The compactness number of a compact topological space I

by

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Abstract. We generalize the notion supercompactness as defined by J. de Groot [6].

1. Introduction. Alexander's well known subbase lemma states that a topological space is compact if and only if it possesses an open subbase $\mathcal{U}$ such that each covering of X by elements of $\mathcal{U}$ contains a subcovering of finitely many elements of $\mathcal{U}$. This lemma suggests the following definition: for a compact Hausdorff space we define the *compactness number* $\mathrm{cmpn}(X)$ of X in the following manner

$\mathrm{compn}(X) \leqslant k$ $(k \in \omega)$ if X has an open subbase $\mathcal{U}$ such that each covering of X by elements of $\mathcal{U}$ has a subcovering of at most k members,

$\mathrm{cmpn}(X) = k (k \in \omega)$ if $\mathrm{cmpn}(X) \leqslant k$ and $\mathrm{cmpn}(X) \nless k$,

$\mathrm{cmpn}(X) = \infty$ if $\mathrm{cmpn}(X)$ is not finite.

This definition of compactness number enables us to distinguish between compact Hausdorff spaces in compactness type. Clearly $\mathrm{cmpn}(X) = 1$ iff $|X| = 1$ and $\mathrm{cmpn}(X) = 2$ iff X is supercompact (in the sense of de Groot [6]) and contains more than one point. In van Douwen & van Mill [4] it was shown that the one point compactification of the Cantor tree ${}^{\omega}_{<}2 \cup {}^{\omega}2$ (cf. Rudin [9]) has compactness number 3 (this fact was also proved independently by the first author of the present paper). In this paper we answer some obvious questions. We show that for each $k \geqslant 1$ there is a compact Hausdorff space X_k which has compactness number k; moreover βN, the Čech–Stone compactification of the natural numbers, has compactness number ∞.

The last years much time has been spent to prove that certain compact Hausdorff spaces are supercompact (cf. Strok & Szymański [10]; cf. also van Douwen [3]) and also that certain compact Hausdorff spaces are not supercompact (cf. Bell [1], [2], van Douwen & van Mill [4], van Mill [7]). The first examples of nonsupercompact compact Hausdorff spaces were given by Bell [1]. The results in this paper generalize some of the results in [1] and [4].

This paper is organized as follows: in Section 2 we prove a combinatorial result, which then is used in Section 3 to construct the examples and to prove that $\operatorname{cmpn}(\beta X) = \infty$ if X is not pseudocompact.

In Section 4 we collect some questions we cannot answer at the moment.

2. Combinatorics. Let N denote the set of natural numbers; $\mathscr{P}(N)$ is the powerset of N. If A is a set and $\varkappa$ is any cardinal, define

$$[A]^{\varkappa} = \{B\subset A|\ |B| = \varkappa\},$$
$$[A]^{<\varkappa} = \{B\subset A|\ |B|<\varkappa\},$$
$$[A]^{\leqslant\varkappa} = \{B\subset A|\ |B|\leqslant\varkappa\}.$$

A collection of sets $\mathscr{C}$ is called an *independent family* if for each pair of disjoint finite subsets $\mathscr{F}$ and $\mathscr{H}$ of $\mathscr{C}$ the set $\bigcap\mathscr{F}-\bigcup\mathscr{H}$ is infinite. The existence of an independent family of cardinality $\mathfrak{c}$ of subsets of N was first proved by G. Fichtenholz and L. Kantorovitch [5].

2.1. Definitions. Let $n\geqslant 1$. Let $\mathscr{A} = \{A_\gamma|\ \gamma\in\Gamma\}$ and $\mathscr{B} = \{B_\gamma|\ \gamma\in\Gamma\}$ be two collections of sets such that $A_\gamma\subset B_\gamma$ for each $\gamma\in\Gamma$. We call $\mathscr{A}$ *independent over* $\mathscr{B}$ if for each pair of disjoint finite subsets F and G of Γ the set $\bigcap_{\gamma\in F} A_\gamma-\bigcup_{\gamma\in G} B_\gamma$ is infinite. In addition T is called an *n-transversal* on $\mathscr{A}/\mathscr{B}$ if

(a) $T\subset\bigcup\mathscr{A}$;

(b) $|T\cap\bigcap\mathscr{F}| = 1$ for each $\mathscr{F}\in[\mathscr{A}]^n$;

(c) $|T\cap\bigcap\mathscr{F}| = \varnothing$ for each $\mathscr{F}\in[\mathscr{B}]^{n+1}$.

2.2. Lemma. *Let $n\geqslant 1$. Let $\{A_{\alpha i}|\ \alpha<\omega_1,\ 1\leqslant i\leqslant n\}$ and $\{B_\alpha|\ \alpha<\omega_1\}$ be two collections of subsets of N such that for each $\alpha<\omega_1$ we have that $\bigcup_{i=1}^{n} A_{\alpha i}\subset B_\alpha$ and $\{\bigcup_{i=1}^{n} A_{\alpha i}|\ \alpha<\omega_1\}$ is independent over $\{B_\alpha|\ \alpha<\omega_1\}$. Then there exists an uncountable subset $\mathscr{M}$ of ω_1 and for each $\alpha\in\mathscr{M}$ an n_α with $1\leqslant n_\alpha\leqslant n$ such that $\{A_{\alpha n_\alpha}|\ \alpha\in\mathscr{M}\}$ is independent over $\{B_\alpha|\ \alpha\in\mathscr{M}\}$.*

