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A TOPOLOGICAL PROPERTY ENJOYED BY NEAR POINTS BUT NOT BY LARGE POINTS

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Let **H** denote the halfline $[0, \infty)$. A point $p \in \beta\mathbf{H} - \mathbf{H}$ is called a near point if p is in the closure of some countable discrete closed subspace of **H**. In addition, a point $p \in \beta\mathbf{H} - \mathbf{H}$ is called a large point if p is not in the closure of a closed subset of **H** of finite Lebesgue measure. We will show that for every autohomeomorphism φ of $\beta\mathbf{H} - \mathbf{H}$ and for each near point p we have that $\varphi(p)$ is not large. In addition, we establish, under CH, the existence of a point $x \in \beta\mathbf{H} - \mathbf{H}$ such that for each autohomeomorphism φ of $\beta\mathbf{H} - \mathbf{H}$ the point $\varphi(x)$ is neither large nor near.

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near point	remote point	P-set	CH
far point	large point	βX	$[0, \infty)$

0. Definitions and conventions

All spaces are completely regular and X^* is the Čech–Stone remainder of the Čech–Stone compactification βX of X. A point $p \in X^*$ is called a

(a) *remote point* if $p \notin \mathrm{cl}_{\beta X} E$ for any nowhere dense subset $E \subset X$;

(b) *near point* if $p \in \mathrm{cl}_{\beta X} D$ for some closed discrete subset $D \subset X$.

Let **H** denote the halfline $[0, \infty)$. A point $p \in \mathbf{H}^*$ is called a

(c) *large point* if $p \notin \mathrm{cl}_{\beta \mathbf{H}} F$ for any closed set $F \subset \mathbf{H}$ of finite Lebesgue measure.

Let $\mathscr{R}$ denote the set of all remote points, $\mathscr{L}$ denote the set of all large points and $\mathscr{N}$ denote the set of all near points of $\mathbf{H}^*$.

A set $B \subset X$ is called a P-*set* provided that $B \cap \bar{F} = \emptyset$ for every F_σ subset $F \subset X - B$.

A point of a space is called a *sub cutpoint* if it is a cutpoint of some closed connected subspace.

A point p of a space X is called a *super sub cutpoint* if there is a closed connected $K \subset X$ with $p \in K$ and a neighborhood U of K such that whenever K' is a closed connected set with $K \subset K' \subset U$ then p is a cutpoint of K'.

As usual, μ denotes Lebesgue measure of **H**.

Points in βX are sometimes identified with z-ultrafilters on X. If $U \subset X$, then $\text{Ex}(U) = \beta X - \text{cl}_{\beta X}(X - U)$.

1. Introduction

It is a classical result in Fine and Gillman [7] (due to Eberlein) that there is a large point in $\mathbf{H}^*$. Indeed, If $\mathcal{G} = \{G \subset H : \mu(\mathbf{H} - G) < \infty\}$, then each point of $\bigcap_{G \in \mathcal{G}} \text{cl}_{\beta\mathbf{H}} G$ is a large point. Large points obviously have the property that they are not near, i.e. $\mathcal{L} \cap \mathcal{N} = \emptyset$. It is clear that for each autohomeomorphism ϕ of $\beta\mathbf{H}$ we have that $\phi[\mathcal{N}] = \mathcal{N}$. However, it is not clear that the same result holds for autohomeomorphisms of $\mathbf{H}^*$. This suggests an obvious question and trying to solve this question we found the following partial answer.

1.1. Theorem. *Any near point of* $\mathbf{H}^*$ *is a super sub cutpoint while no large point is a super sub cutpoint.*

This result does not solve the above question but it shows that for any autohomeomorphism ϕ of $\mathbf{H}^*$ and for any near point $x \in \mathbf{H}^*$ the image $\phi(x)$ of x is "small" in the sense that it is in the closure of some closed subset of $\mathbf{H}$ of finite Lebesgue measure (as a consequence, $\inf\{\varepsilon \geq 0 : \exists \text{ closed } A \subset \mathbf{H} \text{ with } \mu(A) \leq \varepsilon \text{ and } \phi(x) \in \text{cl}_{\beta\mathbf{H}} A\} = 0$).

