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# ON NOWHERE DENSE CCC *P*-SETS

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ABSTRACT. We prove that no compact Hausdorff space can be covered by nowhere dense ccc $P$-sets. As an application it follows that if $X$ is a compact Hausdorff space with a nonisolated $P$-point then $X \times K$ is not homogeneous for any compact ccc space $K$.

**1. Introduction.** All spaces under discussion are Tychonoff.

A subset $B$ of a space $X$ is called a *P-set* whenever the intersection of countably many neighborhoods of $B$ is again a neighborhood of $B$. It is known that no compact space of $\pi$-weight $\omega_1$ can be covered by nowhere dense $P$-sets [**KvMM**]. In addition, there is a compact space of weight $\omega_2$ which can be covered by nowhere dense $P$-sets [**KvMM**]. In this note we will show that no compact space can be covered by nowhere dense ccc $P$-sets. As a consequence it follows that if $X$ is a compact space with a nonisolated $P$-point then $X \times K$ is not homogeneous for any compact ccc space $K$.

**2. Independent matrices.** Let $X$ be a space. An indexed family $\{A_j^i\colon i \in I, j \in J\}$ is called an *I by J independent matrix* for $X$ provided that

(a) each $A_j^i$ is an open $F_\sigma$;

(b) if $i \in I$ and $j_0, j_1$ are distinct elements of $J$ then $A_{j_0}^i \cap A_{j_1}^i = \varnothing$;

(c) if $F \subset I$ is finite and $\varphi\colon F \to J$ then $\bigcap_{i \in F} A_{\varphi(i)}^i \neq \varnothing$.

This concept, in a slightly different form, is due to Kunen.

In [**vM$_1$**] it was shown that each compact space in which each nonempty $G_\delta$ has nonempty interior contains an $\omega_1$ by $\omega_1$ independent matrix. We need a generalization of this result. As usual, a space is called ccc if each pairwise disjoint collection of nonempty open sets is countable. A space is nowhere ccc if no point has a ccc neighborhood.

2.1. THEOREM. *Suppose that $X$ is nowhere ccc. Then $X$ contains an $\omega_1$ by $\omega_1$ independent matrix.*

PROOF. For each finite subset $F \subset \omega_1$ (possibly empty) we will define an open $F_\sigma$, $C_F \subset X$, such that

(i) $C_{F \cup \{\alpha\}} \subset C_F$ for all $\max F < \alpha < \omega_1$;

(ii) $C_{F \cup \{\alpha\}} \cap C_{F \cup \{\beta\}} = \varnothing$ if $\max F < \alpha < \beta < \omega_1$

(as usual, an ordinal is the set of smaller ordinals; we define $\max \varnothing = -1$).

We will induct on the cardinality of $F$. Define $C_\varnothing = X$.

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Suppose that we have defined $C_F$ for all $F \subset \omega_1$ of cardinality $n$. Let $\{C_{F\cup\{\alpha\}}: \max F < \alpha < \omega_1\}$ be a "faithfully indexed" collection of pairwise disjoint nonempty open $F_\sigma$'s of $C_F$. This completes the induction.

FACT. $C_F \cap C_G \neq \varnothing \to (F \subset G) \vee (G \subset F)$.

We induct on the cardinality of $|F| + |G|$. If $|F| + |G| = 1$ then there is nothing to prove. Suppose that we have proved the Fact for all finite sets $F, G \subset \omega_1$ satisfying $|F| + |G| \leqslant n - 1$. Now take finite sets $S, T \subset \omega_1$ so that $|S| + |T| \leqslant n$. Define $S' = S - \{\max S\}$. By (i) we have that $C_S \subset C_{S'}$ and consequently $C_{S'} \cap C_T \neq \varnothing$. By induction hypothesis, $S' \subset T$ or $T \subset S'$. If $T \subset S'$ then we are done, so we may assume that $S' \subset T$. Define $T' = T - \{\max T\}$. By precisely the same argumentation we may conclude that $T' \subset S$. Then clearly

$$(S \cap T) \cup \{\max S\} = S \quad \text{and} \quad (S \cap T) \cup \{\max T\} = T.$$

If $\max S \in T$ or $\max T \in S$ then there is nothing to prove. So assume that this is not true. Then by (ii) we have that $C_S \cap C_T = \varnothing$, which is a contradiction.

