The compactness number of a compact topological space I

by

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Abstract. We generalize the notion supercompactness as defined by J. de Groot [6].

1. Introduction. Alexander's well known subbase lemma states that a topological space is compact if and only if it possesses an open subbase $U$ such that each covering of $X$ by elements of $U$ contains a subcovering of finitely many elements of $U$. This lemma suggests the following definition: for a compact Hausdorff space we define the compactness number $\text{cmpn}(X)$ of $X$ in the following manner

- $\text{cmpn}(X) \leq k$ ($k \in \omega$) if $X$ has an open subbase $U$ such that each covering of $X$ by elements of $U$ has a subcovering of at most $k$ members,
- $\text{cmpn}(X) = k(k \in \omega)$ if $\text{cmpn}(X) \leq k$ and $\text{cmpn}(X) \leq k$,
- $\text{cmpn}(X) = \infty$ if $\text{cmpn}(X)$ is not finite.

This definition of compactness number enables us to distinguish between compact Hausdorff spaces in compactness type. Clearly $\text{cmpn}(X) = 1$ iff $|X| = 1$ and $\text{cmpn}(X) = 2$ iff $X$ is supercompact (in the sense of de Groot [6]) and contains more than one point. In van Douwen & van Mill [4] it was shown that the one point compactification of the Cantor tree $2^\omega$ (cf. Rudin [9]) has compactness number $3$ (this fact was also proved independently by the first author of the present paper). In this paper we answer some obvious questions. We show that for each $k \geq 1$ there is a compact Hausdorff space $X_k$ which has compactness number $k$; moreover $\beta X_k$, the Čech–Stone compactification of the natural numbers, has compactness number $\infty$.

The last years much time has been spent to prove that certain compact Hausdorff spaces are supercompact (cf. Strokol & Szymański [10]; cf. also van Douwen [3]) and also that certain compact Hausdorff spaces are not supercompact (cf. Bell [1], [2], van Douwen & van Mill [4], van Mill [7]). The first examples of nonsupercompact compact Hausdorff spaces were given by Bell [1]. The results in this paper generalize some of the results in [1] and [4].
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This paper is organized as follows: in Section 2 we prove a combinatorial result, which then is used in Section 3 to construct the examples and to prove that \( \text{cmpn}(\beta X) = \infty \) if \( X \) is not pseudocompact.

In Section 4 we collect some questions we cannot answer at the moment.

2. Combinatorics. Let \( N \) denote the set of natural numbers; \( \mathscr{P}(N) \) is the powerset of \( N. \) If \( A \) is a set and \( x \) is any cardinal, define

\[
\begin{align*}
\mathcal{A}^x &= \{ B \subseteq A \mid |B| = x \}, \\
\mathcal{A}^{<x} &= \{ B \subseteq A \mid |B| < x \}, \\
\mathcal{A}^{\leq x} &= \{ B \subseteq A \mid |B| \leq x \}.
\end{align*}
\]

A collection of sets \( \mathscr{G} \) is called an independent family if for each pair of disjoint finite subsets \( \mathscr{F} \) and \( \mathscr{H} \) of \( \mathscr{G} \) the set \( \mathcal{F} \cap \mathcal{H} \) is infinite. The existence of an independent family of cardinality \( n \) of subsets of \( N \) was first proved by G. Fichtenholz and L. Kantorovich [5].

2.1. Definitions. Let \( n \geq 1. \) Let \( \mathcal{A} = (A_\gamma \mid \gamma \in \Gamma) \) and \( \mathcal{B} = (B_\gamma \mid \gamma \in \Gamma) \) be two collections of sets such that \( A_\gamma \subseteq B_\gamma \) for each \( \gamma \in \Gamma. \) We call \( \mathcal{A} \) independent over \( \mathcal{B} \) if for each pair of disjoint finite subsets \( F \) and \( G \) of \( \Gamma \) the set \( \bigcap _{\gamma \in F} A_\gamma - \bigcap _{\gamma \in G} B_\gamma \) is infinite. In addition \( T \) is called an \( n \)-transversal on \( \mathcal{A}\mathcal{B} \) if

(a) \( T \subseteq \bigcup \mathcal{A} \); 
(b) \( |T \cap \bigcap \mathcal{F}| = 1 \) for each \( \mathcal{F} \subseteq [\mathcal{A}]^n ; \)
(c) \( |T \cap \bigcap \mathcal{F}| = 0 \) for each \( \mathcal{F} \subseteq [\mathcal{B}]^{n+1} \).