Proof. The proof is by induction. The case $n = 1$ is obvious. Assume the lemma is true for n and let $A_\alpha = \bigcup_{i=1}^{n+1} A_{\alpha i}$. The $A_{\alpha n_\alpha}$'s are now constructed inductively. Assume we have chosen $\mathscr{M}_\alpha$ and $\mathscr{W}_\alpha$ for $\alpha<\beta<\omega_1$ such that

(1) $\mathscr{M}_\alpha\cup\mathscr{W}_\alpha\subset\omega_1$, $\mathscr{M}_\alpha\cap\mathscr{W}_\alpha = \varnothing$ and $\mathscr{W}_\alpha$ is co-countable in ω_1;

(2) $\gamma<\alpha$ implies that $\mathscr{M}_\gamma$ is properly contained in $\mathscr{M}_\alpha$ and $\mathscr{W}_\alpha\subset\mathscr{W}_\gamma$;

(3) for all disjoint finite subsets F and G of $\mathscr{M}_\alpha$ and all disjoint finite subsets H and K of $\mathscr{W}_\alpha$, $(\bigcap_{\gamma\in F} A_{\gamma n+1}\cap\bigcap_{\gamma\in H} A_\gamma)-\bigcap_{\gamma\in G\cup K} B_\gamma$ is infinite.

If $\mathscr{M}_\beta$ and $\mathscr{W}_\beta$ can now be constructed such that (1), (2) and (3) hold then $\{A_{\alpha n+1}|\ \alpha\in\bigcup_{\beta<\omega_1}\mathscr{M}_\beta\}$ will be independent over $\{B_\alpha|\ \alpha\in\bigcup_{\beta<\omega_1}\mathscr{M}_\beta\}$. To this end, observe that $\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ is again co-countable. For each $\gamma\in\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ define $C_\gamma := \bigcup_{i=1}^{n} A_{\gamma i}$.

If there exists an uncountable subset $\mathscr{P}$ of $\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ such that $\{C_\gamma|\ \gamma\in\mathscr{P}\}$ is independent over $\{B_\gamma|\ \gamma\in\mathscr{P}\}$ then by our inductive hypothesis for n we shall obtain what we want inside of $\mathscr{P}$. Therefore assume that for each uncountable subset $\mathscr{P}$ of $\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ there exist disjoint finite subsets $F_{\mathscr{P}}$ and $G_{\mathscr{P}}$ of $\mathscr{P}$ such that

$$\Big|\bigcap_{\gamma\in F_{\mathscr{P}}} C_\gamma-\bigcup_{\gamma\in G_{\mathscr{P}}} B_\gamma\Big|<\omega.$$

Striving for a contradiction, assume that for each $\delta\in\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ and each co-countable subset $\mathscr{P}$ of $\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ there exist disjoint finite subsets F_δ and G_δ of $\bigcup_{\alpha<\beta}\mathscr{M}_\alpha$ and disjoint finite subsets H_δ and K_δ of $\mathscr{P}$ with

$$\Big|(A_{\delta n+1}\cap\bigcap_{\gamma\in F_\delta} A_{\gamma n+1}\cap\bigcap_{\gamma\in H_\delta} A_\gamma)-\bigcup_{\gamma\in G_\delta\cup K_\delta} B_\gamma\Big|<\omega.$$

Choose an uncountable subset $\mathscr{R}$ of $\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ and for each $\delta\in\mathscr{R}$ a F_δ, G_δ, H_δ and K_δ as above with $\{H_\delta|\ \delta\in\mathscr{R}\}\cup\{K_\delta|\ \delta\in\mathscr{R}\}$ being a mutually disjoint collection and such that

$$\mathscr{R}\cap(\bigcup_{\delta\in\mathscr{R}} H_\delta\cup\bigcup_{\delta\in\mathscr{R}} K_\delta) = \varnothing.$$

The set $\mathscr{R}$ can be constructed inductively using the preceding assumption. Since there are only countably many pairs of disjoint finite subsets of $\bigcup_{\alpha<\beta}\mathscr{M}_\alpha$ it follows that there must be two disjoint finite subsets F and G of $\bigcup_{\alpha<\beta}\mathscr{M}_\alpha$ and an uncountable subset $\mathscr{P}$ of $\mathscr{R}$ such that for each $\delta\in\mathscr{P}$ we have that

$$\Big|(A_{\delta n+1}\cap\bigcap_{\gamma\in F} A_{\gamma n+1}\cap\bigcap_{\gamma\in H_\delta} A_\gamma)-\bigcup_{\gamma\in G\cup K_\delta} B_\gamma\Big|<\omega.$$

For this $\mathscr{P}$ there exist disjoint finite subsets $F_{\mathscr{P}}$ and $G_{\mathscr{P}}$ of $\mathscr{P}$ with $|\bigcap_{\gamma\in F_{\mathscr{P}}} C_\gamma-\bigcup_{\gamma\in G_{\mathscr{P}}} B_\gamma|<\omega$. Since

$$\bigcap_{\gamma\in F_{\mathscr{P}}} A_\gamma\subset\bigcap_{\gamma\in F_{\mathscr{P}}} C_\gamma\cup\bigcup_{\delta\in F_{\mathscr{P}}} A_{\delta n+1}$$

it follows that

$$\Big|(\bigcap_{\gamma\in F} A_{\gamma n+1}\cap\bigcap\{A_\gamma|\ \gamma\in F_{\mathscr{P}}\cup\bigcup_{\delta\in F_{\mathscr{P}}} H_\delta\})-\bigcup\{B_\gamma|\ \gamma\in G\cup G_{\mathscr{P}}\cup\bigcup_{\delta\in F_{\mathscr{P}}} K_\delta\}\Big|<\omega.$$