Let us notice that our result implies that $\mathbf{H}^*$ is not homogeneous and that we found a topological property enjoyed by some but not all points of $\mathbf{H}^*$ and that the points involved are easily described. For earlier results implying that $\mathbf{H}^*$ is not homogeneous see [9, 2, 5].

By a result of van Douwen [4] each nonpseudocompact space of countable π-weight has a remote point; as a consequence $\mathcal{R} \neq \emptyset$. He has asked whether $\phi(\mathcal{R}) = \mathcal{R}$ for each autohomeomorphism ϕ of $\mathbf{H}^*$ [4]. We were unable to answer this question but we found the following partial answer.

1.2. Theorem (CH). *There is a point* $x \in \mathcal{R}$ *such that* $\phi(x) \in \mathcal{R}$ *for each autohomeomorphism* ϕ *of* $\mathbf{H}^*$.

We are indebted to Eric van Douwen for some helpful comments.

2. Standard subcontinua of $\mathbf{H}^*$

Let I denote the closed unit interval $[0, 1]$ and let $\pi : \omega \times I \to \omega$ be the projection. Since π is perfect the Stone extension $\beta\pi$ maps $(\omega \times I)^*$ onto ω^*. Also, π is monotone which implies that $\beta\pi$ is monotone. This is well known (see for example [11]). For convenience we will include a sketch of the proof. Take $x \in \omega^*$ and assume that $\beta\pi^{-1}[\{x\}]$ is not connected. Take nonempty disjoint clopen sets A, $B \subset \beta\pi^{-1}[\{x\}]$ whose union is $\beta\pi^{-1}[\{x\}]$. By compactness we can find open sets $U(A)$, $U(B) \subset \omega \times I$ so that $A \subset \text{Ex}(U(A))$, $B \subset \text{Ex}(U(B))$ and $\text{Ex}(U(A)) \cap \text{Ex}(U(B)) = \emptyset$. Since $x \in \text{cl}_{\beta\omega}\, \pi[U(A)] \cap \text{cl}_{\beta\omega}\, \pi[U(B)]$ the set

$$E = \{n < \omega \colon U(A) \cap (\{n\}xI) \neq \emptyset \quad \text{and} \quad U(B) \cap (\{n\} \times I) \neq \emptyset\}$$

is infinite since it is an element of x (x is an ultrafilter!). Since $\text{Ex}(U(A)) \cap \text{Ex}(U(B)) = \emptyset$ the intersection $U(A) \cap U(B)$ has compact closure in $\omega \times I$. Hence, without loss of generality $U(A) \cap U(B) = \emptyset$. Now, for each $n \in E$ take $x_n \in (\{n\} \times I) - (U(A) \cup U(B))$ and let $p \in (\omega \times I)^*$ be a cluster point of $\{x_n \colon n < \omega\}$. Then

$$p \in \beta\pi^{-1}[\{x\}] - (\text{Ex}(U(A)) \cup \text{Ex}(U(B))),$$

which is a contradiction.

Since $\beta\omega$ is totally disconnected it follows that for each subcontinuum C of $(\omega \times I)^*$ there is a point $x \in \omega^*$ so that $C \subset \beta\pi^{-1}[\{x\}]$. Hence sets of the form $\beta\pi^{-1}[\{x\}]$ are maximal subcontinua of $(\omega \times I)^*$. Since these subcontinua of $(\omega \times I)^*$ play a fundamental role in the remaining part of this paper this section is devoted to study them.

The proof of the following fact is trivial and hence is omitted.

2.1. Fact. $\beta\pi^{-1}[\{p\}] = \bigcap_{P \in p} \text{cl}_{\beta(\omega \times I)}(\bigcup_{n \in P} \{n\} \times I)$ for each $p \in \beta\omega$.

The following fact is due to Mioduszewski [11]. For completeness we will include the proof.