2.2. Lemma. Let \( n \geq 1. \) Let \( \{ A_\alpha \mid \alpha < \omega_1, 1 \leq \alpha \leq n \} \) and \( \{ B_\alpha \mid \alpha < \omega_1 \} \) be two collections of subsets of \( N \) such that for each \( \alpha < \omega_1 \) we have that \( \bigcup _{\alpha \leq n} A_\alpha = B_\alpha \) and \( \bigcup _{\alpha \leq n} A_\alpha \) is independent over \( \{ B_\alpha \mid \alpha < \omega_1 \} \). Then there exists an uncountable subset \( \mathcal{M} \) of \( \omega_1 \) and for each \( \alpha \in \mathcal{M} \) an \( n_\alpha \) with \( 1 \leq n_\alpha \leq n \) such that \( \{ A_\alpha \mid \alpha \in \mathcal{M} \} \) is independent over \( \{ B_\alpha \mid \alpha < \omega_1 \} \).

Proof. The proof is by induction. The case \( n = 1 \) is obvious. Assume the lemma is true for \( n \) and let \( A_\alpha = \bigcup _{1 \leq \alpha \leq n} A_\alpha. \) The \( A_\alpha \)'s are now constructed inductively. Assume we have chosen \( \mathcal{M}_\alpha \) and \( \mathcal{W}_\alpha \) for \( \alpha < \beta < \omega_1 \) such that

(1) \( \mathcal{M}_\alpha \cup \mathcal{W}_\alpha = \mathcal{M}_{\alpha+1} \cap \mathcal{W}_{\alpha} = \emptyset \) and \( \mathcal{W}_\alpha = \emptyset \) is co-countable in \( \omega_1 \); 
(2) \( \gamma \in \alpha \) implies that \( \mathcal{M}_\gamma \) is properly contained in \( \mathcal{M}_\alpha \) and \( \mathcal{W}_\alpha \subseteq \mathcal{W}_\gamma \); 
(3) for all disjoint finite subsets \( F \) and \( G \) of \( \mathcal{M}_\alpha \) and all disjoint finite subsets \( H \) and \( K \) of \( \mathcal{W}_\alpha \) \( (\bigcap _{\gamma \in F} A_\gamma - \bigcap _{\gamma \in H} B_\gamma \) is infinite.

If \( \mathcal{M}_\alpha \) and \( \mathcal{W}_\beta \) can now be constructed such that (1), (2) and (3) hold then \( \{ A_\alpha \mid \alpha \in \bigcup \mathcal{M}_\beta \} \) will be independent over \( \{ B_\alpha \mid \alpha \in \bigcup \mathcal{M}_\beta \} \). To this end, observe that \( \bigcap \mathcal{W}_\alpha \) is again co-countable. For each \( \gamma \in \bigcap \mathcal{W}_\alpha \) define \( C_\gamma = \bigcup _{\gamma \in \mathcal{M}_\beta} A_\gamma \).

If there exists an uncountable subset \( \mathcal{P} \) of \( \bigcap \mathcal{W}_\alpha \) such that \( \{ C_\gamma \mid \gamma \in \mathcal{P} \} \) is independent over \( \{ B_\gamma \mid \gamma \in \mathcal{P} \} \) then by our inductive hypothesis for \( n \) we shall obtain what we want inside of \( \mathcal{P}. \) Therefore assume that for each uncountable subset \( \mathcal{P} \) of \( \bigcap \mathcal{W}_\alpha \) there exist disjoint finite subsets \( F \) and \( G \) of \( \mathcal{P} \) such that

\[
| \bigcap _{\gamma \in F} A_\gamma - \bigcap _{\gamma \in G} B_\gamma | < \omega.
\]

Striving for a contradiction, assume that for each \( \delta \in \bigcap \mathcal{W}_\alpha \) and each co-countable subset \( \mathcal{P} \) of \( \bigcap \mathcal{W}_\alpha \) there exist disjoint finite subsets \( F_\delta \) and \( G_\delta \) of \( \bigcup \mathcal{M}_\alpha \) and disjoint finite subsets \( H_\delta \) and \( K_\delta \) of \( \mathcal{P} \) with

\[
| \bigcap _{\gamma \in F_\delta} A_\gamma - \bigcap _{\gamma \in G_\delta} B_\gamma | < \omega.
\]

Choose an uncountable subset \( \mathcal{R} \) of \( \bigcap \mathcal{W}_\alpha \) and for each \( \delta \in \mathcal{R} \) a \( F_\delta, G_\delta, H_\delta \) and \( K_\delta \) as above with \( \{ H_\delta \mid \delta \in \mathcal{R} \} \) and \( \{ K_\delta \mid \delta \in \mathcal{R} \} \) being a mutually disjoint collection and such that

\[
\mathcal{R} \cap (\bigcup _{\delta \in \mathcal{R}} H_\delta \cup \bigcup _{\delta \in \mathcal{R}} K_\delta) = \emptyset.
\]