Let $f\colon \omega_1 \to \omega_1 \times \omega_1$ be onto and one-to-one. Define $U^\alpha_\beta = \bigcup \{C_{F\cup\{f^{-1}(\langle\alpha,\beta\rangle)\}}: \max F < f^{-1}(\langle\alpha, \beta\rangle)$ and $f[F] \cap (\{\alpha\} \times \omega_1) = \varnothing\}$. Notice that $C_{\{f^{-1}(\langle\alpha,\beta\rangle)\}} \subset U^\alpha_\beta$. We claim that $\{U^\alpha_\beta: \alpha, \beta < \omega_1\}$ is an $\omega_1$ by $\omega_1$ independent matrix for $X$. First observe that each $U^\alpha_\beta$ is an open $F_\sigma$ being the union of at most countably many open $F_\sigma$'s.

Now, let us assume that $U^\alpha_\beta \cap U^\alpha_\gamma \neq \varnothing$ for some $\beta \neq \gamma$. Withot loss of generality assume that $f^{-1}(\langle\alpha, \beta\rangle) < f^{-1}(\langle\alpha, \gamma\rangle)$. There are finite sets $F_0, F_1 \subset \omega_1$ so that

(a) $C_{F_0\cup\{f^{-1}(\langle\alpha,\beta\rangle)\}} \cap C_{F_1\cup\{f^{-1}(\langle\alpha,\gamma\rangle)\}} \neq \varnothing$;

(b) $\max F_0 < f^{-1}(\langle\alpha, \beta\rangle)$ and $f[F_0] \cap (\{\alpha\} \times \omega_1) = \varnothing$;

(c) $\max F_1 < f^{-1}(\langle\alpha, \gamma\rangle)$ and $f[F_1] \cap (\{\alpha\} \times \omega_1) = \varnothing$.

Since $f^{-1}(\langle\alpha, \gamma\rangle) \notin F_0 \cup \{f^{-1}(\langle\alpha, \beta\rangle)\}$, by the Fact, $F_0 \cup \{f^{-1}(\langle\alpha, \beta\rangle)\} \subset F_1 \cup \{f^{-1}(\langle\alpha, \gamma\rangle)\}$. Therefore $f^{-1}(\langle\alpha, \beta\rangle) \in F_1$, since $f^{-1}(\langle\alpha, \beta\rangle) \neq f^{-1}(\langle\alpha, \gamma\rangle)$. However, this contradicts (c).

Take $\alpha_1, \ldots, \alpha_n < \omega_1$ so that $\alpha_i \neq \alpha_j$ for $i \neq j$. In addition, take $\beta_i < \omega_1$ $(i \leqslant n)$ arbitrarily. Put $\gamma_i = f^{-1}(\langle\alpha_i, \beta_i\rangle)$ and without loss of generality assume that $\gamma_1 < \gamma_2 < \cdots < \gamma_n$. Then $C_{\{\gamma_1,\ldots,\gamma_n\}} \subset U^{\alpha_1}_{\beta_1} \cap \cdots \cap U^{\alpha_n}_{\beta_n}$, and since $C_{\{\gamma_1,\ldots,\gamma_n\}} \neq \varnothing$ we find that $U^{\alpha_1}_{\beta_1} \cap \cdots \cap U^{\alpha_n}_{\beta_n} \neq \varnothing$. □

**3. The first application.** A point $x \in X$ is called a *weak P-point* provided that $x \notin \bar{F}$ for each countable $F \subset X - \{x\}$. Kunen **[K]** proved that there is a weak $P$-point in $\omega^*$ $(= \beta\omega\setminus\omega)$. Subsequently van Mill **[vM₁]** proved that there is a weak $P$-point in each compact $F$-space of weight $2^\omega$ in which each nonempty $G_\delta$ has nonempty interior (an *F-space* is a space in which each cozero set is $C^*$-embedded). Bell **[B]** has since shown that the weight condition is superfluous. Using Theorem 2.1 by precisely the same technique as in **[vM$_1$]** we obtain the following generalization.

3.1. THEOREM. *Each compact nowhere ccc F-space contains a weak P-point.*

**4. The main result.** In this section we derive our main result. The techniques of proof used in the following lemma is the same as in **[vM$_1$]**, **[vM$_2$]**.

