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WHEN $U(\kappa)$ CAN BE MAPPED ONTO $U(\omega)$

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ABSTRACT. $U(\kappa)$ can be mapped onto $U(\omega)$ iff $\mathrm{cf}(\kappa) = \omega$ or $\kappa > 2^\omega$.

0. Introduction. In this note we show that $U(\kappa)$ can be mapped onto $U(\omega)$ if and only if $\mathrm{cf}(\kappa) = \omega$ or $\kappa \geqslant 2^\omega$. As a consequence it follows that CH is equivalent to the statement that $U(\omega_1)$ can be mapped onto $U(\omega)$. That $U(\omega)$ is not always a continuous image of $U(\omega_1)$ is known, **[B]**, however, as far as I know, it was unknown that $U(\omega)$ is not a continuous image of $U(\omega_1)$ under $\neg$CH.

1. Conventions. Cardinals carry the discrete topology. If κ is a cardinal then $\beta\kappa$ denotes the Čech-Stone compactification of κ. The subspace

$$\{p \in \beta\kappa\colon \text{if } P \in p \text{ then } |P| = \kappa\}$$

of $\beta\kappa$ is denoted by $U(\kappa)$. It is easy to see that $U(\kappa)$ is compact. For more information on $\beta\kappa$ and $U(\kappa)$ see **[CN]**.

2. The construction.

2.1. LEMMA. *If* $\mathrm{cf}(\kappa) = \omega$ *then* $U(\kappa)$ *can be mapped onto* $U(\omega)$.

PROOF. Let $\kappa = \Sigma_{n<\omega}\kappa_n$ where, for each n, $\kappa_n < \kappa$. Define $f\colon \kappa \to \omega$ by $f(\alpha) = n$ iff $\alpha \in \kappa_n$ and let $\beta f\colon \beta\kappa \to \beta\omega$ be the Stone extension of f. It is routine to verify that $\beta f(U(\kappa)) = U(\omega)$. □

2.2. REMARK. This lemma is known of course, see for example **[vD]**.

2.3. LEMMA. *If* $\kappa \geqslant 2^\omega$ *then* $U(\kappa)$ *can be mapped onto* $U(\omega)$.

PROOF. Let $\{A_\alpha\colon \alpha < 2^\omega\}$ be a (faithfully indexed) partition of κ into 2^ω subsets of cardinality κ. Define $f\colon \kappa \to 2^\omega$ by $f(\alpha) = \mu$ iff $\alpha \in A_\mu$ and let $\beta f\colon \beta\kappa \to \beta(2^\omega)$ be the Stone extension of f. It is routine to verify that $\beta f(U(\kappa)) = \beta(2^\omega)$. Since $U(\omega)$ has clearly weight 2^ω and since $\beta(2^\omega)$ maps onto each compact space of weight at most 2^ω, we conclude that $U(\kappa)$ can be mapped onto $U(\omega)$. □

2.4. LEMMA. *If* $\omega < \mathrm{cf}(\kappa) \leqslant \kappa < 2^\omega$ *then* $U(\omega)$ *is not a continuous image of* $U(\kappa)$.

PROOF. Suppose, to the contrary, that f maps $U(\kappa)$ onto $U(\omega)$. Since there is clearly a compactification of ω with $I = [0,1]$ as remainder, there is a map g from $U(\omega)$ onto I. Let $h\colon U(\kappa) \to I$ be the composition of f and g. In addition, let $\bar{h}\colon \beta\kappa \to I$ extend h.

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Take $s \in I$ arbitrarily. Then $g^{-1}(\{s\})$ is a nonempty G_δ in $U(\omega)$ and consequently has nonempty interior, [CN, 14.17]. Therefore, $f^{-1}g^{-1}(\{s\})$ has nonempty interior (in $U(\kappa)$) and consequently we can find a subset $E \subset \kappa$ so that

$$\varnothing \neq \bar{E} \cap U(\kappa) \subset f^{-1}g^{-1}(\{s\}).$$

CLAIM. If $n < \omega$ then $|\{\alpha \in E\colon \bar{h}(\alpha) \notin (s - 1/n, s + 1/n)\}| < \kappa$. Suppose, to the contrary, that $F = \{\alpha \in E\colon \bar{h}(\alpha) \notin (s - 1/n, s + 1/n)\}$ has cardinality κ. Take a point $x \in \bar{F} \cap U(\kappa)$. By continuity of $\bar{h}$, the point $\bar{h}(x) \notin (s - 1/n, s + 1/n)$. This implies that $x \in (\bar{E} \cap U(\kappa)) - f^{-1}g^{-1}(\{s\})$, which is impossible.

Since $\mathrm{cf}(\kappa) > \omega$ the claim implies that we can find $\kappa_s \in E$ so that $\bar{h}(\kappa_s) = s$.

This is a contradiction since $\kappa < 2^\omega = |I|$. $\square$

2.5. COROLLARY. CH *is equivalent to the statement that* $U(\omega_1)$ *can be mapped onto* $U(\omega)$.

PROOF. Since ω_1 has uncountable cofinality this immediately follows from Lemmas 2.3 and 2.4. $\square$

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