The set \( \mathcal{R} \) can be constructed inductively using the preceding assumption. Since there are only countably many pairs of disjoint finite subsets of \( \bigcup \mathcal{M}_\alpha \) it follows that there must be two disjoint finite subsets \( F \) and \( G \) of \( \bigcup \mathcal{M}_\alpha \) and an uncountable subset \( \mathcal{P} \) of \( \mathcal{R} \) such that for each \( \delta \in \mathcal{P} \) we have that

\[
| \bigcap _{\gamma \in F_\delta} A_\gamma - \bigcap _{\gamma \in G_\delta} B_\gamma | < \omega.
\]

For this \( \mathcal{P} \) there exist disjoint finite subsets \( F_\delta \) and \( G_\delta \) of \( \mathcal{P} \) with \( | \bigcap _{\gamma \in F_\delta} A_\gamma - \bigcap _{\gamma \in G_\delta} B_\gamma | < \omega. \)

Since

\[
| \bigcap _{\gamma \in F} A_\gamma - \bigcap _{\gamma \in G} B_\gamma | < \omega,
\]

it follows that

\[
| \bigcap _{\gamma \in F} A_\gamma - \bigcap _{\gamma \in G} B_\gamma | < \omega.
\]

This contradicts (3) since \( F \) and \( G \) are disjoint finite subsets of some \( \mathcal{M}_\alpha \) for \( \alpha < \beta \) and \( F_\delta \cup \bigcup _{\delta \in \mathcal{R}} H_\delta \) and \( G_\delta \cup \bigcup _{\delta \in \mathcal{R}} K_\delta \) are disjoint finite subsets of \( \bigcup \mathcal{M}_\alpha \).
Consequently choose $\delta \in \bigcap_{s \leq t} \mathcal{W}_s$ and a co-countable subset $\mathcal{W}_s$ of $\bigcap_{s \leq t} \mathcal{W}_s$ such that for disjoint finite subsets $F$ and $G$ of $\bigcup_{s \leq t} \mathcal{M}_s$ and disjoint finite subsets $H$ and $K$ of $\mathcal{W}_s$, we have that

$$[(A_{m+1} \cap \bigcap_{s \leq t} \mathcal{M}_s) \cap \bigcup_{r \leq t} B_r] = \omega.$$ 

Since $\delta \in \bigcap_{s \leq t} \mathcal{W}_s$ it is also true that

$$[(A_{m+1} \cap \bigcap_{s \leq t} \mathcal{M}_s) \cap \bigcup_{r \leq t} B_r] = \omega.$$ 

Hence defining $\mathcal{M}_s := \bigcup_{a \in \beta} \mathcal{M}_a \cup \{\delta\}$ we see that $\mathcal{M}_s$ and $\mathcal{W}_s$ satisfy (1), (2) and (3).

This completes the proof.

We need another lemma.

2.3. Lemma. Let $n \geq 1$. Let $\{A_a\}_{a < \omega_1}$ and $\{B_a\}_{a < \omega_1}$ be two collections of subsets of $S$ such that for each $a < \omega_1$ we have that $A_a \subset B_a$ and $\{A_a : a < \omega_1\}$ is independent over $\{B_a : a < \omega_1\}$. Then there exist $(a_i : i < \omega) \in \mathcal{A}$ and a $T \subset N$ with $T$ an $\alpha$-transversal on $\{A_a \subset \omega\} / \{B_a \subset \omega\}$.

Proof. If $n = 1$ then proceed as follows; if for all $a \in A$, we have that $\{|a| \leq \omega\} \subseteq \omega$ then $\{|a| \leq \omega\} \subset \omega$. Thus there exist infinitely many $\beta > 0$ with $A_\beta \subset B_\beta$. Let $A_0 := A_\beta$ and $\mathcal{M}_0 := \omega_1 \setminus \{0\}$ and $a_0 := 0$.

Assume that we have chosen $\{a_0, \ldots, a_m\}$, $\{a_0, \ldots, a_m\}$ and

$$\{\{g \in \mathcal{H} \mid H \in \{0, \ldots, m\}\}$$

such that

1. $0 \leq i \leq m$ implies that $a_i \in \mathcal{M}_{a_{i-1}} \setminus \mathcal{M}_m$ (\$a_m = \omega_1\$),
2. $\mathcal{M}_{a_{m+1}} \subset \cdots \subset \mathcal{M}_{a_{m+1}} \subset \mathcal{M}_m$ and $\mathcal{M}_m = \omega_1$,
3. $t_0 \in \bigcap_{i \leq t} A_{a_i} \cap \bigcap_{i < \omega} B_{a_i} \setminus \bigcup_{i < \omega} \mathcal{M}_{a_{i+1}}$.