2.2. Fact. Let $\langle n, x_n \rangle \in \{n\} \times (0, 1)$ $(n < \omega)$ and let $p \in \omega^*$. Then $\beta\pi^{-1}[\{p\}]$ intersects $\{\langle n, x_n \rangle \colon n < \omega\}^*$ in precisely one point and this point is a cutpoint of $\beta\pi^{-1}[\{p\}]$.

Proof. By Fact 2.1 it is clear that $\{\langle n, x_n \rangle \colon n < \omega\}^* \cap \beta\pi^{-1}[\{p\}] \neq \emptyset$ so assume it contains at least two distinct points, say a and b. There are disjoint sets A, $B \subset \{\langle n, x_n \rangle \colon n < \omega\}$ so that $a \in A^*$ and $b \in B^*$. Then $p \in \text{cl}_{\beta\omega}\, \pi(A) \cap \text{cl}_{\beta\omega}\, \pi(B)$ and since $\pi(A) \cap \pi(B) = \emptyset$ this is a contradiction.

Let $x(p)$ be the unique point in $\{\langle n, x_n \rangle \colon n < \omega\}^* \cap \beta\pi^{-1}[\{p\}]$ and define $U_0 = \bigcup_{n<\omega} \{n\} \times [0, x_n)$ and $U_1 = \bigcup_{n<\omega} \{n\} \times (x_n, 1]$ respectively. Put

$$U_i' = \text{Ex}(U_i) \cap \beta\pi^{-1}[\{p\}] \quad (i \in 2).$$

Then $U_i' \neq \emptyset$ $(i \in 2)$ and $U_0' \cup U_1' \cup \{x(p)\} = \beta\pi^{-1}[\{p\}]$ since

$$\text{Ex}(U_0) \cup \text{Ex}(U_1) \cup \{\langle n, x_n \rangle \colon n < \omega\}^* = (\omega \times I)^*.$$

Hence $x(p)$ cuts $\beta\pi^{-1}[\{p\}]$. $\square$

Since $\omega \times I$ can be embedded as a closed subspace of $\mathbf{H}$ the remainder $(\omega \times I)^*$ can be embedded as a closed subspace of $\mathbf{H}^*$. A subcontinuum B of $\mathbf{H}^*$ for which there is a closed embedding $\phi: \omega \times I \to \mathbf{H}$ and a point $p \in \omega^*$ so that $B = \beta\phi(\beta\pi^{-1}[\{p\}])$ is called a *standard subcontinuum*.

The proof of the following fact is trivial.

2.3. Fact. A subcontinuum $B \subset \mathbf{H}^*$ is standard iff there is a discrete sequence $\{I_n : n < \omega\}$ of pairwise disjoint nontrivial (faithfully indexed) closed intervals of $\mathbf{H}$ and a point $p \in \omega^*$ so that

$$B = \bigcap_{P \in p} \mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n \in P} I_n\Big). \quad \square$$

We can now prove an important Lemma.

2.4. Lemma. *Let $K \subset \mathbf{H}^*$ be a proper subcontinuum and let U be a neighborhood of K. Then there is a standard subcontinuum B of $\mathbf{H}^*$ so that $K \subset B \subset U$.*

Proof. By compactness of $\beta\mathbf{H}$ we may assume that $U = \mathrm{Ex}(V)$ where V is a discrete union of pairwise disjoint nonempty open intervals in $\mathbf{H}$. Let us assume that $V = \bigcup_{n<\omega} V_n$. For each $n < \omega$ take some closed interval $D_n \subset V_n$ so that

$$K \subset \mathrm{cl}_{\beta\mathbf{H}}(\bigcup_{n<\omega} D_n).$$

Observe that $\bigcup_{n<\omega} D_n$ is homeomorphic to $\omega \times I$ and hence that the connectedness of K implies that there is a $p \in \omega^*$ so that

$$K \subset B = \bigcap_{P \in p} \mathrm{cl}_{\beta\mathbf{H}}(\bigcup_{n \in P} D_n).$$

Since B is a standard subcontinuum and $B \subset U$ the desired result follows. $\square$

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is in two steps.

