A boojum and other snarks

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ABSTRACT

A snark is an extremally disconnected, rigid space in which every nowhere dense set is closed. We give many snarks, inluding (under CH) a boojum -- a hereditarily Lindelöf, hereditarily extremally disconnected, nonseparable snark.

INTRODUCTION

In this paper we describe a machine which constructs rigid extremally disconnected nodec spaces (a space is called *nodec* (cf. [vD_1]) provided each nowhere dense subset is closed). As an application we find a topology on the reals which is larger than the usual topology and which is extremally disconnected, rigid, and nodec. This space, unfortunately, is not normal. Starting, however, from a Luzin space, our machine produces a hereditarily Lindelöf, hereditarily extremally disconnected, nonseparable, rigid, nodec space Y. From these one can derive many other bad properties of Y; for example, every perfect closed set in Y is open.

1. DEFINITIONS

1.1. A *perfect* space is one with no isolated points; a perfect set likewise. That is, our perfect sets are not, as sometimes happens, required to be closed.

1.2. A *nodec* space is one in which every nowhere dense set is closed (and hence discrete).

1.3. A *retractifiable* space is one in which every closed set is a retract.

1.4. A *rigid* space is one with only one autohomeomorphism (to wit, the identity).

1.5. An *incompressible* space is not homeomorphic to any proper subspace.

1.6. A map f from X onto Y is *irreducible* if f maps no closed proper subset of X onto Y.

1.7. If X is a space then eX is the (essentially unique) extremally disconnected space which admits a perfect irreducible map onto X. This map is called π_X. For further details see [Wo] or [CN, p. 57].

1.8. A *Luzin* space is an uncountable perfect space in which every nowhere set is countable.

1.9. An *L-space* is hereditarily Lindelöf but not separable.

1.10. If X is a space and $A \subset X$ then $p \in X$ is *remote* from A if for every $C \subset A$, if C is nowhere dense relative to A then $p \notin cl_X C$.

1.11. βX is the Čech-Stone compactification of X; $X^* = \beta X \setminus X$.

1.12. p is a *remote point* of X if $p \in X^*$ and p is remote from X.

1.13. X is *ω-bounded* if every countable set in X has compact closure.

1.14. A set $A \subset X$ is *C^*-embedded* in X every $f: A \to [0,1]$ extends over X.

1.15. $w(X)$ is the *weight* of X, the least cardinality of a basis for X.

1.16. $\pi(X)$ is the *π-weight* of X, the least cardinality of a π-basis for X, where a π-basis for X is a collection $\mathscr{B}$ of nonempty open sets such that every nonempty open subset of X contains a member of $\mathscr{B}$.

1.17. A *snark* is an extremally disconnected, rigid nodec space.

1.18. A *boojum* is a hereditarily Lindelöf, hereditarily extremally disconnected, nonseparable snark.

2. SOME SNARKS

All spaces are completely regular.

Our first result is an example-generating one;

2.1. THEOREM: *Let X be a perfect space such that $w(\beta eX) \leq c$ and for each $p \in X, \pi_X^{-1}(p)$ is infinite. Then there is a perfect, extremally disconnected, rigid space Y and a one-to-one map f from Y onto X, with the property that $D \subset Y$ is nowhere dense iff $f[D]$ is nowhere dense. If, moreover, for each $p \in X$, there is an infinite, ω-bounded set of points in $\pi_X^{-1}(p)$ which are remote from $eX\text{-}\pi_X^{-1}(p)$, then Y may be taken to be nodec.*

We shall see later that the hypothesis of the second clause of the theorem is very often satisfied.

PROOF: Fix X satisfying the hypotheses. For $p \in X$, fix a countable discrete set $D^p \subset \pi_X^{-1}(p)$ and set $E^p = \overline{D^p} - D^p$; if $\pi_X^{-1}(p)$ has an ω-bounded infinite set B^p of points remote from $eX - \pi_X^{-1}(p)$, take $D^p \subset B^p$; then $E^p \subset B^p$. Note that $E^p \approx \omega^*$.