Upon completion of the inductive step $T = \{t_0 \in H \in \{0\}^\omega\}$ will be an $\alpha$-transversal on $\{A_a \subset \omega\} / \{B_a \subset \omega\}$. This is true since for all $H \in \{0\}^\omega$ we have that

$$T \cap \bigcap_{i < \omega} A_{a_i} = \{t_0\}$$

and for $\alpha$ $H \in \{0\}^\omega$ that $T \cap \{H \in \mathcal{M}_m\} = \emptyset$. Clearly $T \subset \mathcal{M}_m$.

Choose $a_{m+1} \in \mathcal{M}_m$. Enumerate $\{H \mid H \in \{0, \ldots, m+1\}\}$ and $m+1 \in H$ as $(H_i)_{i \leq m}$ for each $j$. For each $i < j$, such that $1 \leq i < j$, choose an uncountable subset $\mathcal{M}_0 \subset \mathcal{M}_0$ and a $t_0 \in \bigcap_{i \leq t} A_{a_i} \setminus \{H_i \mid H_i \in \mathcal{H} \} \cap \{B_{a_i} \mid 0 \leq i \leq m, i \notin H \} \cap \{B_{a_i} \mid \beta \in \mathcal{M}_m\}$ such that

$$t_0 \notin \bigcap_{i \leq t} A_{a_i} \cup \bigcup_{i \leq t} B_{a_i}$$

which would contradict independence.

Let $\mathcal{M}_{m+1} := \mathcal{M}_0$. Then $\{a_0, \ldots, a_{m+1}\}$, $\{\mathcal{M}_0, \ldots, \mathcal{M}_{m+1}\}$ and

$$\{t_m \mid H \in \{0, \ldots, m+1\}\}$$

satisfy (1), (2) and (3). \(\blacksquare\)

We now can prove the main result in this section. We remained the reader of the following theorem of F. P. Ramsey [8]: If $r$ and $l$ are two positive integers and the collection $\{W_j : 1 \leq j \leq l\}$ satisfies $[N]^r \subseteq \emptyset$, there then exists an infinite $A \subset N$ and an $s$ with $1 \leq s \leq l$ such that $[A]^l \subseteq W_s$.

2.4. Theorem. Let $n \geq 2$. Let $\mathcal{F} \subset \mathcal{P}(N)$ and let $g : \mathcal{P}(N) \to [\mathcal{P}(N)]^{\omega_1}$ such that for all $A \in \mathcal{P}(N)$ we have that $A = \cup g(A)$. Then there is a collection $\mathcal{F} \subset \mathcal{P}(N)$ $\mathcal{F}$ and for each $H \in \mathcal{F}$ there is a $g(H)$ such that

(i) $\bigcap_{H \in \mathcal{F}} = \emptyset$;

(ii) for all $A \in \mathcal{P}(H)$ we have that $\bigcap_{H \in \mathcal{F}} = \emptyset$.

Proof. Let $n = 2$. Choose two disjoint non-empty subsets $H$ and $K$ of $N$. Choose $g(H) = \emptyset$ and $g(K) = \emptyset$. Let $\mathcal{F} := \{H, K\}$.

So assume that $n > 2$. Let $\{A_0, \ldots, A_m\}$ be an uncountable independent family of $N$. Pick an uncountable subset $\mathcal{M} \subset \omega$ and an $m \in \omega$ such that for each $a \in \mathcal{M}$, $|a(A_a)| = m$. For each $a \in \mathcal{M}$ let $g(a) = \{A_{a_0}, \ldots, A_{a_m}\}$.

Lemma 2.2 followed by Lemma 2.3 yields $(a_i : i < \omega) \in \mathcal{M}$, for each $i < \omega$ an $a_i$ with $1 \leq i \leq m$ and $T \subset N$ such that $T \cap \{i \leq i \leq n-1\}$ and $\bigcap_{i \leq t} A_{a_i} \cap \{H \mid H \in \mathcal{M}_m\}$.

Moreover $[A_{a_m}] \leq \omega$ has finite intersections infinite.