3.1. Fact. If $p \in \mathcal{N}$, then p is a super sub cutpoint.

Proof. Without loss of generality $p \in \mathbf{N}^*$. Define $A_n = [n - \frac{1}{4}, n + \frac{1}{4}]$ $(n > 0)$ and put

$$B = \bigcap_{P \in p} \mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n \in P} A_n\Big).$$

Then B is a standard subcontinuum and $p \in B$. Let $U = \mathbf{H} - \{n + \frac{1}{2} : n \in \mathbf{N}\}$. Then $\mathrm{Ex}(U)$ is a neighborhood of B. Let K be a subcontinuum of $\mathbf{H}^*$ so that $B \subset K \subset$

$\mathrm{Ex}(U)$. Let x^+ be the unique point in the intersection

$$\bigcap_{P\in p} \mathrm{cl}_{\beta\mathbf{H}}\{t+\tfrac{1}{4}\colon t\in P\}.$$

Similarly, let x^- be the unique point in the intersection

$$\bigcap_{P\in p} \mathrm{cl}_{\beta\mathbf{H}}\{t-\tfrac{1}{4}\colon t\in P\}.$$

Then $\{x^+, x^-\}\subset B\subset K$ which implies that K intersects both $\mathrm{Ex}(\bigcup_{n\in\mathbf{N}}(n-\frac{1}{2}, n))$ and $\mathrm{Ex}(\bigcup_{n\in\mathbf{N}}(n, n+\frac{1}{2}))$. By similar arguments as in the proof of Fact 2.2 it follows that p cuts K. $\square$

3.2. Fact. If $p\in\mathscr{L}$, then p is not a super sub cutpoint.

Proof. By Lemma 2.4 we need only to show that p is not a cutpoint of any standard subcontinuum of $\mathbf{H}^*$. So let $D_n=[a_n, b_n]$ $(n<\omega)$ be a discrete sequence of closed intervals of $\mathbf{H}$ so that $a_n<b_n<a_{n+1}$ $(n<\omega)$. Assume that for some free ultrafilter q on ω

$$p\in K=\bigcap_{Q\in q}\mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n\in Q} D_n\Big).$$

Define

$$A=\bigcup\Big\{\bigcap_{Q\in q}\mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n\in Q}E_n\Big)\colon E_n=[a_n, t_n]\text{ for some } a_n<t_n<b_n \text{ and } \bigcup_{n<\omega}E_n\notin p\Big\}$$

and

$$B=\bigcup\Big\{\bigcap_{Q\in q}\mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n\in Q}F_n\Big)\colon F_n=[t_n, b_n]\text{ for some } a_n<t_n<b_n \text{ and } \bigcup_{n<\omega}F_n\notin p\Big\}$$

respectively.

Claim 1. Both A and B are nonempty and connected.

We will only show that A is nonempty and connected. Take $t_n\in(a_n, b_n)$ so that $\mu([a_n, t_n])<2^{-n}$ and define $E_n=[a_n, t_n]$. Then $\bigcup_{n<\omega}E_n$ is closed and $\mu(\bigcup_{n<\omega}E_n)<\infty$. Hence $\bigcup_{n<\omega}E_n\notin p$. Consequently

$$D=\bigcap_{Q\in q}\mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n\in Q}E_n\Big)\subset A$$

and since trivially D is nonempty we find that A is nonempty.

Since A is the union of standard subcontinua we need only show that the standard subcontinua described in the definition of A pairwise intersect. In fact the intersection of all these subcontinua is nonempty since they all contain the unique point in the intersection

$$\bigcap_{Q\in q} \mathrm{cl}_{\beta\mathbf{H}}\{a_n : n \in Q\}.$$

Claim 2. $A \cup B$ is dense in K.