For $p \in X$, set

$T(p) = \{u \in \omega^* \mid \exists$ discrete sequence $\{p_n \mid n \in \omega\} \subset \beta eX$ such that $\{p\} = \bigcap_{A \in u} cl_{\beta eX} \{p_n \mid n \in A\}\}$.

Since $w(\beta eX) \leq c$ one sees easily that $|T(p)| \leq c$. Since $|X| \leq 2^c$ we can list X as $\{p_\alpha \mid \alpha < \kappa\}$ where $\kappa \leq 2^c$. Pick, by induction on α, a point $y_\alpha \in E^{p_\alpha}$ so that

$$T(y_\alpha) \not\subset \bigcup_{\beta<\alpha} T(y_\beta).$$

This is possible since for each $u \in \omega^*$ we have that there is some $y \in E^{p_\alpha}$ such that $u \in T(y)$ and

$$|\bigcup_{\beta<\alpha} T(y_\beta)| < 2^c = |\omega^*|.$$

Set $Y = \{y_\alpha \mid \alpha < \kappa\}$ and $f = \pi_X \upharpoonright Y$; that is, $f(y_\alpha) = x_\alpha$ $(\alpha < \kappa)$. Clearly, f is one-to-one and onto; also, Y is dense in eX since π_X is irreducible, and $D \subset Y$ is nowhere dense iff $f[D]$ is nowhere dense. Suppose that $\varphi: Y \to Y$ is an autohomeomorphism. Then φ extends to an autohomeomorphism $\beta\varphi: \beta Y \to \beta Y = \beta eX$. But if $\alpha < \mu < \kappa$, $\beta\varphi(y_\alpha) \neq y_\mu$ and conversely since $T(y_\alpha) \neq T(y_\mu)$ and for each $T \subset \omega^*$,

$$T(p) = T$$

is a topological property of p. We conclude that φ is the identity.

Now suppose that for each $\alpha < \kappa$, y_α is remote from $eX - \pi_X^{-1}(x_\alpha)$. Then y_α is remote from $Y - \{y_\alpha\}$; that is, since this holds for each $y \in Y$, Y is nodec. □

3. LOTS OF SPACES HAVE ENOUGH REMOTE POINTS

In the following lemma we find some spaces which satisfy the hypothesis of the second clause of Theorem 2.1.

3.1. LEMMA: *Let X be a first countable perfect space of countable π-weight; or assume CH, and let X be a first countable space with at most c regular open sets. Then $w(\beta eX) \leq c$ and for each $p \in X$, $\pi_X^{-1}(p)$ contains an infinite ω-bounded set of points which are remote from $eX - \pi_X^{-1}(p)$.*

PROOF: Assume first that X has countable π-weight. Then eX is separable and hence $w(\beta eX) \leq c$. Fix $p \in X$. Then

$$Z = \beta eX - \pi_X^{-1}(p)$$

is locally compact and σ-compact, and $Z^* = \pi_X^{-1}(p)$. Clearly $\pi[Z] = X - \{p\}$, and hence the conclusion follows from [vD_2, 4.2] and the following

FACT: Let X be locally compact and σ-compact. Then the set of remote points of X is ω-bounded.

For take $E \subset X^*$ to be a countable set of remote points. Fix $p \in \bar{E} - E$ and assume that p is not remote; i.e. p is in the closure of some nowhere dense $D \subset X$. By [vMM, 4.1]

$$F = cl_{\beta X} D \cap X^*$$

is a P-set of X^*; i.e. the intersection of countably many neighborhoods of F is again a neighborhood of F. Then $F \cap \bar{E} = \emptyset$, a contradiction (this result is due to van Douwen; see [vD_2, 11.2]).

Now assume CH and let X be a first countable space which has at most c regular open sets. Since βeX is (homeomorphic to) the Stone space of the boolean algebra of regular open sets of X it follows that $w(\beta eX) \le c$ (see [CN, p. 57]). By [KvMM, 1.3] for each $p \in X$ the set of remote points of

$$Z = \beta eX - \pi_X^{-1}(p)$$

is infinite and the Fact implies that it is ω-bounded. □

3.2. COROLLARY: *There is a topology on the reals which is finer than the usual topology and which is extremally disconnected, rigid, and nodec and which moreover has countable π-weight; each nowhere dense set of the usual topology is nowhere dense and hence closed and discrete in the new topology.* □

3.3. REMARK: The condition that X is first countable in Lemma 3.1 can be weakened considerably. In fact, one only needs that each point of X is in the closure of countably many pairwise disjoint open sets and that $eX - \pi_X^{-1}(p)$ is nonpseudocompact for each $p \in X$.