This contradicts (3) since F and G are disjoint finite subsets of some $\mathscr{M}_{\alpha_0}$ for $\alpha_0<\beta$ and $F_{\mathscr{P}}\cup\bigcup_{\delta\in F_{\mathscr{P}}} H_\delta$ and $G_{\mathscr{P}}\cup\bigcup_{\delta\in F_{\mathscr{P}}} K_\delta$ are disjoint finite subsets of $\bigcap_{\alpha<\beta}\mathscr{W}_\alpha\subset\mathscr{W}_{\alpha_0}$.

Consequently choose $\delta \in \bigcap_{\alpha<\beta} \mathscr{W}_\alpha$ and a co-countable subset $\mathscr{W}_\beta$ of $\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ such that for disjoint finite subsets F and G of $\bigcup_{\alpha<\beta}\mathscr{M}_\alpha$ and disjoint finite subsets H and K of $\mathscr{W}_\beta$ we have that

$$|(A_{\delta n+1}\cap\bigcap_{\gamma\in F}A_{\gamma n+1}\cap\bigcap_{\gamma\in H}A_\gamma)-\bigcup_{\gamma\in G\cup K}B_\gamma|=\omega .$$

Since $\delta\in\bigcap_{\alpha<\beta}\mathscr{W}_\alpha$ it is also true that

$$|(\bigcap_{\gamma\in F}A_{\gamma n+1}\cap\bigcap_{\gamma\in H}A_\gamma)-(B_\delta\cup\bigcup_{\gamma\in G\cup K}B_\gamma)|=\omega .$$

Hence defining $\mathscr{M}_\beta := \bigcup_{\alpha<\beta}\mathscr{M}_\alpha\cup\{\delta\}$ we see that $\mathscr{M}_\beta$ and $\mathscr{W}_\beta$ satisfy (1), (2) and (3). This completes the proof. ■

We need another lemma.

2.3. LEMMA. *Let $n\geq 1$. Let $\{A_\alpha|\ \alpha<\omega_1\}$ and $\{B_\alpha|\ \alpha<\omega_1\}$ be two collections of subsets of N such that for each $\alpha<\omega_1$ we have that $A_\alpha\subset B_\alpha$ and $\{A_\alpha|\ \alpha<\omega_1\}$ is independent over $\{B_\alpha|\ \alpha<\omega_1\}$. Then there exist $\{\alpha_i|\ i<\omega\}\subset\omega_1$ and a $T\subset N$ with T an n-transversal on $\{A_{\alpha_i}|\ i<\omega\}/\{B_{\alpha_i}|\ i<\omega\}$.*

Proof. If $n=1$ then proceed as follows; if for all $y\in A_0$ we have that $|\{\beta<\omega_1|\ y\notin B_\beta\}|\leq\omega$ then $|\{\beta<\omega_1|\ A_0\not\subset B_\beta\}|<\omega_1$. Thus there exist infinitely many $\beta>0$ with $A_0\subset B_\beta$, which is a contradiction. Choose $t_0\in A_0$ and $\mathscr{M}_0\subset\omega_1$ with $|\mathscr{M}_0|=\omega_1$ and $t_0\notin\bigcup_{\alpha\in\mathscr{M}_0}B_\alpha$. Let $\alpha_0:=0$. If $n>1$ then proceed as follows; let $\mathscr{M}_0:=\omega_1-\{0\}$ and $\alpha_0:=0$.

Assume that we have chosen $\{\alpha_0,\dots,\alpha_m\}$, $\{\mathscr{M}_0,\dots,\mathscr{M}_m\}$ and

$$\{t_H|\ H\in[\{0,\dots,m\}]^n\}$$

such that

(1) $0\leq i\leq m$ implies that $\alpha_i\in\mathscr{M}_{i-1}-\mathscr{M}_i$ ($\mathscr{M}_{-1}=\omega_1$),

(2) $\mathscr{M}_m\subset\mathscr{M}_{m-1}\subset\dots\subset\mathscr{M}_0\subset\mathscr{M}_{-1}$ and $|\mathscr{M}_m|=\omega_1$,

(3) $t_H\in\bigcap_{i\in H}A_{\alpha_i}-(\bigcup\{B_{\alpha_i}|\ 0\leq i\leq m,\ i\notin H\}\cup\bigcup\{B_\beta|\ \beta\in\mathscr{M}_{\max H}\})$.

Upon completion of the inductive step $T=\{t_H|\ H\in[\omega]^n\}$ will be an n-transversal on $\{A_{\alpha_i}|\ i<\omega\}/\{B_{\alpha_i}|\ i<\omega\}$. This is true since for all $H\in[\omega]^n$ we have that $T\cap\bigcap_{i\in H}A_{\alpha_i}=\{t_H\}$ and for all $H\in[\omega]^{n+1}$ that $T\cap\bigcap_{i\in H}B_{\alpha_i}=\emptyset$. Clearly $T\subset\bigcup_{i<\omega}A_{\alpha_i}$.