Let $g(T) := \{A_0 \cap T \mid T \subset N \}$ and $W_j := \{F \cap N : j \leq n \} = \emptyset$. Thus $[N]^n \subseteq \emptyset$. F. P. Ramsey's theorem [8] supplies an infinite $A \subset N$ and an $s$ with $1 \leq s \leq l$ such that $[A]^n \subseteq W_s$. Choose $n-1$ distinct elements from $A$: without loss of generality let them be $1, \ldots, n-1$. Define $\mathcal{F} := \{T \mid T \cap \{i \leq i \leq n-1\}$ and $\bigcap_{i \leq t} A_{a_i} \cap \{H \mid H \in \mathcal{M}_m\}$.

Since $T$ is an $n-2$ transversal, $\bigcap_{H \in \mathcal{F}} = \emptyset$. So $\bigcap_{H \in \mathcal{F}} = \emptyset$. Then $\bigcap_{H \in \mathcal{F}} = \emptyset$ for all $H \in \mathcal{F}$.

3. Spaces with finite and infinite compactness number. In the introduction we defined the compactness number $\text{comp}(X)(X)$ in terms of X in an open subbase. This can of course also be defined in a dual form $\text{comp}(X)(X) \leq k (k \in \omega)$ if $X$ admits a closed subbase $\mathcal{S}$ such that for all $A \mathcal{S}$ with $\bigcap_{H \in \mathcal{F}} = \emptyset$ there is a $\mathcal{M} \subset X$ such that $[\mathcal{M}]^r = \emptyset$ and $\text{comp}(X) = \infty$ if for each closed subbase $\mathcal{S}$ for $X$ and for each $k \in \omega$ there is an $\mathcal{M} \subset X$ with $\bigcap_{H \in \mathcal{M}} = \emptyset$. We prefer to work with closed subbases.

We start with some auxiliary results. The easy proofs are left to the reader.

3.1. Proposition. Let $\{X_0 : a \in \omega\}$ be a collection of compact Hausdorff spaces.

Then $\text{comp}(\prod_{a \in \omega} X_a) \leq \sup \{\text{comp}(X_a) : a \in \omega\}$.
3.2. Lemma. Let \( X \) be a compact Hausdorff space for which \( k = \text{cmpn}(X) \) is finite. Then there is a closed subspace \( Y \) for \( X \) which is closed under arbitrary intersections and which in addition realizes \( k \), i.e. for all \( A \subseteq Y \) with \( \bigcap A = \emptyset \) there is an \( \mathcal{F} \in [A]^{k} \) such that \( \bigcap \mathcal{F} = \emptyset \).

We now can prove a simple but useful fact.

3.3. Theorem. Let \( X \) be a compact Hausdorff space and let \( A \) be an open and closed subspace of \( X \). Then \( \text{cmpn}(A) \leq \text{cmpn}(X) \).

Proof. If \( \text{cmpn}(X) = \infty \), then this is a triviality; therefore assume that \( \text{cmpn}(X) \) is finite. Let \( \mathcal{F} \) be a closed subspace for \( X \), closed under arbitrary intersections, which realizes \( \text{cmpn}(X) \). Define \( \mathcal{A} := \{ \mathcal{S} \in \mathcal{F} | \mathcal{S} \subseteq A \} \). We claim that \( \mathcal{A} \) is a closed subspace for \( A \). If this is the case, then clearly \( \text{cmpn}(A) \leq \text{cmpn}(X) \).

Indeed, let \( a \in A \) and let \( C \subseteq A \) be a closed subset not containing \( a \). Then \((X - A) \cup \{a\} \) and \( C \) are disjoint closed subsets of \( X \). By the compactness of \( X \) and by the fact that \( \mathcal{F} \) is closed under arbitrary intersections, there is a finite \( \mathcal{F} \subseteq \mathcal{F} \) such that \( C \subseteq \mathcal{F} \) and \( \bigcup \mathcal{F} \cap ((X - A) \cup \{a\}) = \emptyset \). Hence \( \mathcal{F} \subseteq \mathcal{A} \) which implies that \( \mathcal{A} \) is a closed subspace for \( A \).

3.4. Corollary. Let \( X_k (k \in N) \) be a sequence of compact Hausdorff spaces for which \( \text{cmpn}(X_k) = k (k \in N) \). Let \( Y \) be the disjoint topological sum of the \( X_k \)’s. Then every compactification of \( Y \) has infinite compactness number.

The following theorem gives a wide class of compact Hausdorff spaces with infinite compactness number. Recall that two subsets \( A \) and \( B \) of \( X \) are called completely separated provided that there is a continuous function \( f: X \to I \) such that \( f(A) = 0 \) and \( f(B) = 1 \). The fact is easily verified. If \( U \) and \( V \) are two completely separated subsets of the Tychonoff space \( X \) then there is a zero-set \( Z \) of \( X \) with \( U \cap \text{int}_X \text{cl}_X(Z) \) and \( Z \cap V = \emptyset \).