4. A BOOJUM

Henceforth we assume CH.

It has been pointed out to us that a space with the properties (1) and (3) of our boojum and which in addition is an L-space has been discovered by Tall ([T], p. 282).

4.1. LEMMA: *There is a first countable Luzin space X of cardinality and weight ω_1 which has no separable open sets.* □

For a proof we refer the reader to [vDTW].

Note that, since X is hereditarily Lindelöf, X has only c regular open sets and hence satisfies the hypotheses of Lemma 3.1.

Accordingly, fix Y and $f: Y \to X$ as in section 2. Then Y is a very bad space. In particular,

(1) *If $A \subset Y$, following are equivalent:*
  (a) *A is discrete;*
  (b) *A is at most countable;*
  (c) *A is nowhere dense;*
  (d) *A is closed and discrete.*

(2) *Y is hereditarily retractifiable.*

(3) *Y is hereditarily extremally disconnected.*

(4) *Every subspace of Y is the free union of its isolated points and its perfect kernel (= largest perfect subspace).*

(5) *Every perfect subspace of Y is rigid, incompressible and retractifiable.*

By (1),

(6) *Y is a nonseparable Luzin space and hence an L-space (i.e. a hereditarily Lindelöf nonseparable space).*

(7) *Every perfect closed subspace of Y is open.*
(8) *Every subspace of Y is C^*-embedded.*

The reader can easily cook up many more improbable properties of Y in the same spirit.

PROOFS: Note that every perfect set of Y is dense in an open set of eX, since Y is nodec. Hence once we prove (1) and (2), the other properties follow easily. However, (1) is easy; for recall that A is nowhere dense iff $f[A]$ is, and note that A is countable iff $f[A]$ is. By hypothesis, (*b*) and (*c*) are equivalent for subspaces of X; hence also for Y. Moreover $(c) \Leftrightarrow (d) \Leftrightarrow (a)$ follows since Y is nodec.

(2) is also easy. We give the proof that Y is retractifiable; the proof for subspaces is the same. If $A \subset X$ is closed then $A = B \cup C$ where B is clopen and $C = \{c_n \mid n \in \omega\}$ is closed and discrete. Since Y is Lindelöf we may fix a discrete collection $\{D_n \mid n \in \omega\}$ of clopen sets with $c_n \in D_n \subset Y - B$.

Then put

$$r(x) = \begin{cases} c_n \text{ if } x \in D_n; \\ x \text{ if } x \in B; \\ c_0 \text{ otherwise.} \end{cases}$$

Then r retracts Y onto B. □

5. NOTES

In fact, the assumption of CH in section 4 is not necessary; one only needs to assume that there is a first countable nonseparable Luzin space. For then one shows easily that there is one of small enough weight and that such a space has enough remote points.

The trick used in the proof of theorem 2.1 ensure the rigidity of Y is similar to one used by Comfort & Negrepontis ([CN, 16.18]); both tricks trace their ancestry back to Frolík [F].

We have seen that, under CH, there is an extremally disconnected L-space; in fact, a nodec one. This suggests the question of whether there is an extremally disconnected nodec S-space (an S-space is hereditarily separable and not Lindelöf) since extremally disconnected S-spaces can exist ([W]). The answer however is no.

5.1. PROPOSITION: *Each hereditarily separable nodec space is Lindelöf.*

PROOF: Let X be a hereditarily separable nodec space and let $\mathscr{U}$ be an open cover of X. Since X is separable, there is a countable $\mathscr{E} \subset \mathscr{U}$ such that $\cup \mathscr{E}$ is dense. Then $X - \cup \mathscr{E}$ is nowhere dense, hence discrete, hence countable. We conclude that $\mathscr{U}$ has a countable subcover. □

Finally, note that our boojum is Luzin and that Luzin spaces do not exist under $MA + \neg CH$ (see [K]).

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