Choose $\alpha_{m+1}\in\mathscr{M}_m$. Enumerate $\{H|\ H\in[\{0,\dots,m+1\}]^n$ and $m+1\in H\}$ as $\{H_j|\ 1\leq j\leq r\}$. For each j such that $1\leq j\leq r$ choose an uncountable subset $\mathscr{P}_j$ of $\mathscr{M}_m$ and a $t_{H_j}\in\bigcap\{A_{\alpha_i}|\ i\in H_j\}-(\bigcup\{B_{\alpha_i}|\ 0\leq i\leq m,\ i\notin H_j\}\cup\bigcup\{B_\beta|\ \beta\in\mathscr{P}_j\})$ such that if $1\leq j<k\leq r$, then $\mathscr{P}_k\subset\mathscr{P}_j$. For if this could not be achieved then there would exist a j with $1\leq j\leq r$ and infinitely many $\beta\notin\{\alpha_i|\ i\in H_j\}$ such that

$$\bigcap_{i\in H_j}A_{\alpha_i}\subset\bigcup\{B_{\alpha_i}|\ 0\leq i\leq m,\ i\notin H_j\}\cup B_\beta$$

which would contradict independence.

Let $\mathscr{M}_{m+1}:=\mathscr{P}_r$. Then $\{\alpha_0,\dots,\alpha_{m+1}\}$, $\{\mathscr{M}_0,\dots,\mathscr{M}_{m+1}\}$ and

$$\{t_H|\ H\in[\{0,\dots,m+1\}]^n\}$$

satisfy (1), (2) and (3). ■

We now can prove the main result in this section. We remaind the reader of the following theorem of F. P. Ramsey [8]: If r and l are two positive integers and the collection $\{W_j:\ 1\leq j\leq 1\}$ satisfies $[N]^r=\bigcup_{j=1}^{l}W_j$, then there exists an infinite $A\subseteq N$ and an s with $1\leq s\leq l$ such that $[A]^r\subseteq W_s$.

2.4. THEOREM. *Let $n\geq 2$. Let $\mathscr{T}\subset\mathscr{P}(N)$ and let $g\colon\mathscr{P}(N)\to[\mathscr{T}]^{<\omega}$ such that for all $A\in\mathscr{P}(N)$ we have that $A=\bigcup g(A)$. Then there is a collection $\mathscr{H}\in[\mathscr{P}(N)]^n$ and for each $H\in\mathscr{H}$ there is a $G_H\in g(H)$ such that*

(i) $\bigcap\mathscr{H}=\emptyset$;

(ii) *for all $\mathscr{B}\in[\{G_H|\ H\in\mathscr{H}\}]^{n-1}$ we have that* $\bigcap\mathscr{B}\neq\emptyset$.

Proof. For $n=2$ choose two disjoint non-empty subsets H and K of N. Choose $G_H\in g(H)-\{\emptyset\}$ and $G_K\in g(K)-\{\emptyset\}$. Let $\mathscr{H}:=\{H,K\}$.

So assume that $n>2$. Let $\{A_\alpha|\ \alpha<\omega_1\}$ be an uncountable independent family of subsets of N. Pick an uncountable subset $\mathscr{M}$ of ω_1 and an $m<\omega$ such that for each $\alpha\in\mathscr{M}$, $|g(A_\alpha)|=m$. For each $\alpha\in\mathscr{M}$ let $g(A_\alpha)=\{A_{\alpha 1},\dots,A_{\alpha m}\}$.

Lemma 2.2 followed by Lemma 2.3 yields $\{\alpha_i|\ i<\omega\}\subset\mathscr{M}$, for each $i<\omega$ an m_i with $1\leq m_i\leq m$ and a $T\subset N$ with T an $n-2$ transversal on $\{A_{\alpha_i m_i}|\ i<\omega\}/\{A_{\alpha_i}|\ i<\omega\}$. Moreover $\{A_{\alpha_i m_i}|\ i<\omega\}$ has finite intersections infinite.

Let $g(T)=\{G_1,\dots,G_l\}$ and $W_j:=\{F\in[N]^{n-2}|\ T\cap\bigcap_{i\in F}A_{\alpha_i m_i}\in G_j\}$ $(1\leq j\leq l)$. Thus $[N]^{n-2}=\bigcup_{j=1}^{l}W_j$. F. P. Ramsey's theorem [8] supplies an infinite $A\subset N$ and an s with $1\leq s\leq l$ such that $[A]^{n-2}\subset W_s$. Choose $n-1$ distinct elements from A; without loss of generality let them be $1,\dots,n-1$. Define $\mathscr{H}:=\{T\}\cup\{A_{\alpha_i}|\ 1\leq i\leq n-1\}$ and let $G_T:=G_s$ and $G_{A_{\alpha_i}}:=A_{\alpha_i m_i}$. Since T is an $n-2$ transversal, $\bigcap\mathscr{H}=\emptyset$. Since $\bigcap\{G_{A_{\alpha_i}}|\ 1\leq i\leq n-1\}\neq\emptyset$ and $[\{1,\dots,n-1\}]^{n-2}\subset W_s$, all $n-1$ fold intersections of the G_H's for $H\in\mathscr{H}$ are non-empty. ■

3. Spaces with finite and infinite compactness number. In the introduction we defined the compactness number $\mathrm{cmpn}(X)$ of X in terms of an open subbase. This can of course also be defined in a dual form; $\mathrm{cmpn}(X)\leq k$ ($k\in\omega$) if X admits a closed subbase $\mathscr{S}$ such that for all $\mathscr{M}\subset\mathscr{S}$ with $\bigcap\mathscr{M}=\emptyset$ there is an $\mathscr{W}\in[\mathscr{M}]^k$ such that $\bigcap\mathscr{W}=\emptyset$ and $\mathrm{cmpn}(X)=\infty$ if for each closed subbase $\mathscr{S}$ for X and for each $k\in N$ there is an $\mathscr{M}\subset\mathscr{S}$ with $\bigcap\mathscr{M}=\emptyset$ while $\bigcap\mathscr{W}\neq\emptyset$ for all $\mathscr{W}\in[\mathscr{M}]^{k-1}$. We prefer to work with closed subbases.