3.5. Theorem. If \( X \) is a non-pseudocompact space and if \( Y \) is a compact Hausdorff space which can be mapped continuously onto \( X \), then \( \text{cmpn}(Y) = \infty \).

Proof. Let \( X \) be a non-pseudocompact space and let \( Y \) be a compact Hausdorff space which admits a continuous surjection \( g: Y \to X \). Assume that \( \text{cmpn}(Y) = m \) and let \( \mathcal{F} \) be a closed subspace for \( Y \), closed under finite intersections, which realizes this fact. Let \( C = \{c_n | n \in N\} \) be a subset of \( X \) for which there exists a continuous map \( f \) from \( X \) to \( R \) with \( f(c_n) = n \). Define \( C_n := \{x \in X | n < f(x) < n + 1\} \).

Then \( \mathcal{F} := \{C_n | n \in N\} \) is a disjoint collection of cozero-sets of \( X \) with \( c_n \in C_n \) and such that for each \( A \subseteq N \) the set \( \{c_n | n \in A \} \) and \( X - \bigcup C_n \) are completely separated.

For each \( A \subseteq N \) choose a zero-set \( Z_A \subseteq X \) such that \( \text{cl}_X(\{c_n | n \in A\}) \subseteq \text{int}_X \text{cl}_X(Z_A) \) and \( Z_A \subseteq \bigcup C_n \).

Moreover for each \( A \subseteq N \) choose a finite \( \mathcal{F}_A \subseteq \mathcal{F} \) such that \( g^{-1}(\text{int}_X \text{cl}_X(Z_A)) \subseteq \bigcup \mathcal{F}_A \subseteq g^{-1}(\text{int}_X \text{cl}_X(Z_A)) \).

For each \( n \in N \) let \( d_n := g^{-1}(c_n) \); let \( D := \{d_n | n \in N\} \). Let \( \mathcal{F} := \{f_{|D} \cap D \} \subseteq \mathcal{F}_A \) and \( A \subseteq N \).

and define \( \mathfrak{g}: \mathcal{P}(N) \to [\mathcal{F}]^m \) by \( \mathfrak{g}(A) := \{f_{|D} \cap D \} \subseteq \mathcal{F}_A \).

Then clearly \( A = \bigcup \mathfrak{g}(A) \). Now, by Theorem 2.4, there is an \( \mathcal{F} \in [\mathcal{P}(N)]^{m+1} \) and for each \( H \subseteq \mathcal{F} \) there is a \( G_H = \mathfrak{g}(H) \) such that

(i) \( \bigcap \mathcal{F} = \emptyset \)

(ii) for all \( \mathcal{F} \subseteq \{G_H, H \subseteq \mathcal{F}\} \) we have that \( \bigcap \mathcal{F} = \emptyset \).

For each \( H \subseteq \mathcal{F} \) choose \( S_H \in \mathcal{F}_H \) such that \( G_H = f_{|S_H} \cap D \). The contradiction: \( \{S_H | H \subseteq \mathcal{F}\} \) contradicts \( \text{cmpn}(Y) = m \), since

(a) \( \bigcap S_H = g^{-1}(\text{cl}_X(Z_H)) = g^{-1}(\text{cl}_X(Z_H)) \)

(b) \( \text{cl}_X(\bigcap (\bigcup C_n)) = \emptyset \)

(c) \( \mathfrak{g}(I) = \{f_{|D} \cap D \} \subseteq \mathcal{F}_A \).

Arriving at this contradiction, we conclude that \( \text{cmpn}(Y) = \infty \).

Remark. With the same technique it can be shown that if \( X \) is a non-pseudo-compact space then \( \beta X \) is not a continuous image of a closed neighborhood retract of a space \( Y \) with \( \text{cmpn}(Y) < \infty \).

We shall now construct the examples \( X_k (k \in N) \) which were announced in the introduction; first we give some definitions.

Let \( X \) be a set; a subset \( \mathcal{F} \subseteq \mathcal{P}(X) \) is called a linked system if any two of its members meet. A maximal linked system \( \mathcal{F} \subseteq \mathcal{P}(X) \), or briefly mls, is a linked system not properly contained in any other linked system \( \mathcal{F}' \subseteq \mathcal{P}(X) \).