4.1. LEMMA. *No compact nowhere ccc space can be covered by ccc $P$-sets.*

PROOF. Let $X$ be a compact nowhere ccc space. Clearly $X$ is not finite, so there is a collection $\{V_n: n < \omega\}$ of (faithfully indexed) pairwise disjoint nonempty open $F_\sigma$ subsets of $X$. For each $n < \omega$ let $\{U_\alpha^i(n): \alpha < \omega_1, i < \omega\}$ be an $\omega_1$ by $\omega$ independent matrix for $V_n$ (Theorem 1.1). Notice that each $U_\alpha^i(n)$ is an open $F_\sigma$ of $X$. Put $\mathfrak{F} = \{A \subset X: \forall n < \omega\ \forall i \leqslant n\ \exists \alpha < \omega_1$ such that $U_\alpha^i(n) \subset A\}$. It is clear that $\mathfrak{F}$ has the finite intersection property, so there is an $x \in \bigcap_{F \in \mathfrak{F}} \overline{F}$. We claim that $x \notin K$ for each ccc $P$-set $K$. Indeed, let $K \subset X$ be any ccc $P$-set. Since $K$ is ccc for each $n < \omega$ and for each $i \leqslant n$ there is an $\alpha(n, i) < \omega_1$ so that

$$U_{\alpha(n,i)}^i(n) \cap K = \varnothing.$$

Put $F = \bigcup_{n<\omega} \bigcup_{i \leqslant n} U_{\alpha(n,i)}^i(n)$. Then $F \in \mathfrak{F}$ and $F$ is an open $F_\sigma$ being the union of countably many open $F_\sigma$'s. Also, $F \cap K = \varnothing$. Since $K$ is a $P$-set, it also follows that $\overline{F} \cap K = \varnothing$. We conclude that $x \notin K$. □

We now come to our main result.

4.2. THEOREM. *No compact space can be covered by ccc nowhere dense $P$-sets.*

PROOF. Let $X$ be a compact space and suppose that $X$ can be covered by ccc nowhere dense $P$-sets. Let $U \subset X$ be nonempty and open and suppose that $U$ is ccc. Let $B$ be a nowhere dense $P$-set meeting $U$. Since $B \cap U$ is nowhere dense in $U$ the fact that $U$ is ccc implies that there is a countable family $\mathcal{G}$ of compact subsets of $U - B$ so that $\bigcup \mathcal{G}$ is dense in $U$. However, this is impossible since $B$ is a $P$-set. So $U$ is not ccc. But now the assumption that $X$ can be covered by ccc nowhere dense $P$-sets contradicts Lemma 4.1. □

**5. Another application.** A space $X$ is called *homogeneous* provided that for all $x$, $y \in X$ there is an autohomeomorphism $\varphi$ from $X$ onto $X$ mapping $x$ onto $y$. It is well known that although $X$ is not homogeneous the product $X \times K$ can be homogeneous for certain $K$ (for example, let $X$ be a convergent sequence and let $K$ be the Cantor set). This makes the following straightforward corollary to Theorem 4.2 of some interest.

5.1. COROLLARY. *Let $X$ be a compact space having a nonisolated $P$-point. Then $X \times K$ is not homogeneous for any compact ccc nonempty space $K$.*

PROOF. Let $x$ be a nonisolated $P$-point of $X$. Then $\{x\} \times K$ is a ccc nowhere dense $P$-set of $X \times K$. Take any $\langle x, y \rangle \in \{x\} \times K$. By Theorem 4.2 there is a point $\langle p, q \rangle \in X \times K$ so that $\langle p, q \rangle \notin E$ for any nowhere dense ccc $P$-set $E \subset X \times K$. It is clear that no autohomeomorphism of $X \times K$ can map $\langle x, y \rangle$ onto $\langle p, q \rangle$. □

**6. Questions.** Since there is a compact space $X$ of weight $\omega_2$ which can be covered by nowhere dense $P$-sets (which all have to have cellularity at most $\omega_2$), Theorem 4.2 suggests the following question:

6.1. QUESTION. *Is there a compact space $X$ which can be covered by nowhere dense $P$-sets of cellularity at most $\omega_1$?*

Since Frankiewicz and Mills **[FM]** have shown that Con(ZFC + $\omega^*$ can be covered by nowhere dense $P$-sets) the question naturally arises whether it is consistent that $\omega^*$ can be covered by nowhere dense $P$-sets of cellularity at most $\omega_1$. Let us answer this question.

6.2. PROPOSITION. *$\omega^*$ cannot be covered by nowhere dense P-sets of cellularity at most $\omega_1$.*

PROOF. Under CH the result follows from **[KvMM]**. So assume $\neg$CH. Kunen **[K]** proved that (in ZFC) there is a $2^\omega$ by $2^\omega$ independent matrix of clopen subsets of $\omega^*$. Since $\omega_1 < 2^\omega$ we can use the same proof as in Lemma 4.1 to get a point $x \in \omega^*$ so that $x \notin B$ for any $P$-set $B$ of cellularity at most $\omega_1$. $\square$

Let us finally notice that Proposition 5.1 suggests the following question.

6.3. QUESTION. *Let $X$ be a compact space having a nonisolated P-point and let $K$ be compact. Is $X \times K$ not homogeneous?*

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