We start with some auxiliary results. The easy proofs are left to the reader.

3.1. PROPOSITION. *Let X_α ($\alpha\in\varkappa$) be a collection of compact Hausdorff spaces. Then* $\mathrm{cmpn}(\prod_{\alpha\in\varkappa}X_\alpha)\leq\sup\{\mathrm{cmpn}(X_\alpha)|\ \alpha\in\varkappa\}$. ■

3.2. LEMMA. *Let X be a compact Hausdorff space for which $k = \mathrm{cmpn}(X)$ is finite. Then there is a closed subbase $\mathcal{S}$ for X which is closed under arbitrary intersections and which in addition realizes k, i.e. for all $\mathcal{M} \subset \mathcal{S}$ with $\bigcap \mathcal{M} = \emptyset$ there is an $\mathcal{W} \in [\mathcal{M}]^k$ such that $\bigcap \mathcal{W} = \emptyset$.* ∎

We now can prove a simple but useful fact.

3.3. THEOREM. *Let X be a compact Hausdorff space and let A be an open and closed subspace of X. Then $\mathrm{cmpn}(A) \leqslant \mathrm{cmpn}(X)$.*

Proof. If $\mathrm{cmpn}(X) = \infty$, then this is a triviality; therefore assume that $\mathrm{cmpn}(X)$ is finite. Let $\mathcal{S}$ be a closed subbase for X, closed under arbitrary intersections, which realizes $\mathrm{cmpn}(X)$. Define $\mathcal{A} := \{S \in \mathcal{S} \mid S \subset A\}$. We claim that $\mathcal{A}$ is a closed subbase for A. If this is the case, then clearly $\mathrm{cmpn}(A) \leqslant \mathrm{cmpn}(X)$.

Indeed, let $a \in A$ and let $C \subset A$ be a closed subset not containing a. Then $(X-A) \cup \{a\}$ and C are disjoint closed subsets of X. By the compactness of X and by the fact that $\mathcal{S}$ is closed under arbitrary intersections, there is a finite $\mathcal{F} \subset \mathcal{S}$ such that $C \subset \bigcup \mathcal{F}$ and $\bigcup \mathcal{F} \cap ((X-A) \cup \{a\}) = \emptyset$. Hence $\mathcal{F} \subset \mathcal{A}$ which implies that $\mathcal{A}$ is a closed subbase for A. ∎

3.4. COROLLARY. *Let X_k $(k \in N)$ be a sequence of compact Hausdorff spaces for which $\mathrm{cmpn}(X_k) = k$ $(k \in N)$. Let Y be the disjoint topological sum of the X_k's. Then every compactification of Y has infinite compactness number.* ∎

The following theorem gives a wide class of compact Hausdorff spaces with infinite compactness number. Recall that two subsets A and B of X are called *completely separated* provided that there is a continuous function $f\colon X \to I$ such that $f[A] = 0$ and $f[B] = 1$. The following fact is easily verified. If U and V are two completely separated subsets of the Tychonoff space X then there is a zero-set Z of X with $U \subset \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}(Z)$ and $Z \cap V = \emptyset$.

3.5. THEOREM. *If X is a non-pseudocompact space and if Y is a compact Hausdorff space which can be mapped continuously onto βX, then $\mathrm{cmpn}(Y) = \infty$.*

Proof. Let X be a non-pseudocompact space and let Y be a compact Hausdorff space which admits a continuous surjection $g\colon Y \to \beta X$. Assume that $\mathrm{cmpn}(Y) = m$ and let $\mathcal{S}$ be a closed subbase for Y, closed under finite intersections, which realizes this fact. Let $C = \{c_n \mid n \in N\}$ be a subset of X for which there exists a continuous map f from X to R with $f(c_n) = n$. Define

$$C_n := \{x \in X \mid n - \tfrac{1}{2} < f(x) < n + \tfrac{1}{2}\}.$$

Then $\mathcal{C} := \{C_n \mid n \in N\}$ is a disjoint collection of cozero-sets of X with $c_n \in C_n$ and such that for each $A \subset N$ the set $\{c_n \mid n \in A\}$ and $X - \bigcup_{n \in A} C_n$ are completely separated. For each $A \subset N$ choose a zero-set $Z_A \subset X$ such that

$$\mathrm{cl}_{\beta X}(\{c_n \mid n \in A\}) \subset \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}(Z_A) \quad \text{and} \quad Z_A \subset \bigcup_{n \in A} C_n.$$

Moreover for each $A \subset N$ choose a finite $\mathcal{S}_A \subset \mathcal{S}$ such that

$$g^{-1}[\mathrm{cl}_{\beta X}(\{c_n \mid n \in A\})] \subset \bigcup \mathcal{S}_A \subset g^{-1}[\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}(Z_A)].$$