Let $U \subset \mathbf{H}$ be open so that $\mathrm{Ex}(U) \cap K \neq \emptyset$. We may assume that there is an infinite $E \subset \omega$ so that $U = \bigcup_{n\in E} U_n$, where $U_n \subset [a_n, b_n]$ is nonempty and open. For each $n < \omega$ pick points $s_n, t_n \in (a_n, b_n)$ so that

(a) $s_n < t_n$ and $\mu([s_n, t_n]) < 2^{-n}$;

(b) if $n \in E$, then $\{s_n, t_n\} \subset U_n$.

Then $\mu(\bigcup_{n<\omega} [s_n, t_n]) < \infty$ so that $\bigcup_{n<\omega} [s_n, t_n] \notin p$. Consequently, either $\bigcup_{n<\omega} [a_n, s_n] \in p$ or $\bigcup_{n<\omega} [t_n, b_n] \in p$. So, without loss of generality $\bigcup_{n<\omega} [a_n, s_n] \notin p$. Then $A \cap \mathrm{Ex}(U)$ contains the unique point in the intersection

$$\bigcap_{Q\in q} \mathrm{cl}_{\beta\mathbf{H}}\{s_n : n \in Q\}.$$

So, to complete the proof that p is not a cutpoint of K we have only to show that there is a $z \in A^- \cap B^- - \{p\}$.

For each $n < \omega$, pick a finite set $G_n \subset (a_n, b_n)$ such that if $U \subset (a_n, b_n)$ is an interval disjoint from G_n, then $\mu(U) < 2^{-n}$. Let

$$\mathscr{F} = \Big\{ \bigcup_{n<\omega} [t_n, u_n] : a_n \leq t_n \leq u_n \leq b_n \text{ and } \bigcap_{Q\in q} \mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n\in Q} [a_n, t_n]\Big) \subset A \\ \text{and } \bigcap_{Q\in q} \mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n\in Q} [u_n, b_n]\Big) \subset B \Big\}.$$

It is easily seen that $\mathscr{F}$ has the finite intersection property, since $p \in \mathrm{cl}_{\beta\mathbf{H}} F$ for each $F \in \mathscr{F}$. Now take $\bigcup_{n<\omega} [t_n, u_n] \in \mathscr{F}$ and $Q \in q$. Then

$$\sum_{n\in Q} \mu([t_n, u_n]) = \infty$$

since $\bigcup_{n\in Q} [t_n, u_n] = F \cap \bigcup_{n\in Q} [a_n, b_n] \in p$, whence for infinitely many $n \in Q$ we have that $\mu([t_n, u_n]) > 2^{-n}$. We conclude that

$$G_n \cap [t_n, u_n] \neq \emptyset$$

for infinitely many $n \in Q$. This implies that there is a point

$$z \in \bigcap_{F\in\mathscr{F}} \mathrm{cl}_{\beta\mathbf{H}} F \cap \mathrm{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n<\omega} G_n\Big) \cap K.$$

Notice that $z \neq p$ since $\mu(\bigcup_{n<\omega} G_n) = 0$.

Claim 3. $z \in A^- \cap B^-$.

We will only show that $z \in A^-$. Let $V \subset \mathbf{H}$ be open so that $z \in \text{Ex}(V)$. Since $\bigcup_{n<\omega} G_n \subset \bigcup_{n<\omega} (a_n, b_n)$ we may assume without loss of generality that $V^- \subset \bigcup_{n<\omega} (a_n, b_n)$. Put $E = \{n < \omega : V \cap (a_n, b_n) \neq \emptyset\}$. For each $n \in E$ let $s_n = \inf(V \cap (a_n, b_n))$. In addition, for $n \notin E$ take $s_n \in (a_n, b_n)$ so that $\mu([a_n, s_n]) < 2^{-n}$.

Case 1. $\bigcup_{n<\omega} [a_n, s_n] \in p$.

