

A PEANO CONTINUUM HOMEOMORPHIC TO ITS OWN SQUARE BUT NOT TO ITS COUNTABLE INFINITE PRODUCT

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ABSTRACT. We give an example of a Peano continuum X with $X^2 \approx X$ but $X^\infty \not\approx X$.

In [2], Fajtlowicz, Holsztyński, Mycielski and Węglorz ask whether there is a (metrizable) continuum X with $X^2 \approx X$ (as a consequence, $X^n \approx X$ for all $n \in \mathbb{N}$) but $X^\infty \not\approx X$. The aim of this note is to construct a Peano continuum with these properties. Our basic tool is a simple but interesting result due to van Douwen [1, 17.1]: if X is a locally compact space homeomorphic to its own square, then X has a compactification homeomorphic to its own square.

Let E be the topological product of countably many copies of the 1-sphere S^1 and let B be the topological sum of $\mathbb{N} \times E$ and a one-point space. Clearly $B^2 \approx B$. As usual, I denotes $[0, 1]$ and Q denotes the Hilbert cube. Put $Y = B \times [0, 1) \times Q$.

LEMMA 1. $Y^2 \approx Y$.

PROOF. $Y^2 = (B \times [0, 1) \times Q) \times (B \times [0, 1) \times Q) \approx (B \times B) \times ([0, 1) \times [0, 1)) \times (Q \times Q) \approx B \times ([0, 1) \times I) \times Q \approx B \times [0, 1) \times Q = Y$. \square

Since Y is clearly locally compact, by van Douwen's result previously cited, Y has a compactification homeomorphic to its own square. It turns out that the compactification of Y we get from the proof of van Douwen's result is a Peano continuum X not homeomorphic to its countable infinite product. For the reader's convenience we repeat van Douwen's argument and give the proof that the resulting compactification is Peano.

LEMMA 2. Y has a Peano compactification X with $X^2 \approx X$.

PROOF. The compactification X of Y is the inverse limit of a suitable inverse sequence $(a_n Y, f_n)_n$ of compactifications $a_n Y$ of Y , where for each $n \in \mathbb{N}$, f_n is a mapping from $a_{n+1} Y$ onto $a_n Y$ with $f_n|_Y = \text{id}_Y$. As is well known, and easy to prove, this implies that $f_n(a_{n+1} Y - Y) = a_n Y - Y$. In order to show that $X^2 \approx X$ we will construct the inverse sequence in such a way that for each $n \in \mathbb{N}$ there is a homeomorphism $h_n: a_{n+1} Y \rightarrow a_n Y \times a_n Y$ so that the diagram

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$$\begin{array}{ccccccc}
 a_1 Y & \xleftarrow{f_1} & a_2 Y & \xleftarrow{f_2} & a_3 Y & \leftarrow & \dots \\
 & & h_1 \downarrow & & h_2 \downarrow & & \\
 & & a_1 Y \times a_1 Y & \xleftarrow{f_1 \times f_1} & a_2 Y \times a_2 Y & \leftarrow & \dots
 \end{array}$$

commutes. As remarked by van Douwen [1, 17.1], the construction is straightforward. Let $a_1 Y$ be the one-point compactification of Y and let h be any homeomorphism from Y onto Y^2 (Lemma 1). There is a compactification $a_2 Y$ of Y so that h extends to a homeomorphism $h_1: a_2 Y \rightarrow a_1 Y \times a_1 Y$. Since $a_1 Y$ is the smallest compactification of Y , the identity mapping on Y extends to a mapping $f_1: a_2 Y \rightarrow a_1 Y$. Similarly, there is a compactification $a_3 Y$ of Y such that h extends to a homeomorphism $h_2: a_3 Y \rightarrow a_2 Y \times a_2 Y$. Define $f_2: a_3 Y \rightarrow a_2 Y$ by $f_2 = h_1^{-1} \circ (f_1 \times f_1) \circ h_2$. It is easily seen that $f_2|_Y = \text{id}_Y$. In the same way we define $a_4 Y, h_3$ and f_3 , etc.