For each $n \in N$ let $d_n \in g^{-1}[\{c_n\}]$; let $D := \{d_n \mid n \in N\}$. Let

$$\mathcal{T} := \{fg[S \cap D] \mid S \in \mathcal{S}_A \text{ and } A \subset N\}$$

and define $\bar{g}\colon \mathcal{P}(N) \to [\mathcal{T}]^{<\omega}$ by

$$\bar{g}(A) := \{fg[S \cap D] \mid S \in \mathcal{S}_A\}.$$

Then clearly $A = \bigcup \bar{g}(A)$. Now, by Theorem 2.4, there is an $\mathcal{H} \in [\mathcal{P}(N)]^{m+1}$ and for each $H \in \mathcal{H}$ there is a $G_H \in \bar{g}(H)$ such that

(i) $\bigcap \mathcal{H} = \emptyset$,

(ii) for all $\mathcal{B} \in [\{G_H \mid H \in \mathcal{H}\}]^m$ we have that $\bigcap \mathcal{B} \neq \emptyset$.

For each $H \in \mathcal{H}$ choose $S_H \in \mathcal{S}_H$ such that $G_H = fg[S_H \cap D]$. The contradiction: $\{S_H \mid H \in \mathcal{H}\}$ contradicts $\mathrm{cmpn}(Y) = m$, since

(a) $\bigcap_{H \in \mathcal{H}} S_H \subset \bigcap_{H \in \mathcal{H}} g^{-1}[\mathrm{cl}_{\beta X}(Z_H)] = g^{-1}[\bigcap_{H \in \mathcal{H}} \mathrm{cl}_{\beta X}(Z_H)] = g^{-1}[\mathrm{cl}_{\beta X}(\bigcap_{H \in \mathcal{H}} Z_H)]$ $\subset g^{-1}[\mathrm{cl}_{\beta X}(\bigcap_{H \in \mathcal{H}} (\bigcup_{n \in \mathcal{H}} C_n)] = \emptyset$,

(b) let $\mathcal{H}^1 \in [\mathcal{H}]^m$ and $n \in \bigcap_{H \in \mathcal{H}'} G_H = \bigcap_{H \in \mathcal{H}'} fg[S_H \cap D]$.

Then $d_n \in \bigcap_{H \in \mathcal{H}'} S_H$.

Arriving at this contradiction, we conclude that $\mathrm{cmpn}(Y) = \infty$. ∎

Remark. With the same technique it can be shown that if X is a non-pseudocompact space then βX is not a continuous image of a closed neighborhood retract of a space Y with $\mathrm{cmpn}(Y) < \infty$.

We shall now construct the examples X_k $(k \geqslant 1)$ which were announced in the introduction; first we give some definitions.

Let X be a set; a subset $\mathcal{L} \subset \mathcal{P}(X)$ is called a *linked system* if any two of its members meet. A *maximal linked system* $\mathcal{L} \subset \mathcal{P}(X)$, or briefly mls, is a linked system not properly contained in any other linked system $\mathcal{L}' \subset \mathcal{P}(X)$.

Define

$$\lambda N := \{\mathcal{L} \subset \mathcal{P}(N) \mid \mathcal{L} \text{ is an mls}\}$$

(recall that N is the set of natural numbers). For all $A \subset N$ define $A^+ \subset \lambda N$ by

$$A^+ := \{\mathcal{M} \in \lambda N \mid A \in \mathcal{M}\}.$$

The collection $\{A^+ \mid A \subset N\}$ is taken as a closed subbase for a topology on λN. It is known, cf. de Groot [6], Verbeek [11], that λN is a supercompact totally disconnected separable Hausdorff space; the subbase $\{A^+ \mid A \subset N\}$ realizes 2. The space λN is called the *superextension* of N. For convenience we will recall some properties of λN and of the subbase $\{A^+ \mid A \subset N\}$. The proof of the following lemma can be found in Verbeek [11].

3.6. LEMMA. *Let $\mathcal{M}_0, \mathcal{M}_1 \in \lambda N$. Then*

(a) $\mathcal{M}_0 \neq \mathcal{M}_1$ *iff* $\exists M_i \in \mathcal{M}_i$ $(i \in \{0, 1\})$: $M_0 \cap M_1 = \emptyset$,

(b) *if* $A \subset N$ *then* $A \in \mathcal{M}_0$ *or* $N - A \in \mathcal{M}_0$,

(c) $A \cap B = \emptyset \Rightarrow A^+ \cap B^+ = \emptyset$,

(d) *if* $\mathcal{L} \subset \mathcal{P}(N)$ *is linked then there is an* $\mathcal{M} \in \lambda N$: $\mathcal{L} \subset \mathcal{M}$,

(e) *the mapping* i: $N \to \lambda N$ *defined by* $i(n) := \{A \subset N|\ n \in A\}$ *is an embedding*,

(f) *the closure in* λN *of* $i[N]$ *is equivalent to* βN. ■

We will always indentify N and $i[N]$. Then notice that $B^+ \cap N = B$ for all $B \subset N$. If $A \subset \lambda N$ then define $I(A) \subset \lambda N$ by

$$I(A) := \bigcap \{M^+|\ M \subset N \text{ and } A \subset M^+\}.$$

We need a simple lemma.

3.7. Lemma. *If* $\mathcal{M} \in I(A)$ *then for all* $M \in \mathcal{M}$ *there is an* $\mathcal{A} \in A$ *such that* $M \in \mathcal{A}$.