Then $\bigcup_{n<\omega} [s_n, b_n] \notin p$ since $\mu(\{s_n : n < \omega\}) = 0$. For each $n < \omega$ take $s'_n \in (a_n, s_n)$ so that $\mu([a_n, s'_n]) < 2^{-n}$. Then $\bigcup_{n<\omega} [a_n, s'_n] \notin p$ since it has finite measure. We conclude that

$$F = \bigcup_{n<\omega} [s'_n, s_n] \in \mathscr{F}.$$

Since $F \cap V = \emptyset$, $p \in \text{cl}_{\beta\mathbf{H}} F$ and $p \in \text{Ex}(V)$ we have derived a contradiction.

Case 2. $\bigcup_{n<\omega} [a_n, s_n] \notin p$.

For each $n < \omega$ let $s'_n \in (a_n, b_n)$ so that $s_n < s'_n < b_n$ while moreover $\mu([s_n, s'_n]) < 2^{-n}$. Since $\mu(\bigcup_{n<\omega} [s_n, s'_n]) < \infty$ it follows that

$$\bigcup_{n<\omega} [a_n, s'_n] \notin p.$$

For each $n \in E$ take a point $v_n \in [a_n, s'_n] \cap V$. Let $v_n = s'_n$ if $n \notin E$. Then

$$\bigcap_{Q \in q} \text{cl}_{\beta\mathbf{H}}\Big(\bigcup_{n \in Q} [a_n, s'_n]\Big) \subset A$$

and it contains the unique point x in the intersection

$$\bigcap_{Q \in q} \text{cl}_{\beta\mathbf{H}}\{v_n : n \in Q\}.$$

By construction $x \in \text{Ex}(V)$ which shows that $V \cap A \neq \emptyset$. □

4. The structure of $\mathscr{L}$

A space X is called a *Parovičenko space* provided that

(a) X is a compact zero-dimensional space of weight 2^ω without isolated points;

(b) each nonempty G_δ in X has nonempty interior;

(c) every two disjoint open F_σ's of X have disjoint closures.

Parovičenko [13] proved that CH implies that every Parovičenko space is homeomorphic to ω^*. In van Douwen and van Mill [6] it was shown that the statement "each Parovičenko space is homeomorphic to ω^*" is equivalent to CH.

In this section we will show that $\mathscr{L}$ is a Parovičenko space and that $\mathscr{L}$ is a P-set in $\mathbf{H}^*$.

4.1. Theorem. (a) *$\mathscr{L}$ is a Parovičenko space;*

(b) *$\mathscr{L}$ is a nowhere dense (closed) P-set of* $\mathbf{H}^*$.

Proof. That $\mathscr{L} \neq \emptyset$ follows, as remarked in the introduction, from [7]. Since each countable closed subspace of $\mathbf{H}$ is of measure zero it follows that whenever F is a discrete union of closed intervals the intersection

$$\text{cl}_{\beta\mathbf{H}} F \cap \mathscr{L}$$

is clopen in $\mathscr{L}$. We conclude that $\mathscr{L}$ is zero-dimensional. We further claim that $\mathscr{L}$ is closed in $\mathbf{H}^*$, whence $\mathscr{L}$ is compact. Indeed, suppose that $x \notin L$. Then there is a closed set $E \subset \mathbf{H}$ of finite Lebesgue measure which has x in its closure. There is an open set $U \subset \mathbf{H}$ which contains E and which is also of finite Lebesgue measure. Then $\text{Ex}(U)$ is a neighborhood of x which misses $\mathscr{L}$. Since $\mathscr{L}$ is clearly infinite it follows that $\mathscr{L}$ is a compact zero-dimensional space of weight 2^ω. It is straightforward to verify that $\mathscr{L}$ has no isolated points. We leave this to the reader. We conclude that $\mathscr{L}$ satisfies (a).

L satisfies (c) since $\mathbf{H}^*$ satisfies (c), [8, 2.7].