Define \( \mathcal{L} \mathcal{N} := \{\mathcal{F} \subseteq \mathcal{P}(N) | \mathcal{F} \text{ is an mls}\} \) (recall that \( N \) is the set of natural numbers). For all \( A \subseteq N \) define \( A^* \subseteq \mathcal{L} \mathcal{N} \) by \( A^* := \{\mathcal{F} \subseteq \mathcal{L} \mathcal{N} | A \subseteq \mathcal{F}\} \).

The collection \( \{A^* | A \subseteq N\} \) is taken as a closed subspace for a topology on \( \mathcal{L} \mathcal{N} \). It is known, cf. de Groot [6], Verbeek [11], that \( \mathcal{L} \mathcal{N} \) is a supercompact totally disconnected separable Hausdorff space; the subspace \( \{A^* | A \subseteq N\} \) realizes 2. The space \( \mathcal{L} \mathcal{N} \) is called the superextension of \( N \). For convenience we will recall some properties of \( \mathcal{L} \mathcal{N} \) and of the subspace \( \{A^* | A \subseteq N\} \). The proof of the following lemma can be found in Verbeek [11].

3.6. Lemma. Let \( \mathcal{F}_0, \mathcal{F}_1 \subseteq \mathcal{L} \mathcal{N} \). Then

(a) \( \mathcal{F}_0 \neq \mathcal{F}_1 \) if \( \exists M \subseteq \mathcal{F}_1 \) (i.e. \( i \in \{0, 1\} \)): \( M \cap \mathcal{F}_0 = \emptyset \).

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Assume that any \( n - 1 \) members of \( \mathcal{M} \) meet. If \( n \in \{1, 2, \ldots, k\} \) then clearly any \( n \) members of \( \mathcal{M} \) meet. Therefore assume that \( n > k \). Let \( L^+ \cap X_k \in \mathcal{M}(i \in \{1, 2, \ldots, n\}) \) and take for each \( i \in \{1, 2, \ldots, k+1\} \) a point

\[
\mathcal{M}_i \cap (L_i^+ \cap X_k).
\]

Define \( \mathcal{A} := \{i \in \mathcal{A}_i \}^1 \) and \( \mathcal{A} := \{i \in \mathcal{A}_i \}^1 \). Moreover, let

\[
Z := \bigcap_{i \in \mathcal{A}_i} \mathcal{A}_i \cap (A \cup \{1\}).
\]

We claim that this set is nonvoid. Indeed, the system

\[
\mathcal{A} := \{M \in \mathcal{N} \mid \exists B \in \mathcal{B}: B \cap M \neq \emptyset\} \cup \{M \in \mathcal{N} \mid \exists A \in \mathcal{A} \cap (A \cup \{1\}) \neq M\}
\]

is clearly linked, and consequently, by Lemma 3.6(d), there is a point \( \mathcal{N} \in \mathcal{N} \) such that \( \mathcal{A} \subseteq \mathcal{N} \). Then obviously \( \mathcal{N} \subseteq Z \).

Next, observe that \( Z \in \bigcap_{i \in \mathcal{A}_i} \mathcal{A}_i \cap (A \cup \{1\}) \) and hence if \( Z \cap X_k \neq \emptyset \) we have proved Claim 2.

We prove even more: the set \( Z \) is contained in \( X_k \). To this end, let \( \mathcal{N} \in Z \) and let \( V_i \subseteq \mathcal{N} \) for all \( i \in \mathcal{A}_i \) such that \( V_i \cap X_k \neq \emptyset \) for all \( i \in \mathcal{A}_i \). We will derive a contradiction, showing that \( \mathcal{N} \subseteq X_k \).

Fix \( i \in \mathcal{A}_i \) and define \( D_i := \{j \in \mathcal{A}_i \mid V_i \cap X_k \} \). Let us prove that \( |D_i| \neq k \). Indeed, suppose that \( |D_i| \neq k \). Choose distinct \( j_0, j_1, \ldots, j_{k-1} \in \{1, 2, \ldots, k\} \). Then, since \( \mathcal{N} \cap \bigcap_{i \in \mathcal{A}_i} \mathcal{A}_i \cap (A \cup \{1\}) \) is closed, which is impossible.

Now, as \( |D_i| \neq k \) for all \( i \in \mathcal{A}_i \) there is an index \( j_0 \in D_i \). Then \( V_i \cap X_k \) for all \( i \in \mathcal{A}_i \), this is a contradiction. 

**Claim 3.** If \( Y \) is a compact Hausdorff space which can be mapped continuously onto \( X_k \), then \( \text{cmap}(Y) \neq k \). In particular \( \text{cmap}(X_k) = k \).