Proof. Suppose, to the contrary, that there is an $\mathcal{M} \in I(A)$ and an $M \in \mathcal{M}$ such that $M \notin \mathcal{A}$ for all $\mathcal{A} \in A$. Then, by Lemma 3.6(b), $N \setminus M \in \mathcal{A}$ for all $\mathcal{A} \in A$. Hence $A \subset (N \setminus M)^+$ and consequently

$$A \subset I(A) \subset (N \setminus M)^+,$$

this is a contradiction, since $\mathcal{M} \in I(A)$. ■

We now can construct the examples.

3.8. Example. A sequence of compact Hausdorff spaces X_k $(k \geqslant 2)$ with the following properties:

(a) $\operatorname{cmpn}(X_k) = k$ $(k \geqslant 2)$,

(b) if Y is a compact Hausdorff space which can be mapped continuously onto X_k, then $\operatorname{cmpn}(Y) \geqslant k$ $(k \geqslant 2)$.

Indeed, define

$$X_k := \{\mathcal{M} \in \lambda N|\ \forall \mathcal{B} \in [\mathcal{M}]^k\colon (\bigcap \mathcal{B} = \emptyset \Rightarrow \exists B \in \mathcal{B}\colon 1 \in B)\}.$$

Notice that $N \subset X_k$ $(k \geqslant 2)$.

Claim 1. X_k *is closed in* λN, *so that* X_k *is compact, Hausdorff and totally disconnected. Therefore, as* $N \subset X_k$ *also* $\beta N \subset X_k$.

Indeed, take $\mathcal{M} \in \lambda N - X_k$. Let $\mathcal{B} \in [\mathcal{M}]^k$ such that $\bigcap \mathcal{B} = \emptyset$ and for all $B \in \mathcal{B}$: $1 \notin B$. Then $U = \bigcap_{B \in \mathcal{B}} B^+$ is a neighborhood of $\mathcal{M}$ which misses X_k (notice that $\mathcal{B}$ is finite and also that each set of the form M^+ is open and closed in λN, cf. Lemma 3.6(c)(b)). ■

Claim 2. $\operatorname{cmpn}(X_k) \leqslant k$.

Define $\mathcal{T}_k := \{M^+ \cap X_k|\ M \subset N\}$. Then clearly $\mathcal{T}_k$ is a closed subbase for X_k. Let $\mathcal{L} \subset \mathcal{T}_k$ be a subsystem such that for all $\mathcal{B} \in [\mathcal{L}]^k$: $\bigcap \mathcal{B} \neq \emptyset$. We will prove that $\mathcal{L}$ has the finite intersection property and consequently, by Claim 1, $\bigcap \mathcal{L} \neq \emptyset$. This suffices to prove the claim. The proof is by induction.

Assume that any $n-1$ members of $\mathcal{L}$ meet. If $n \in \{1, 2, \ldots, k\}$ then clearly any n members of $\mathcal{L}$ meet. Therefore assume that $n > k$. Let $L_i^+ \cap X_k \in \mathcal{L}$ $(i \in \{1, 2, \ldots, n\})$ and take for each $i \in \{1, 2, \ldots, k+1\}$ a point

$$\mathcal{M}_i \in \bigcap_{\substack{j \leqslant n \\ j \neq i}} (L_j^+ \cap X_k).$$

Define $\mathcal{B} := [\{\mathcal{M}_i|\ i \leqslant k+1\}]^k$ and $\mathcal{A} := [\{\mathcal{M}_i|\ i \leqslant k+1\}]^2$. Moreover, let

$$Z := \bigcap_{B \in \mathcal{B}} I(B) \cap \bigcap_{A \in \mathcal{A}} I(A \cup \{1\}).$$

We claim that this set is nonvoid. Indeed, the system

$$\mathcal{P} := \{M \subset N|\ \exists B \in \mathcal{B}\colon B \subset M^+\} \cup \{M \subset N|\ \exists A \in \mathcal{A}\colon A \cup \{1\} \subset M^+\}$$

clearly is linked, and consequently, by Lemma 3.6(d), there is a point $\mathcal{W} \in \lambda N$ such that $\mathcal{P} \subset \mathcal{W}$. Then obviously $\mathcal{W} \in Z$.

Next, observe that $Z \subset \bigcap_{B \in \mathcal{B}} I(B) \subset \bigcap_{i \leqslant n} L_i^+$ and hence if $Z \cap X_k \neq \emptyset$ we have proved Claim 2.

We prove even more; the set Z is contained in X_k. To this end, let $\mathcal{V} \in Z$ and let $V_i \in \mathcal{V}$ $(i \leqslant k)$ such that $\bigcap_{i \leqslant k} V_i = \emptyset$ and $1 \notin V_i$ for all $i \leqslant k$. We will derive a contradiction, showing that $\mathcal{V} \in X_k$.

Fix $i \leqslant k$ and define $D_i := \{j \leqslant k+1|\ V_i \in \mathcal{M}_j\}$. Let us prove that $|D_i| \geqslant k$. Indeed, suppose that $|D_i| < k$. Choose distinct $j_0, j_1 \in \{1, 2, \ldots, k+1\} - D_i$. Then, since $\mathcal{V} \in Z \subset I(\{\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, 1\})$, by Lemma 3.7 it follows that $V_i \in \mathcal{M}_{j_0}$ or $V_i \in \mathcal{M}_{j_1}$ or $1 \in V_i$, which is impossible.