We will now show that $\mathscr{L}$ satisfies (b). Indeed, let G be any nonempty G_δ in $\mathscr{L}$. We may assume that

$$G = \bigcap_{n<\omega} \text{Ex}(U_n) \cap \mathscr{L},$$

where $U_{n+1} \subset U^-_{n+1} \subset U_n \subset \mathbf{H}$, $\mu(U_n - U^-_{n+1}) = \infty$ and $\mu(U^-_{n+1} - U_n) < \infty$ for all $n < \omega$. It is easy to construct an open set $V \subset \mathbf{H}$ such that $\mu(V - U^-_n) < \infty$ for all $n < \omega$ while in addition $\mu(V) = \infty$, hence $\emptyset \neq \text{Ex}(V) \cap \mathscr{L}$. In addition, it is trivial to verify that

$$\text{Ex}(V) \cap \mathscr{L} \subset \bigcap_{n<\omega} \text{Ex}(U_n) \cap \mathscr{L} = G,$$

which proves that $\mathscr{L}$ satisfies (b).

(b) That $\mathscr{L}$ is nowhere dense is trivial. Hence we need only prove that $\mathscr{L}$ is a P-set of $\mathbf{H}^*$. Indeed, let F be an F_σ disjoint from $\mathscr{L}$. Assume that $F = \bigcup_{n<\omega} F_n$, where each F_n is closed in $\mathbf{H}^*$ $(n < \omega)$. For each $n < \omega$ take an open set $U_n \subset \mathbf{H}$ such that

(i) $U^-_{n+1} \subset U_n$;
(ii) $\mu(\mathbf{H} - U_n) < \infty$;
(iii) $\text{Ex}(U_n) \cap F_n = \emptyset$.

It is trivial to find an open set $V \subset \mathbf{H}$ such that

(i)′ $\mu(\mathbf{H} - \text{V}) < \infty$;
(ii)′ $V - U_n$ is bounded for each $n < \omega$.

Then $\text{Ex}(V)$ is a neighborhood of $\mathscr{L}$ which misses F. □

3.2. Corollary (CH). *$\mathscr{L}$ is homeomorphic to ω^*.*

We do not know whether $\mathscr{L}$ is homeomorphic to ω^* in ZFC.

5. Proof of Theorem 0.2

We start with a simple Lemma.

5.1. Lemma. *Let X be a locally compact σ-compact space and let A be a closed subspace of X. Then $cl_{\beta X} A \cap X^*$ is a P-set of X^*.*

Proof. Let F be an F_σ of X^* disjoint from $A^* = cl_{\beta X} A \cap X^*$. Assume that $F = \bigcup_{n<\omega} F_n$ where each F_n is closed in $\mathbf{H}$. For each $n < \omega$ take a neighborhood U_n of A such that

(i) $U_{n+1}^- \subset U_n$;

(ii) $\text{Ex}(U_n) \cap F_n = \emptyset$.

Since X is σ-compact, so is A. So assume that $A = \bigcup_{n<\omega} A_n$, where the A_n's are compact. For each $n < \omega$ let V_n be an open subset of X such that $A_n \subset V_m \subset U_n$ while in addition V_n^- is compact. Let $V = \bigcup_{n<\omega} V_n$. Then $\text{Ex}(V)$ is a neighborhood of A^* which misses F. □

We can now prove Theorem 1.2.

5.2. Proof of Theorem 1.2. By a result of Kunen, van Mill and Mills [10], CH is equivalent to the statement that no compact space of weight 2^ω can be covered by nowhere dense closed P-sets. Since $\mathbf{H}^*$ has weight 2^ω, and since we assume CH, we find that there is a point $x \in \mathbf{H}^*$ such that $x \notin K$ for each nowhere dense closed P-set $K \subset \mathbf{H}^*$. If ϕ is any autohomeomorphism of H the point $\phi(x)$ is also not contained in any nowhere dense closed P-set of $\mathbf{H}^*$. So it suffices to prove that x is a remote point. By Woods, [14, 2.11], the family

$$\mathcal{A} = \{cl_{\beta \mathbf{H}} D \cap \mathbf{H}^* : D \text{ is nowhere dense in } \mathbf{H}\}$$

consists of nowhere dense subsets of $\mathbf{H}^*$. Also, by Lemma 5.1, $\mathcal{A}$ consists of P-sets. Therefore $x \notin \bigcup \mathcal{A}$. We conclude that x is a remote point. □

5.3. Question. *Is Theorem 1.2 true in* ZFC?

6. The structure of $\mathcal{N}$

One might easily conjecture that $\mathcal{N}$ is connected. This is not true however as the results in this section show. In fact, we will prove that $\mathcal{N}$ is zero-dimensional under CH. The proof presented here is due to Eric van Douwen and is much simpler than our original proof.

