WHEN $U(\kappa)$ CAN BE MAPPED ONTO $U(\omega)$

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Abstract. $U(\kappa)$ can be mapped onto $U(\omega)$ iff $\text{cf}(\kappa) = \omega$ or $\kappa > 2^\omega$.

0. Introduction. In this note we show that $U(\kappa)$ can be mapped onto $U(\omega)$ if and only if $\text{cf}(\kappa) = \omega$ or $\kappa > 2^\omega$. As a consequence it follows that CH is equivalent to the statement that $U(\omega_1)$ can be mapped onto $U(\omega)$. That $U(\omega)$ is not always a continuous image of $U(\omega_1)$ is known, [B], however, as far as I know, it was unknown that $U(\omega)$ is not a continuous image of $U(\omega_1)$ under $\neg$CH.

1. Conventions. Cardinals carry the discrete topology. If $\kappa$ is a cardinal then $\beta\kappa$ denotes the Čech-Stone compactification of $\kappa$. The subspace

$$\{ p \in \beta\kappa : \text{ if } P \in p \text{ then } |P| = \kappa \}$$

of $\beta\kappa$ is denoted by $U(\kappa)$. It is easy to see that $U(\kappa)$ is compact. For more information on $\beta\kappa$ and $U(\kappa)$ see [CN].

2. The construction.

2.1. Lemma. If $\text{cf}(\kappa) = \omega$ then $U(\kappa)$ can be mapped onto $U(\omega)$.

Proof. Let $\kappa = \Sigma_{n<\omega} \kappa_n$ where, for each $n$, $\kappa_n < \kappa$. Define $f: \kappa \to \omega$ by $f(\alpha) = n$ iff $\alpha \in \kappa_n$ and let $\beta f: \beta\kappa \to \beta\omega$ be the Stone extension of $f$. It is routine to verify that $\beta f(U(\kappa)) = U(\omega)$. \hfill \Box

2.2. Remark. This lemma is known of course, see for example [vD].

2.3. Lemma. If $\kappa > 2^\omega$ then $U(\kappa)$ can be mapped onto $U(\omega)$.

Proof. Let $\{ A_\alpha : \alpha < 2^\omega \}$ be a (faithfully indexed) partition of $\kappa$ into $2^\omega$ subsets of cardinality $\kappa$. Define $f: \kappa \to 2^\omega$ by $f(\alpha) = \mu$ iff $\alpha \in A_\mu$ and let $\beta f: \beta\kappa \to \beta(2^\omega)$ be the Stone extension of $f$. It is routine to verify that $\beta f(U(\kappa)) = \beta(2^\omega)$. Since $U(\omega)$ has clearly weight $2^\omega$ and since $\beta(2^\omega)$ maps onto each compact space of weight at most $2^\omega$, we conclude that $U(\kappa)$ can be mapped onto $U(\omega)$. \hfill \Box

2.4. Lemma. If $\omega < \text{cf}(\kappa) < \kappa < 2^\omega$ then $U(\omega)$ is not a continuous image of $U(\kappa)$.

Proof. Suppose, to the contrary, that $f$ maps $U(\kappa)$ onto $U(\omega)$. Since there is clearly a compactification of $\omega$ with $I = [0,1]$ as remainder, there is a map $g$ from $U(\omega)$ onto $I$. Let $h: U(\kappa) \to I$ be the composition of $f$ and $g$. In addition, let $\bar{h}: \beta\kappa \to I$ extend $h$.

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Take $s \in I$ arbitrarily. Then $g^{-1}((s))$ is a nonempty $G_{\delta}$ in $U(\omega)$ and consequently has nonempty interior, [CN, 14.17]. Therefore, $f^{-1}g^{-1}((s))$ has nonempty interior (in $U(\kappa)$) and consequently we can find a subset $E \subset \kappa$ so that

$$\emptyset \neq \overline{E} \cap U(\kappa) \subset f^{-1}g^{-1}((s)).$$

CLAIM. If $n < \omega$ then $\{|(\alpha \in E: \overline{h}(\alpha) \notin (s - 1/n, s + 1/n)}| < \kappa$. Suppose, to the contrary, that $F = \{\alpha \in E: \overline{h}(\alpha) \notin (s - 1/n, s + 1/n)\}$ has cardinality $\kappa$. Take a point $x \in \overline{F} \cap U(\kappa)$. By continuity of $\overline{h}$, the point $\overline{h}(x) \notin (s - 1/n, s + 1/n)$. This implies that $x \in (\overline{E} \cap U(\kappa)) - f^{-1}g^{-1}((s))$, which is impossible.

Since $\text{cf}(\kappa) > \omega$ the claim implies that we can find $\kappa \in E$ so that $\overline{h}(\kappa) = s$.

This is a contradiction since $\kappa < 2^\omega = |I|$. □

2.5. Corollary. CH is equivalent to the statement that $U(\omega_1)$ can be mapped onto $U(\omega)$.

PROOF. Since $\omega_1$ has uncountable cofinality this immediately follows from Lemmas 2.3 and 2.4. □

References


[vD] E. K. van Douwen, Transfer of information about $\beta \mathbb{N} - \mathbb{N}$ via open remainder maps (to appear).

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