is a nonempty G_δ, hence is a nonempty open set, so $\{f^{\leftarrow}\{f(x)\}\colon x \in C\}$ is a disjoint open family of cardinality ω_2. Hence D is not dense. This contradiction shows that $2^\omega = \omega_1$.

If. Let Π be the product of 2^{ω_1} discrete two-point spaces. Let D be a discrete space of cardinality ω_2. Then the weight of βD equals $2^{\omega_2} = 2^{\omega_1}$, hence we may assume $\beta D \subseteq \Pi$.

Π has a dense subset A of cardinality ω_1. Since $\omega_1^\omega = \omega_1$, there is a subset C of Π with $|C| = \omega_1$ such that every sequence in A has a limit point in C.

Let Π_δ be Π, retopologized by making the G_δ-subsets of Π a base. It is easy to see that C is dense in Π_δ.

Retopologize the subspace $C \cup D$ of Π_δ by making the points of $C - D$ isolated, call the resulting space X. Clearly X is a P-space in which C is dense and in which D is a closed discrete set. So in order to prove that X is normal it suffices

to show that disjoint subsets of D have disjoint neighborhoods in X. Indeed, disjoint subsets of D have disjoint closures in βD, hence have disjoint neighborhoods in Π, hence in Π_δ, hence in $C \cup D$ as subspace of Π_δ, hence in X. $\square$

REFERENCES

[vD₁] E. K. van Douwen, *A basically disconnected normal space* Φ *with* $|\beta\Phi - \Phi| = 1$, Canad. J. Math. **31** (1979), 911–914.

[vD₂] _____, *Cardinal functions on compact F-spaces and on weakly countably complete Boolean algebras* (to appear).

[vD₃] _____, *Functions from the integers to the integers and topology* (to appear).

[vDvM] E. K. van Douwen and J. van Mill, *Parovičenko's characterization of* $\beta\omega - \omega$ *implies* CH, Proc. Amer. Math. Soc. **72** (1978), 539–541.

[GH] L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. **82** (1956), 366–391.

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

[H] R. W. Heath, *Separability and* $\aleph_1$*-compactness*, Colloq. Math. **12** (1964), 11–14.

[J] F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. **43** (1937), 671–677.

[K] S. Koppelberg, *Homomorphic images of* σ*-complete Boolean algebras*, Proc. Amer. Math. Soc. **51** (1975), 171–175.

[L] A. Louveau, *Caractérisation des sous-espaces compacts de* βN, Bull. Sci. Math. (2) **97** (1973), 259–263.

[MR] E. K. van Douwen, J. D. Monk and M. Rubin, *Some questions about Boolean algebras*, Algebra Universalis (to appear).

[P] C. W. Proctor, *A separable pseudonormal nonmetrizable Moore space*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **18** (1970), 179–181.

INSTITUTE FOR MEDICINE AND MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701 (Current address of E. K. van Douwen)

WISKUNDIG SEMINARIUM, VRIJE UNIVERSITEIT, AMSTERDAM, THE NETHERLANDS

Current address (J. van Mill): Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803