An F-space is a space in which every cozero-set is C*-embedded. It is known, [8, 2.7], that X is an F-space if X is noncompact, Lindelöf and locally compact (for an easier proof of this fact see [12, 3.1]).

If $U \subset X$, let Bd U denote the boundary of U.

6.1. Proposition. *Let X be a normal F-space. For each $\alpha < \omega_1$ let K_α be a closed zero-dimensional subspace of X and let $K = \bigcup_{\alpha<\omega_1} K_\alpha$. Then for each pair of disjoint closed subsets F, $G \subset X$ there exists an open set $U \subset X$ with $F \subset U$, $U^- \cap G = \emptyset$ and* $\mathrm{Bd}\, U \cap K = \emptyset$.

Proof. By induction we construct open F_σ's U_α and V_α $(\alpha < \omega_1)$ so that

(i) $F \subset U_\alpha$, $G \subset V_\alpha$ and $U_\alpha^- \cap V_\alpha^- = \emptyset$;

(ii) $K_\alpha \subset U_\alpha \cup V_\alpha$;

(iii) If $\alpha < \beta$, then $U_\alpha^- \subset U_\beta$ and $V_\alpha^- \subset V_\beta$.

Since X is an F-space any two disjoint open F_σ's have disjoint closures. This easily implies that the above inductive construction can be carried out. Now define $U = \bigcup_{\alpha<\omega_1} U_\alpha$. □

6.2. Corollary (CH). *$\{p \in \mathbf{H}^*: \exists$ closed zero-dimensional $F \subset \mathbf{H}$ such that $p \in \mathrm{cl}_{\beta\mathbf{H}} F\}$ is zero-dimensional.*

6.2. Corollary to Corollary (CH). *$\mathcal{N}$ is zero-dimensional.*

6.3. Question. *Is $\mathcal{N}$ zero-dimensional in* ZFC?

7. Discussion and questions

We have shown that for each near point $x \in \mathbf{H}^*$ and for each autohomeomorphism ϕ of $\mathbf{H}^*$ we have that $\phi(x)$ is not large. This suggests the following question:

7.1. Question. Is $\phi(\mathcal{N}) = \mathcal{N}$ for each autohomeomorphism ϕ of $\mathbf{H}^*$?

Notice that there is an autohomeomorphism ϕ of $\mathbf{H}^*$ such that $\phi(\mathcal{L}) \cap \mathcal{L} = \emptyset$. For completeness let us also add (see the introduction) that van Douwen has asked whether $\phi(\mathcal{R}) = \mathcal{R}$ for each autohomeomorphism ϕ of $\mathbf{H}^*$. As a consequence of Theorem 1.2, there is no autohomeomorphism ϕ of $\mathbf{H}^*$ for which $\phi(\mathcal{R}) \cap \mathcal{R} = \emptyset$, a fact which is far from answering van Douwen's question, but it is an indication that his question might have a positive answer.

Let us also add that interest in $\mathbf{H}^*$ was motivated by Bellamy's [1] and Woods's [14] result that $\mathbf{H}^*$ is an indecomposable continuum (i.e. $\mathbf{H}^*$ is not the union of two nonempty proper subcontinua). Mioduszewski [11] has given an easier proof that $\mathbf{H}^*$ is indecomposable. Van Douwen [5] has investigated the structure of the subcontinua of $\mathbf{H}^*$ and has proved that there are at least five mutually nonhomeomorphic nondegenerate proper subcontinua of $\mathbf{H}^*$.

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