Now, as $|D_i| \geqslant k$ for all $i \leqslant k$ there is an index $i_0 \in \bigcap_{i \leqslant k} D_i$. Then $V_i \in \mathcal{M}_{i_0}$ for all $i \leqslant k$. But as $\mathcal{M}_{i_0} \in X_k$, this is a contradiction. ■

Claim 3. *If* Y *is a compact Hausdorff space which can be mapped continuously onto* X_k, *then* $\operatorname{cmpn}(Y) \geqslant k$. *In particular* $\operatorname{cmpn}(X_k) = k$ $(k \geqslant 2)$.

Let Y be a compact Hausdorff space and let f: $Y \to X_k$ be a continuous surjection. Suppose that $\mathcal{S}$ is any closed subbase of Y which is closed under arbitrary intersections. For each $B \subset N - \{1\}$ choose a finite $\mathcal{F}(B) \subset \mathcal{S}$ such that $\bigcup \mathcal{F}(B) = f^{-1}[B^+ \cap X_k]$. Notice that B^+ is clopen in λN so that $f^{-}[B^+ \cap X_k]$ is clopen in Y too. For each $n \in N - \{1\}$ pick $d_n \in f^{-1}[\{n\}]$. Define a function g: $\mathcal{P}(N - \{1\}) \to [\mathcal{P}(N - \{1\})]^{<\omega}$ by

$$g(B) := \big\{\{i \in N - \{1\}|\ d_i \in F\}|\ F \in \mathcal{F}(B)\big\}.$$

Notice that $g(B) \in [\mathcal{P}(N - \{1\})]^{<\omega}$ and that $B = \bigcup g(B)$. By Theorem 2.4 there is a collection $\mathcal{H} \in [P(\mathcal{N} - \{1\})]^k$ and for each $H \in \mathcal{H}$ there is a $G_H \in g(H)$ such that

(a) $\bigcap \mathcal{H} = \emptyset$,

(b) for all $\mathcal{B} \in [\{G_H|\ H \in \mathcal{H}\}]^{k-1}$ we have that $\bigcap \mathcal{B} \neq \emptyset$.

For each $H \in \mathscr{H}$ take $S(H) \in \mathscr{S}$ such that $\{i \in N-\{1\}|\ d_i \in S(H)\} = G_H$. Notice that for all $\mathscr{B} \in [\{S(H)|\ H \in \mathscr{H}\}]^{k-1}$ we have that $\bigcap \mathscr{B} \neq \emptyset$ and also that

$$\bigcap_{H \in \mathscr{H}} S(H) \subset \bigcap_{H \in \mathscr{H}} f^{-1}[H^+ \cap X_k] = f^{-1}[\bigcap_{H \in \mathscr{H}} (H^+ \cap X_k)].$$

We claim that $\bigcap_{H \in \mathscr{H}} (H^+ \cap X_k) = \emptyset$, which suffices to prove that $\mathrm{cmpn}(Y) \geqslant k$. Indeed, assume that there is an $\mathscr{M} \in \bigcap_{H \in \mathscr{H}} (H^+ \cap X_k)$. Then, as $\mathscr{H} \in [\mathscr{M}]^k$ and as $\bigcap \mathscr{H} = \emptyset$ there is an $H_0 \in \mathscr{H}$ such that $1 \in H_0$, since $\mathscr{M} \in X_k$. Since $\mathscr{H} \subset \mathscr{P}(N-\{1\})$ this is a contradiction. ■

Remark. With the same technique it can be shown that if X_k is a continuous image of a closed neighborhood retract of a compact Hausdorff space Y, then $\mathrm{cmpn}(Y) \geqslant k$.

In view of Corollary 3.4 we have also constructed the following example.

3.9. Example. A noncompact locally compact and σ-compact space X all compactifications of which have infinite compactness number. ■

4. Discussion and questions. The results derived in the present paper suggest many questions. For example, the spaces constructed in Example 3.8 are not first countable and have cardinality $2^{\mathfrak{c}}$; this suggests the question whether there exist first countable spaces with the same properties.

4.1. Question. Is there a sequence of first countable separable compact Hausdorff spaces X_k for which $\mathrm{cmpn}(X_k) = k$ $(k \geqslant 2)$?

If the answer to this question is affirmative, then the Alexandroff one point compactification of the disjoint topological sum of the X_k's would yield a separable first countable space with infinite compactness number.

The problem whether Hausdorff continuous images of supercompact Hausdorff spaces are supercompact, cf. van Douwen and van Mill [4], is still unsolved. The examples (Example 3.8) constructed in this paper suggest a more general question.

4.2. Question. Let X and Y be compact Hausdorff spaces and let $f\colon X \to Y$ be a continuous surjection. Is $\mathrm{cmpn}(Y) \leqslant \mathrm{cmpn}(X)$?

If this is not true, then we still have the following question:

4.3. Question. Let X and Y be compact Hausdorff spaces and let $f\colon X \to Y$ be a continuous surjection. Is $\mathrm{cmpn}(Y) < \infty$ if $\mathrm{cmpn}(X) < \infty$?

There is a countable space no compactification of which is supercompact (cf. van Mill [7]). In view of Example 3.9 this suggests the following:

4.4. Question. Is there a countable space with only one non-isolated point all compactifications of which have infinite compactness number?

Added in proof. C. F. Mill and J. van Mill have recently constructed a non-supercompact Hausdorff continuous image of a supercompact Hausdorff space.

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