Clearly $X = \varprojlim (a_n Y, f_n)$ is a compactification of Y homeomorphic to its own square. It remains to be shown that X is Peano. First observe that $a_1 Y$ is Peano. This implies that $a_2 Y$ is Peano, being homeomorphic to $a_1 Y \times a_1 Y$. Continuing in this way, we see that all the $a_n Y$'s are Peano continua. So X is the inverse limit of an inverse sequence of continua, hence X is itself a continuum. It remains to be shown that X is locally connected. From the local connectivity of the $a_n Y$'s it is easy to derive that X is locally connected provided that the bonding maps f_n are all monotone. But this is clear from the construction. \square

We will now show that $X^\infty \approx X$.

LEMMA 3. $X^\infty \approx X$.

PROOF. Suppose, to the contrary, that h is a homeomorphism from X onto X^∞ . Since Y , being locally compact, is open in X , $h(Y)$ is open in X^∞ . In addition, since B has an isolated point, Y contains an open copy of $[0, 1) \times Q$. Therefore, X^∞ contains an open copy of $[0, 1) \times Q$, say F . Since F is open and nonempty, we can find an integer n and open nonempty sets $U_i \subset X$ ($i \leq n$) so that

$$U_1 \times U_2 \times \dots \times U_n \times X \times X \times \dots \subset F.$$

Hence X is (homeomorphic to) a retract of some open subspace of F , or, equivalently, X is homeomorphic to a retract of some open subset of $[0, 1) \times Q$. Since $[0, 1) \times Q$ is locally contractible, this implies that X is locally contractible. Then Y , and therefore the countable infinite product of circles, is locally contractible, a contradiction. \square

It might be worthwhile to notice that our example has trivial shape, since it is the inverse limit of an inverse sequence consisting of contractible continua. This is of interest, since if X is a nondegenerate AR which is compact and homeomorphic to its own square, then X is homeomorphic to the Hilbert cube, [4], in which case X is homeomorphic to its countable infinite product.

Let us also remark that Taylor's [3] Example can be used instead of the 1-sphere S^1 . By a double inverse limit construction one can then produce a cell-like map $f: Q \rightarrow Y$ so that Y has trivial shape, $Y^2 \approx Y$ but $Y^\infty \approx Y$. This might be of interest,

since there is a topological, and not algebraic, argument why Y is not an AR. If Y were an AR then, by Toruńczyk's [4] result, $Y \approx Q$, so $Y^\infty \approx Y$, which is not the case. We will sketch the construction.

Let $f: Q \rightarrow X$ be a cell-like map which is not a hereditary shape equivalence. We may assume that $X^\infty \approx X$. Let B be the topological sum of $\mathbb{N} \times X$ and a one-point space. Similarly, let B_0 be the topological sum of $\mathbb{N} \times Q$ and a one-point space. Define $\hat{f}: B_0 \xrightarrow{\text{onto}} B$ so that $\hat{f}(n, q) = (n, f(q))$ for all $(n, q) \in \mathbb{N} \times Q$. Put $E = B \times [0, 1) \times Q$ and $E_0 = B_0 \times [0, 1) \times Q$. There clearly exist, by Lemma 1, a homeomorphism $h: E \rightarrow E^2$ and a homeomorphism $h_0: E_0 \rightarrow E_0^2$ so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{h} & E^2 \\ \hat{f} \times \text{id} \uparrow & & \uparrow (\hat{f} \times \text{id}) \times (\hat{f} \times \text{id}) \\ E_0 & \xrightarrow{h_0} & E_0^2 \end{array}$$

commutes. We can then, using these homeomorphisms, run both E and E_0 through the inverse limit construction of Lemma 2. It is routine to verify that all bonding maps are cell-like and that the inverse system we obtain from E_0 consists of AR's. The second inverse limit maps onto the first inverse limit by a cell-like map (since the bonding maps in the second inverse system are hereditary shape equivalences). The second inverse limit is an AR homeomorphic to its own square, so it is homeomorphic to the Hilbert cube. The first inverse limit Y is homeomorphic to its own square and it can be shown that Y is not homeomorphic to its countable infinite product by a method similar to the one used in the proof of Lemma 2 by noting that X is not an ANR.

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