Let \( Y \) be a compact Hausdorff space and let \( f: Y \to X_k \) be a continuous surjection. Suppose that \( \mathcal{C} \) is any closed subbase of \( Y \) which is closed under arbitrary intersections. For each \( B \cap N \neq 1 \) choose a finite \( \mathcal{C}(B) \subseteq \mathcal{C} \) such that \( \bigcup \mathcal{C}(B) = f^{-1}[B \cap X_k] \). Notice that \( B \cap X_k \) is clopen in \( X_k \) and \( f^{-1}[B \cap X_k] \) is clopen in \( Y \). For each \( a \in X_k \) pick \( d_a \in f^{-1}\{a\}(n) \). Define a function \( g: \mathcal{C}(N \cap \{1\}) \to \bigcup \mathcal{C}(B) \) such that

\[
g(B) := \{i \in \mathcal{A} \cap (A \cup \{1\}) \mid d_a \in f^{-1}(B) \}.
\]

Notice that \( g(B) \in \mathcal{C}(N \cap \{1\}) \) and that \( B = \bigcup g(B) \). By Theorem 2.4 there is a collection \( \mathcal{C} \in \mathcal{C}(N \cap \{1\}) \) and for each \( H \in \mathcal{C} \) there is a \( G_H \in g(H) \) such that

\[
\begin{align*}
& (a) \quad \mathcal{C} = \emptyset, \\
& (b) \quad \text{for all } \emptyset \in \{[G_H] \mid H \in \mathcal{C} \}^{\mathcal{C}} \text{ we have that } \emptyset \neq \emptyset.
\end{align*}
\]
For each $H \in \mathcal{H}$ take $S(H) \in \mathcal{F}$ such that $\{i \in N - \{1\} | d_i \in S(H)\} = G_H$.
Notice that for all $\mathcal{F} \in \{S(H) | H \in \mathcal{H}\}$ we have that $\bigcap_{H \in \mathcal{H}} \mathcal{F} = \emptyset$ and also that

$$\bigcap_{H \in \mathcal{H}} S(H) \subseteq \bigcap_{H \in \mathcal{H}} f^{-1}(H^+ \cap X_i) = f^{-1}\left(\bigcap_{H \in \mathcal{H}} (H^+ \cap X_i)\right).$$

We claim that $\bigcap_{H \in \mathcal{H}} (H^+ \cap X_i) = \emptyset$, which suffices to prove that $\text{cmpn}(Y) \geq k$.

Indeed, assume that there is an $\mathcal{F} \in \{\mathcal{F} \cap \mathcal{H}\}$ and as $\bigcap_{H \in \mathcal{H}} = \emptyset$ there is an $H_0 \in \mathcal{H}$ such that $1 \in H_0$, since $\mathcal{H} \in \mathcal{H}$. Since $\mathcal{H} \in \mathcal{P}(N - \{1\})$ this is a contradiction. \[ \blacksquare \]

Remark. With the same technique it can be shown that if $X_i$ is a continuous image of a closed neighborhood retract of a compact Hausdorff space $Y$, then $\text{cmpn}(Y) \geq k$.

In view of Corollary 3.4 we have also constructed the following example.

3.9. Example. A noncompact locally compact and $\sigma$-compact space $X$ all compactifications of which have infinite compactness number. \[ \blacksquare \]

4. Discussion and questions. The results derived in the present paper suggest many questions. For example, the spaces constructed in Example 3.8 are not first countable and have cardinality $\aleph_1$; this suggests the question whether there exist first countable spaces with the same properties.

4.1. Question. Is there a sequence of first countable separable compact Hausdorff spaces $X_k$ for which $\text{cmpn}(X_k) = k (k \geq 1)$?

If the answer to this question is affirmative, then the Alexandrov one point compactification of the disjoint topological sum of the $X_k$'s would yield a separable first countable space with infinite compactness number.

The problem whether Hausdorff continuous images of supercompact Hausdorff spaces are supercompact, cf. van Douwen and van Mill [4], is still unsolved. The examples (Example 3.8) constructed in this paper suggest a more general question.

4.2. Question. Let $X$ and $Y$ be compact Hausdorff spaces and let $f: X \rightarrow Y$ be a continuous surjection. Is $\text{cmpn}(Y) \leq \text{cmpn}(X)$?

If this is not true, then we still have the following question:

4.3. Question. Let $X$ and $Y$ be compact Hausdorff spaces and let $f: X \rightarrow Y$ be a continuous surjection. Is $\text{cmpn}(Y) < \text{cmpn}(X)$?

There is a countable space no compactification of which is supercompact (cf. van Mill [7]). In view of Example 3.9 this suggests the following:

4.4. Question. Is there a countable space with only one non-isolated point all compactifications of which have infinite compactness number?

References